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# C*-algebras hereditarily containing nonzero, square-zero elements

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*Abstract.* We introduce and study the weak Glimm property for C*-algebras, and also a property we shall call ($HS_0$). We show that the properties of being nowhere scattered and residual ($HS_0$) are equivalent for any C*-algebra. Also, for a C*-algebra with the weak Glimm property, the properties of being purely infinite and weakly purely infinite are equivalent. It follows that for a C*-algebra with the weak Glimm property such that the absolute value of every nonzero, square-zero, element is properly infinite, the properties of being (weakly, locally) purely infinite, nowhere scattered, residual ($HS_0$), residual ($HS_t$), and residual (HI) are all equivalent, and are equivalent to the global Glimm property. This gives a partial affirmative answer to the global Glimm problem, as well as certain open questions raised by Kirchberg and Rørdam.

## 1 Introduction and main results

In this article, we consider certain classes of C*-algebras, extending the class of purely infinite C*-algebras and contained in the class of C*-algebras such that every nonzero hereditary C*-subalgebra has nonzero, square-zero, elements (see Figure 1). Our motivation for introducing these classes is to find (partial) answers to the following three open problems.

(1) In [7], Kirchberg and Rørdam introduced and compared three different notions of infiniteness for a C*-algebra (namely, the properties of being strongly purely infinite, purely infinite, or weakly purely infinite). Later in [2], Blanchard and Kirchberg also introduced the notion of being locally purely infinite, as a generalization of weak pure infiniteness. They asked whether these properties are all equivalent for any C*-algebra. Pasnicu and the third author in [10, Theorem 3.10], and the second and third authors in [3, Theorem 2.1], showed that for C*-algebras with topological dimension zero, the properties of being weakly purely infinite, purely infinite, or strongly purely infinite are equivalent.

(2) The global Glimm problem asks if every nowhere scattered C*-algebra has the global Glimm property. (These properties are recalled below.) The other direction is known: a C*-algebra with the global Glimm property is nowhere scattered (see [15]).

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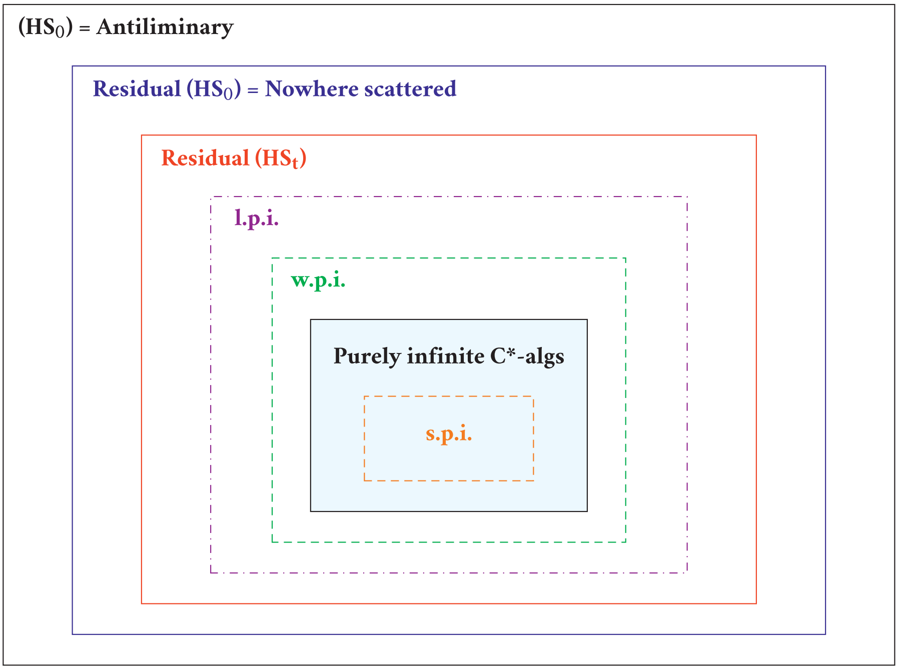


*Figure 1*: Classes of $C^*$-algebras with the property $(HS_0)$.

Thiel and Vilalta in [15, Theorem 7.1] gave a partial affirmative answer to this problem via the Cuntz semigroup.

(3) In [6, Question 4.8], Kirchberg and Rørdam asked whether a $C^*$-algebra $A$ is purely infinite if and only if every nonzero hereditary $C^*$-subalgebra in every quotient of $A$ contains an infinite positive element (i.e., $A$ is residual (HI); see Definition 2.1).

As one can see, these problems are related to two main properties, namely, the global Glimm property and pure infiniteness. We show that $C^*$-algebras with the global Glimm property, in particular, purely infinite $C^*$-algebras, contain nonzero, square-zero, elements. The next natural question is whether nonzero, square-zero, elements are automatically properly infinite and can be chosen to be full.

In this article, we investigate the following three properties for an arbitrary $C^*$-algebra:

(i) the existence of nonzero, square-zero, elements in hereditary $C^*$-subalgebras,
(ii) the fullness of nonzero, square-zero, elements in hereditary $C^*$-subalgebras,
(iii) the proper infiniteness of the absolute values of nonzero, square-zero, elements.

Concerning (i), we define the property $(HS_0)$ (resp., residual $(HS_0)$) which requires that every nonzero hereditary $C^*$-subalgebra (resp., of every quotient) contain a nonzero, square-zero, element (Definition 2.1). We show that property $(HS_0)$ (resp., residual $(HS_0)$) is equivalent to being antiliminary (resp., nowhere scattered).

Concerning (ii), we define the weak Glimm property as a generalization of the global Glimm property (Definition 2.3). We show (Proposition F) that the global

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Glimm property for a Noetherian C*-algebra guarantees the existence of full, square-zero, elements in hereditary C*-subalgebras. In particular, we show that a weakly purely infinite (resp., nowhere scattered) C*-algebra with the weak Glimm property is purely infinite (resp., has the global Glimm property), giving partial affirmative answers to the questions (1) and (2) above.

Finally, assuming (iii) together with the weak Glimm property, we show (Theorem A) that the properties of being (weakly, locally) purely infinite, nowhere scattered, residual ($HS_0$), residual ($HS_t$), and residual (HI) are all equivalent to the global Glimm property, which provides partial answers to the problems (1), (2), and (3) above.

The first main result of this article is as follows. Recall that a C*-algebra is *nowhere scattered* if none of its quotients contains a minimal nonzero hereditary subalgebra (see [16, Definition A]). (Definitions of other concepts are given in Section 2.)

***Theorem A*** *For a* C*-*algebra A with the weak Glimm property and such that the absolute value of every nonzero, square-zero, element is properly infinite, the following statements are equivalent:*

(1) *A is locally purely infinite,*
(2) *A has residual ($HS_t$),*
(3) *A is nowhere scattered,*
(4) *A has residual ($HS_0$),*
(5) *A has the global Glimm property,*
(6) *A is purely infinite,*
(7) *A is weakly purely infinite,*
(8) *A has residual (HI).*

This result has a series of corollaries.

***Corollary B*** *For a* C*-*algebra A in which the absolute value of every nonzero, square-zero, element is properly infinite, the following statements are equivalent:*

(1) *A is purely infinite,*
(2) *A has the global Glimm property,*
(3) *A has the weak Glimm property and residual ($HS_t$),*
(4) *A has the weak Glimm property and residual (HI).*

***Corollary C*** *For a* C*-*algebra A with the weak Glimm property the following statements are equivalent:*

(1) *A is purely infinite,*
(2) *A is weakly purely infinite.*

*Recall that a* C*-*algebra is antiliminary if it contains no nonzero, abelian, positive element. Equivalently, a* C*-*algebra A is antiliminary if every nonzero hereditary* C*-*subalgebra of A is noncommutative (see [1, Definition IV.1.1.6] and [11, p. 191]). A* C*-*algebra A is strictly antiliminary if every quotient of A is antiliminary (see [2, Remark 3.2]).*

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***Corollary D*** *For every* $\mathrm{C}^*$*-algebra A,*

(i) *A has property* ($HS_0$) *if, and only if, it is antiliminary.*
(ii) *A has residual* ($HS_0$) *if, and only if, it is nowhere scattered.*

***Corollary E*** *For every* $\mathrm{C}^*$*-algebra A, the following statements are equivalent:*

(1) *A has the global Glimm property,*
(2) *A is nowhere scattered and has the weak Glimm property,*
(3) *A is antiliminary and has the weak Glimm property.*

*A* $\mathrm{C}^*$*-algebra A with the global Glimm property is known to be nowhere scattered, and hence no hereditary* $\mathrm{C}^*$*-subalgebra of A admits a finite-dimensional irreducible representation (see [16, Theorem 3.1]). Recall that a* $\mathrm{C}^*$*-algebra A is Noetherian (resp., Artinian) if it satisfies the ascending (resp., descending) chain condition for (closed two-sided) ideals. Note that being Noetherian (or Artinian) passes to hereditary* $\mathrm{C}^*$*-subalgebras and quotients, and is preserved under extensions (see [13] for details on chain conditions for* $\mathrm{C}^*$*-algebras). It's worth mentioning that Noetherian (and Artinian)* $\mathrm{C}^*$*-algebras are not necessarily separable (for example, consider* $\mathbb{B}$*, the* $\mathrm{C}^*$*-algebra of bounded operators on a separable infinite-dimensional Hilbert space). Furthermore, every hereditary* $\mathrm{C}^*$*-subalgebra of every Noetherian* $\mathrm{C}^*$*-algebra has a compact primitive spectrum and therefore has a strictly full element (see [10, Lemma 2.1]). This helps us to make a connection between the existence of full, square-zero, elements in all hereditary* $\mathrm{C}^*$*-subalgebras of a Noetherian* $\mathrm{C}^*$*-algebra and the global Glimm property for that* $\mathrm{C}^*$*-algebra (see Proposition F). The existence of full, square-zero, elements has considerable implications on the structure of the unitary group of the* $\mathrm{C}^*$*-algebra (see [15, p. 4714]).*

***Proposition F*** *For a Noetherian* $\mathrm{C}^*$*-algebra A, the following statements are equivalent:*

(1) *A has the global Glimm property,*
(2) *every hereditary* $\mathrm{C}^*$*-subalgebra of A contains a full, square-zero, element.*

***Corollary G*** *For a* $\mathrm{C}^*$*-algebra A which is both Artinian and Noetherian, the following statements are equivalent:*

(1) *A has residual* ($HS_0$)*,*
(2) *A has the global Glimm property.*

## 2 Preliminaries

In this article, by an ideal we always mean a closed two-sided ideal. For an ideal $I$ (resp., $\mathrm{C}^*$-subalgebra $B$) in a $\mathrm{C}^*$-algebra $A$, we write $I \trianglelefteq A$ (resp., $B \leq A$). Also, for a hereditary (resp., and full) $\mathrm{C}^*$-subalgebra $B$ in a $\mathrm{C}^*$-algebra $A$, we write $B \leq_h A$ (resp., $B \leq_{h,f} A$). For a subset $S \subseteq A$, $\overline{ASA}$ is the ideal generated by $S$. We simply write $\overline{AaA}$ when $S = \{a\}$. An element $a \in A$ is called *full* if $A = \overline{AaA}$ (see [1, p. 91]). An element $a \in A^+$ is *strictly full* if $(a - \varepsilon)_+$ is full for some (and so for all sufficiently small) $\varepsilon > 0$ (see [8, p. 46]). A positive element $a$ in a $\mathrm{C}^*$-algebra $A$ is *abelian* if the hereditary $\mathrm{C}^*$-subalgebra $\overline{aAa}$ is commutative. A $\mathrm{C}^*$-algebra is of *type I* if every nonzero quotient

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contains a nonzero, abelian, positive element. An *ideal-quotient* of a $C^*$-algebra $A$ is an ideal of a quotient (equivalently, a quotient of an ideal) of $A$ (see [16]).

We denote by $\mathrm{Prim}(A)$ the set of primitive ideals in a $C^*$-algebra $A$, the latter being considered as a topological space with the hull-kernel topology (see [1]). Throughout this article, the symbols $\otimes$ and $\sim_M$ are used to denote the minimal tensor product and strong Morita equivalence, respectively. The $C^*$-algebra of all compact operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$. When $\mathcal{H}$ is separable and infinite-dimensional, we simply write $\mathcal{K}$ for $\mathcal{K}(\mathcal{H})$.

### 2.1 Purely infinite $C^*$-algebras

Given $a, b \in A^+$, we say that $a$ is Cuntz subequivalent to $b$ (and write $a \precsim b$), if there is a sequence $\{x_k\}_{k=1}^{\infty} \subseteq A$ such that $x_k^* b x_k \to a$, in norm. We say that $a$ and $b$ are *Cuntz equivalent* (and write $a \sim_{cu} b$), if $a \precsim b$ and $b \precsim a$ (see [6]). A positive element $a$ is called *infinite* if there exists a nonzero, positive element $b$ in $A$ such that $a \oplus b \precsim a \oplus 0$ in $\mathbb{M}_2(A)$. If $a$ is not infinite, then we say that a is *finite*. If $a$ is nonzero and if $a \oplus a \precsim a \oplus 0$ in $\mathbb{M}_2(A)$, then $a$ is said to be *properly infinite* (see [6, Definition 3.2]). A $C^*$-algebra $A$ is said to be *purely infinite* (p.i.) if there are no characters on $A$, and for every pair of positive elements $a, b$ in $A$, if $a \in \overline{AbA}$, then $a \precsim b$ (see [6, Definition 4.1]). A $C^*$-algebra $A$ is said to have property *pi-n* if the $n$-fold direct sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite in $\mathbb{M}_n(A)$, for every nonzero, positive element $a$ in $A$. If $A$ is pi-$n$ for some $n$, then $A$ is called *weakly purely infinite* (w.p.i.) (see [7, Definition 4.3]). A $C^*$-algebra $A$ is *strongly purely infinite* (s.p.i.) (see [7, Definition 5.1]) if for every

$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in \mathbb{M}_2(A)^+$$

and every $\varepsilon > 0$, there exist $a_1, a_2 \in A$ such that

$$\left\| \begin{pmatrix} a_1^* & 0 \\ 0 & a_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| < \varepsilon.$$

A $C^*$-algebra $A$ is said to be *locally purely infinite* (l.p.i.) if for every primitive ideal $J$ of $A$ and every element $b \in A^+$ with $\|b + J\| > 0$, there is a nonzero, stable, $C^*$-subalgebra $D$ of the hereditary $C^*$-subalgebra generated by $b$ such that $D$ is not included in $J$ [2, Definition 1.3].

### 2.2 Square-zero elements in a $C^*$-algebra

An element $r$ in a $C^*$-algebra will be called *square-zero* if $r^2 = 0$ (see [15]). We denote by $S_0(A)$ the set of all square-zero elements in the $C^*$-algebra $A$. The existence of nonzero, square-zero, elements implies that the underlying $C^*$-algebra is noncommutative (see [4, Corollary 5]). These elements also play the role of generators for a large class of noncommutative $C^*$-algebras. To be more explicit, for a unital $C^*$-algebra $A$, denote by $\mathrm{Inv}(A)$ the set of all invertible elements in $A$, and set $1 - S_0(A) := \{1 - r \mid r \in S_0(A)\}$. Then the following assertions hold:

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(i) If $A$ is noncommutative, then

$$(1 - S_0(A))^* = (1 - S_0(A))^{-1} = 1 - S_0(A) \subseteq \mathrm{Inv}(A).$$

(ii) If $A$ has no one-dimensional irreducible representation, then

$$C^*(1 - S_0(A)) = A.$$

To see (i), note that, for every $r \in S_0(A)$,

$$(1 - r)(1 + r) = (1 + r)(1 - r) = 1,$$

and hence every element of $1 - S_0(A)$ is invertible, and $(1 - r)^{-1} = (1 + r)$. Moreover, for every $r \in S_0(A)$, $(r^*)^2 = (r^2)^* = 0$, and so $(1 - r)^* = 1 - r^* \in 1 - S_0(A)$. Therefore, the sets $(1 - S_0(A))^*$, $(1 - S_0(A))^{-1}$ and $1 - S_0(A)$ coincide.

Next, let us prove (ii). For every $r \in S_0(A)$, we have

$$r = 1 - (1 - r) \in \mathrm{span}(1 - S_0(A));$$

that is, $S_0(A) \subseteq \mathrm{span}(1 - S_0(A))$. By [5, Theorem 4.3] and [12, Theorems 1.3 and 4.2], we have, by the hypothesis on $A$,

$$A = \overline{A[A, A]A} = C^*([A, A]) = C^*(\mathrm{span}(S_0(A))) = C^*(S_0(A)),$$

where $[A, A]$ is the set of all commutators in $A$. Thus, $C^*(1 - S_0(A)) = A$, as claimed.

Note that though both sets $S_0(A)$ and $1 - S_0(A)$ generate $A$ as a C$^*$-algebra, while $(1 - S_0(A)) \subseteq \mathrm{Inv}(A)$ we only have $S_0(A) \subseteq \overline{\mathrm{Inv}(A)}$. Indeed, if $r$ is a square-zero element of $A$, then its spectral radius is zero and

$$r = \lim_{n\to\infty}(r - \frac{1}{n}1) \in \overline{\mathrm{Inv}(A)}.$$

## 2.3 The weak Glimm property

In this section, we define the property (HS$_0$) and the weak Glimm property for C$^*$-algebras. Everywhere, $A$ is an arbitrary C$^*$-algebra.

***Definition 2.1*** (i) $A$ is said to have property $(HS_0)$ (resp., $(HI)$) if every nonzero hereditary
C$^*$-subalgebra of $A$ contains a nonzero, square-zero (resp., infinite) element.

(ii) $A$ is said to have residual $(HS_0)$ (resp., residual $(HI)$) if every quotient of $A$ has property (HS$_0$) (resp., (HI)).

***Definition 2.2*** (i) $A$ is said to have property $(HS_t)$ if every nonzero hereditary C$^*$-subalgebra of $A$ contains a nonzero, stable, C$^*$-subalgebra.

(ii) $A$ is said to have residual $(HS_t)$ if every quotient of $A$ has property (HS$_t$).

***Remark 2.1*** Define $\mathcal{C}_1$ (resp., $\mathcal{C}_2$, and $\mathcal{C}_3$) to be the class of all C$^*$-algebras which have a nonzero, square-zero, element (resp., an infinite element, a nonzero, stable, C$^*$-subalgebra). These classes are upwards directed. We recall that a class $\mathcal{C}$ of C$^*$-algebras is called *upwards directed* (or upwards hereditary) if whenever $A$ is a C$^*$-algebra

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which contains a $C^*$-subalgebra isomorphic to a $C^*$-algebra in $\mathcal{C}$, then $A \in \mathcal{C}$ (see [9, Definition 5.1]). By [9, Definition 5.2], and Definitions 2.1 and 2.2 above,

(i) $A$ has property ($HS_0$) (resp., (HI), or ($HS_t$)) if and only if all hereditary subalgebras belong to $\mathcal{C}_1$ (resp., $\mathcal{C}_2$, or $\mathcal{C}_3$), and

(ii) $A$ has residual ($HS_0$) (resp., residual (HI), or residual ($HS_t$)) if and only if quotients of hereditary subalgebras belong to $\mathcal{C}_1$ (resp., $\mathcal{C}_2$, or $\mathcal{C}_3$).

We can use the permanence results of Section 5 of [9] to obtain the following permanence properties:

(1) The properties ($HS_0$), (HI), ($HS_t$), residual ($HS_0$), residual (HI), and residual ($HS_t$) are preserved under:

- (i) passage to hereditary $C^*$-subalgebras (see [9, Proposition 5.10]),
- (ii) strong Morita equivalence for separable $C^*$-algebras (see [9, Corollary 5.13]),
- (iii) direct limits of separable $C^*$-algebras (see [9, Proposition 5.9]),
- (iv) crossed products by arbitrary actions of $\mathbb{Z}_2$ (see [9, Corollary 5.6]),
- (v) crossed products by Rokhlin actions of $\mathbb{Z}$ on unital $C^*$-algebras (see [9, Corollary 5.4]).

(2) For an ideal $I \trianglelefteq A$, $A$ has residual ($HS_0$) (resp., residual (HI), or residual ($HS_t$)) if and only if both $I$ and $A/I$ do (see [9, Proposition 5.8]).

(3) $A$ has property ($HS_0$) (resp., property (HI), property ($HS_t$), residual ($HS_0$), residual (HI), or residual ($HS_t$)) if and only if $A \otimes \mathcal{K}$ or $\mathbb{M}_n(A)$ do, for some (all) $n \geq 1$ (see [9, Propositions 5.11 and 5.12]).

A $C^*$-algebra $A$ is said to have the *global Glimm property* if for each natural number $n$, for each positive element $a$ in $A$, and for each $\varepsilon > 0$ there is a $*$-homomorphism $\varphi : \mathbb{M}_n(C_0((0,1])) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ belongs to the ideal of $A$ generated by the image of $\varphi$ (see [7, Definition 4.12]); or, equivalently, if, for every $a \in A^+$ and $\varepsilon > 0$ there exists an element $r \in \overline{aAa}$ with $r^2 = 0$ such that $(a-\varepsilon)_+ \in \overline{ArA}$ (see [15, Theorem 3.6((1) $\iff$ (3))]).

***Definition 2.3*** A $C^*$-algebra $A$ will be said to have the *weak Glimm property*, if for every $\varepsilon > 0$ and $a \in A^+$ with $\{0\} \neq S_0(\overline{aAa})$, there exists $r \in S_0(\overline{aAa})$ such that $(a-\varepsilon)_+ \in \overline{ArA}$.

Clearly, the global Glimm property implies the weak Glimm property. The converse is not necessarily true, even for simple $C^*$-algebras. For example, the $C^*$-algebras $A = \mathbb{C}$ and $B = \mathbb{C} \oplus \mathcal{O}_n$, for the Cuntz algebra $\mathcal{O}_n$ with $2 \leq n \leq \infty$, have the weak Glimm property, but not the global Glimm property ($\mathbb{C}$ has no nonzero, square-zero, element).

## 3 Proofs of the main results

**Proof of Theorem A** First, note that we use the assumption "the weak Glimm property" only to prove the implication (4) $\Rightarrow$ (5) below, and the assumption "the absolute value of every nonzero, square-zero, element is properly infinite" only to prove the implication (5) $\Rightarrow$ (6) below.

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(1) ⇒ (2): Let $Q$ be a quotient of $A$, and let $0 \neq H \leq_h Q$. Since local pure infiniteness passes to hereditary C*-subalgebras and quotients (see [2, Proposition 4.1(iii)]), $H$ is locally purely infinite. In particular, $H$ contains a nonzero, stable, C*-subalgebra $D$, as desired. ($D$ may be chosen to be nonzero in any specified primitive quotient of $H$.)

(2) ⇒ (3): Suppose that $A$ is not nowhere scattered. Then [16, Theorem 3.1((1) ⟺ (6))] implies that $A$ has a GCR irreducible representation, say $\pi : A \to \mathcal{B}(\mathcal{H})$. Thus, $\mathcal{K}(\mathcal{H}) \cap \pi(A) \neq \{0\}$, and in fact $\mathcal{K}(\mathcal{H}) \subseteq \pi(A)$, so

$$\mathcal{K}(\mathcal{H}) \cong \pi^{-1}(\mathcal{K}(\mathcal{H}))/\ker(\pi) \trianglelefteq A/\ker(\pi).$$

Since by hypothesis $A$ has residual ($\mathrm{HS_t}$), and (by Remark 2.1) residual ($\mathrm{HS_t}$) passes to ideals and quotients, $\mathcal{K}(\mathcal{H})$ has property ($\mathrm{HS_t}$). By [2, Proposition 3.1], a simple C*-algebra such that every nonzero hereditary C*-subalgebra contains a nonzero, stable, C*-subalgebra must be purely infinite. This is a contradiction since $\mathcal{K}(\mathcal{H})$ is simple and elementary, and so is of type I [1, Corollary IV.1.2.6], and no C*-algebra of type I is purely infinite (see [6, Proposition 4.4]).

(3) ⇒ (4): Let $Q$ be a quotient of $A$, and let $0 \neq H \leq_h Q$. Note that the property of being nowhere scattered passes to hereditary C*-subalgebras and quotients (by [16, Proposition 4.1]). Since $H$ is nowhere scattered, it is not of type I (see [16, Theorem 3.1((1) ⟺ (3))]), and so is noncommutative [1, Section IV.1.1.4]. But every noncommutative C*-algebra has a nonzero, square-zero, element (Proposition II.6.4.14 of [1]). This shows that $A$ has residual ($\mathrm{HS_0}$).

(4) ⇒ (5): Suppose that $A$ has residual ($\mathrm{HS_0}$), $0 \neq a \in A^+$, and $\varepsilon > 0$. Then $\overline{aAa}$ has a nonzero, square-zero, element. By the weak Glimm property, there is $r \in \overline{aAa}$ with $r^2 = 0$ and $(a-\varepsilon)_+ \in \overline{ArA}$. Thus $A$ has the global Glimm property.

(5) ⇒ (6): Let $0 \neq a \in A^+$ and $\varepsilon > 0$. Since $A$ has the global Glimm property, there is $r \in S_0(\overline{aAa})$ with $(a-\varepsilon)_+ \in \overline{ArA}$. Since $|r| \sim_{cu} rr^* \sim_{cu} r^*r$, the orthogonal elements $x_1 := rr^*$ and $x_2 := r^*r$ are properly infinite, because, by hypothesis, $|r|$ is properly infinite. On the other hand, we have that

$$(a-\varepsilon)_+ \in \overline{Ax_1A} = \overline{Ax_2A},$$

and hence, $(a-\varepsilon)_+ \precsim x_1$ and $(a-\varepsilon)_+ \precsim x_2$, by [6, Proposition 3.5(ii)]. Now, [6, Proposition 3.3((i) ⟺ (iii))] implies that $a$ is properly infinite. This shows that $A$ is purely infinite (see [6, Lemma 4.2]).

(6) ⇒ (1), and (6) ⇒ (7): These implications are known (see [2, p. 465] and [7, Definition 4.1]).

(7) ⇒ (8): Let $A$ be a weakly purely infinite C*-algebra, $Q$ be a quotient of $A$, and $H \leq_h Q$. Then $H$ is weakly purely infinite, and hence pi-$n$ for some $n \in \mathbb{N}$, because being weakly purely infinite passes to hereditary C*-subalgebras and quotients (see [7, Definition 4.3 and Proposition 4.5(ii) and (iii)]). Since $H$ has no nonzero finite-dimensional representation, Glimm's lemma [6, Proposition 4.10] implies the existence of nonzero, pairwise orthogonal, pairwise equivalent, positive, elements $t_1, t_2, \dots, t_n$ in $H$. By [6, Lemmas 2.8 and 2.9],

$$t_1 \otimes 1_n \sim t_1 \oplus t_2 \oplus \cdots \oplus t_n \sim \sum_{i=1}^{n} t_i.$$

Since $H$ is pi-$n$, the element $t_1 \otimes 1_n$ is properly infinite in $\mathbb{M}_n(H)$. But for $x := \sum_{i=1}^n t_i \in H$, $x \sim t_1 \otimes 1_n$. Thus $x$ is properly infinite in $H$.

(8) $\Rightarrow$ (3): Assume that $A$ is not nowhere scattered. Then $A$ has a nonzero elementary ideal-quotient, say $L$ (see [16, Theorem 3.1((1) $\iff$ (5))]). In particular, $L$ is a simple $C^*$-algebra [1, Definition IV.1.2.1]. Since residual (HI) passes to ideals and quotients (by Remark 2.1), the $C^*$-algebra $L$ has residual (HI). Let $0 \neq a \in L^+$ and $L_a := \overline{aLa}$. Since $L$ has residual (HI), there is a properly infinite element $x$ in $L_a$. But, we have that $x \precsim a$ and $a \in L_a = \overline{L_a x L_a}$. Therefore, by [6, Lemma 3.8], $a$ is properly infinite in $L_a$, and so in $L$. Hence $L$ is purely infinite (by [6, Lemma 4.2]), which is a contradiction, as $L$ is of type I (see [6, Proposition 4.4]). ■

***Remark 3.1*** As shown in the proof of Theorem A, for every $C^*$-algebra $A$ (without any extra assumption), (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, it is known that s.p.i. $\Rightarrow$ p.i. $\Rightarrow$ w.p.i. $\Rightarrow$ l.p.i. (see [2, 7]). Therefore, the inclusions in Figure 1 are confirmed. Note that antiliminary (and nowhere scattered) $C^*$-algebras are not necessarily purely infinite. For example, let $A_\infty$ be the universal Glimm algebra (see [11, Section 6.4.2]), and let $\mathcal{Z}$ be the Jiang-Su algebra (see [14]). Then $A_\infty$ is a simple, unital AF-algebra which is antiliminary (see [11, Proposition 6.4.3, Theorem 6.5.7, and Section 8.12.8]), and $\mathcal{Z}$ is a simple, unital $C^*$-algebra which is nowhere scattered (see [16, Remark 9.2]). But these $C^*$-algebras are stably finite (see [1, p. 420] and [14, p. 1]), and hence are not purely infinite. In particular, $A_\infty$ and $\mathcal{Z}$ do not have residual ($HS_t$), because if they had such a property, then since they are simple, they must be purely infinite (by [2, Proposition 3.1]), a contradiction.

**Proof of Corollary B** Note first that, by [2, Remark 2.9(iv)] and [15, Remark 3.2], any purely infinite $C^*$-algebra has the global (and so the weak) Glimm property. Thus the implication (1) $\Rightarrow$ (2) is true. The converse holds, by the proof of the implication (5) $\Rightarrow$ (6) of Theorem A (where the weak Glimm property is not necessary). Thus (1) and (2) are equivalent. Moreover, (locally) purely infinite $C^*$-algebras have residual ($HS_t$), by Remark 3.1, and residual (HI), by [6, Theorem 4.16]. Thus the implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) hold. The converse implications hold by Theorem A. ■

**Proof of Corollary C** Suppose that $A$ is weakly purely infinite, $0 \neq a \in A^+$, and $\varepsilon > 0$. Then $A$ is locally purely infinite (see [2, Proposition 4.11]), and hence $A$ has residual ($HS_0$), by Remark 3.1. Thus $\overline{aAa}$ has a nonzero, square-zero, element. But $A$ has the weak Glimm property, and so there exists an element $r \in \overline{aAa}$ with $r^2 = 0$ such that $(a - \varepsilon)_+ \in \overline{ArA}$. This shows that $A$ has the global Glimm property. Now, by [7, Proposition 4.15], $A$ ■

**Proof of Corollary D** To prove, use [4, Corollary 5] and [16, Theorem 3.1((1) $\iff$ (2))]. ■

**Proof of Corollary E** It is known that a $C^*$-algebra with the global Glimm property is nowhere scattered [15, Theorem 7.1], and so is strictly antiliminary (use [16, Theorem 3.1((1) $\iff$ (2))] – or Corollary D). This proves (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). For the converse implications, suppose that $A$ has the weak Glimm property and is nowhere scattered (resp., antiliminary), and consider $a \in A^+$ and $\varepsilon > 0$. By Corollary D, $A$ has

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residual ($HS_0$) (resp., property ($HS_0$)). Thus $\{0\} \neq S_0(\overline{aAa})$, and hence, by the weak Glimm property, there is $r \in \overline{aAa}$ such that $r^2 = 0$ and $(a-\varepsilon)_+ \in \overline{ArA}$. This shows that $A$ has the global Glimm property. ∎

**Proof of Proposition F** (1) ⇒ (2): Suppose that $A$ has the global Glimm property and $H$ is a hereditary C*-subalgebra of $A$. Then $H$ is a Noetherian C*-algebra with the global Glimm property (see [15, Theorem 3.10]). Since $H$ is Noetherian, $\mathrm{Prim}(H)$ is compact, and so $H$ has a strictly full element $h$ (see [10, Lemma 2.1]). Thus, there exists $\varepsilon > 0$ such that $(h-\varepsilon)_+$ is full in $H$. Since $H$ has the global Glimm property, there is $r \in S_0(\overline{hHh})$ such that $(h-\varepsilon)_+ \in \overline{HrH}$. Hence $H = \overline{H(h-\varepsilon)_+H} \subseteq \overline{HrH}$, and so $H = \overline{HrH}$. Thus, (1) implies (2).

(2) ⇒ (1): This is clear, since if $a \in A^+$, $\varepsilon > 0$, and $A_a := \overline{aAa}$, then $A_a$ has a square-zero element $r$ with $A_a = \overline{A_a r A_a}$. Thus,

$$(a-\varepsilon)_+ \in A_a \subseteq \overline{ArA},$$

and so $A$ has the global Glimm property. ∎

**Proof of Corollary G** By Corollary D, we only need to show (1) ⇒ (2). Note that a Noetherian and Artinian C*-algebra $A$ has a composition series of ideals as $0 = I_0 \trianglelefteq I_1 \trianglelefteq I_2 \trianglelefteq \cdots \trianglelefteq I_n = A$ such that $I_j/I_{j-1}$ is simple for each $1 \leq j \leq n$. Since residual ($HS_0$) passes to ideals and quotients (see Remark 2.1), and the global Glimm property is preserved under extensions [15, Theorem 3.10], it suffices to show that the implication holds for simple C*-algebras. But this is obvious, since if $A$ is simple with residual ($HS_0$), $a \in A^+$, and $\varepsilon > 0$, then there is $r \in S_0(\overline{aAa})$ such that $(a-\varepsilon)_+ \in \overline{ArA} = A$. ∎

The results obtained above give partial positive answers to questions (1)–(3) of Section 1. We recall that question (1) asks whether the four properties s.p.i., p.i., w.p.i., and l.p.i. are always equivalent. Corollary C shows that having the assumption "the weak Glimm property" implies that the two properties p.i. and w.p.i. are equivalent, and Theorem A((1) ⇔ (6) ⇔ (7)) shows that if we also have the assumption "the absolute value of every nonzero, square-zero, element is properly infinite," then the three properties p.i., w.p.i., and l.p.i. are equivalent. Also, question (2) asks if the two properties being nowhere scattered and the global Glimm property are always equivalent. Corollaries E and G show that these two properties are equivalent for all C*-algebras with the weak Glimm property and all Artinian and Noetherian C*-algebras. Finally, question (3) asks whether the two properties p.i. and residual (HI) are always equivalent. Theorem A((6) ⇔ (8)) shows that with the two assumptions "the weak Glimm property" and "the absolute value of every nonzero, square-zero, element is properly infinite," the two properties p.i. and residual (HI) are equivalent.

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