NEW APPROXIMATIONS FOR WIENER INTEGRALS, WITH ERROR ESTIMATES

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1. Introduction. The principal theorem of this paper, a generalization of a theorem given by R. H. Cameron **(2)**, provides a means of approximating certain Wiener integrals to any desired degree of accuracy by an $(n+1)$-fold Riemann integral with sufficiently large n. The generalization is in the use of a general complete orthonormal set of functions, whereas Cameron's paper used only the odd harmonic set.

Let C' be the class of real-valued functions $x(t)$ defined on $[0, 1]$ and such that $x(0) = 0$ and which are continuous except perhaps for one left continuous jump. Let C be the class of continuous members of C'. Finally, let $\{\alpha_n(s): n = 1, 2, 3, \ldots\}$ be a complete orthonormal set of right continuous functions of bounded variation on $[0, 1]$ and normalized to vanish at $s = 1$, and let

$$\gamma_{i,i} = \int_0^1 \left(\int_0^t \alpha_i(s)\, ds \right)^2 dt.$$

In order that $\int_C F[x(\cdot)]\, dx$ can be approximated by the techniques of this paper, the existence of a certain $(n+1)$-fold Riemann integral with integrand dependent both on F and on the α's mentioned above is required. This condition being satisfied, F is given a third-degree "Taylor's expansion with remainder." Specifically for each $x_0(\cdot) \in C$, there are assigned functions

$$K_i(x_0(\cdot)|\, s_1, \ldots, s_i), \qquad i = 1, 2, 3,$$

which are right continuous and of bounded variation **(4**, pp. 345–7**)** in any j $(j \leqslant i)$ of the variables for the other $i - j$ variables fixed. (In the integrals written below, the symbol $\int_0^1$ is used rather than $\int_0^1 (i) \int_0^1$ and $d_{(i)}$ replaces the usual d subscripted with i subscripted s's.) $F[x_0(\cdot) + x(\cdot)]$ is, for each pair $[x_0(\cdot), x(\cdot)] \in C \times C'$, written as

$$F[x_0(\cdot) + x(\cdot)] =$$

$$F[x_0(\cdot)] + \sum_{i=1}^{3} \int_0^1 x(s_1) \ldots x(s_i) d_{(i)}\, K_i(x_0(\cdot)|s_1, \ldots, s_i) + Q[x_0(\cdot), x(\cdot)],$$

the above equation defining $Q[x_0(\cdot), x(\cdot)]$. The right side of this equation is called the third-degree Taylor's expansion of F with remainder about $x_0(\cdot)$.

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If Q satisfies the relation

$$|Q[x_0(\cdot), x(\cdot)]| \leqslant A\left\{\int_0^1 [x(s)]^2\, ds\right\}^2 \exp\left\{B\int_0^1 [x_0(s)]^2\, ds + B\int_0^1 [x(s)]^2\, ds\right\}$$

with $0 \leqslant B < \pi^2/12$, and if the set of α's satisfies an extra condition (shown to hold for all Sturm–Liouville sets (**4**, vol. 2, p. 272), the Fourier sine, and the Haar set) it is shown that the error of the approximation is

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right),$$

and in fact a specific estimate is given for the error. It is also shown that the order

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)$$

for the error is the best possible for general sets of α's satisfying the extra condition. In certain cases, however, it is shown that the error of approximation is

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^2.$$

2. Notation. Let $\{\alpha_n(s): n = 1, 2, 3, \ldots\}$ be a complete orthonormal set of functions of bounded variation on $[0, 1]$ normalized to be right continuous and to vanish at $s = 1$. A slight modification of a result in (**5**, p. 356) shows that $\alpha_n(s)$ can be decomposed into its increasing and decreasing components $\alpha_n^{(1)}(s)$ and $\alpha_n^{(2)}(s)$, each also normalized in this way. Consider the measurable space $(0, 1]$ with the σ-ring generated by half-open intervals

$$(a, b]: 0 \leqslant a < b \leqslant 1.$$

To $a, b]$ assign the signed measure $\alpha_n(b) - \alpha_n(a)$. Thus for $0 \leqslant p < q \leqslant 1$ there exists, for any function Radon measurable for $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$ (of course Borel measurable will suffice), the Radon integral

$$\int_{(p,q]} f(s)\, d\alpha_n(s)$$

(defined as the difference

$$\int_{(p,q]} f(s)\, d\alpha_n^{(1)}(s) - \int_{(p,q]} f(s)\, d\alpha_n^{(2)}(s)$$

of Lebesgue–Stieltjes integrals). This Radon integral will also be denoted by

$$\int_p^q f(s)\, d\alpha_n(s).$$

Again, a slight modification of a result in (**5**, p. 339) proves (via the above-mentioned decomposition of the integrator) that if $f(s)$ is bounded and Radon measurable and if

$$\int_p^q f(s)\, d\alpha_n(s)$$

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exists as Riemann–Stieltjes, then so does it as Radon and the two interpretations yield the same value. The prefix R.S. will sometimes be used to emphasize that an integral is to be interpreted as Riemann–Stieltjes. Now $x(s) \in C'$ is bounded and Borel measurable, so c_n can be defined by

$$c_n = -\int_0^1 x(s)\, d\alpha_n(s).$$

Let

$$\beta_n(t) = \int_0^t \alpha_n(s)\, ds \qquad \text{for } t \in [0, 1];\ n = 1, 2, 3, \ldots,$$

$$\rho(s, t) = \begin{cases} 1 & \text{for } 0 \leqslant s < t \leqslant 1, \\ 0 & \text{for } 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$x^n(t) = \sum_{i=1}^{n} c_i\, \beta_i(t), \qquad n = 1, 2, 3, \ldots,$$

$$\psi_n(\xi, t) = \sum_{i=1}^{n} \xi_i\, \beta_i(t), \qquad n = 1, 2, 3, \ldots,$$

$$\gamma_{i,j} = \int_0^1 \beta_i(t)\, \beta_j(t)\, dt, \qquad i, j = 1, 2, 3, \ldots,$$

$$e_n(\xi) = \pi^{-n/2} \exp(-\xi_1^2 - \ldots - \xi_n^2).$$

It is to be noted that for $\rho(s, t)$ considered as a function of t, the result

$$\rho^n(s, t) = \sum_{i=1}^{n} \alpha_i(s)\, \beta_i(t)$$

can be obtained as follows.

For fixed s, let $\rho^*(s, t)$ be $\rho(s, t)$ modified to be right continuous at s. Then

$$\int_0^1 \rho(s, t)\, d\alpha_i(t) = \int_0^s \rho(s, t)\, d\alpha_i(t) + \int_s^1 \rho(s, t)\, d\alpha_i(t)$$

$$= \text{R.S.} \int_0^s \rho(s, t)\, d\alpha_i(t) + \int_s^1 \rho^*(s, t)\, d\alpha_i(t)$$

(because $\rho(s, t)$ is continuous on $[0, s]$ and $\rho(s, t) = \rho^*(s, t)$ on $(s, 1]$)

$$= \text{R.S.} \int_0^s \rho(s, t)\, d\alpha_i(t) + \text{R.S.} \int_s^1 \rho^*(s, t)\, d\alpha_i(t)$$

(because $\rho^*(s, t)$ is continuous on $[s, 1]$).

Integration by parts gives at once the result

$$\int_0^1 \rho(s, t)\, d\alpha_i(t) = -\alpha_i(s).$$

Again, in connection with n-dimensional Radon integrals, it is to be noted that the symbol $\int_0^1$ is used rather than $\int_0^1 (n) \int_0^1$ and that the usual d subscripted

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with n subscripted s's will be replaced by $d_{(n)}$. As another abbreviation the expression

$$\int_{-\infty}^{\infty} G(f(\xi), n)\, d\mu_n$$

will be used in place of

$$\int_{-\infty}^{\infty}(n) \int_{-\infty}^{\infty} e_n(\xi) G(f(\xi), n)\, d\xi_1 \ldots d\xi_n.$$

If $F[x(\cdot)]$ is defined on C and the expression has meaning, we define

$$I_n(F) = \int_{-\infty}^{\infty} F[\psi_n(\xi, \cdot)]\, d\mu_n$$

and if $F[x(\cdot)]$ is defined on C' and the expression has meaning, we define

$$J_n(F) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \{F[\psi_n(\xi, \cdot) + \rho(s, \cdot)/\sqrt{2} - \rho^n(s, \cdot)/\sqrt{2}] \\ + F[\psi_n(\xi, \cdot) - \rho(s, \cdot)/\sqrt{2} + \rho^n(s, \cdot)/\sqrt{2}]\}\, ds\, d\mu_n.$$

3. The principal theorem. The main results of the paper are contained in the following theorem and its corollary.

THEOREM 6. *Let $F[x(\cdot)]$ be defined on C' and integrable on C and be such that $J_n(F)$ exists as a finite quantity. For each $x_0(\cdot) \in C$ let*

$$K_i(x_0(\cdot)|s_1, \ldots, s_i), \qquad i = 1, 2, 3,$$

be right continuous and of bounded variation (**4**, pp. 345-7) *in any j ($j \leqslant i$) of the variables for the other $i - j$ variables fixed. For each pair $[x_0(\cdot), x(\cdot)] \in C \times C'$ let*

$$(3.1)\quad P[x_0(\cdot), x(\cdot)] = F[x_0(\cdot)] + \sum_{i=1}^{3}\int_0^1 x(s_1)\ldots x(s_i) d_{(i)} K_i(x_0(\cdot)|s_1, \ldots, s_i).$$

For each pair $[x_0(\cdot), x(\cdot)] \in C \times C'$ let $Q[x_0(\cdot), x(\cdot)]$ be defined by the equation

$$(3.2)\qquad F[x_0(\cdot) + x(\cdot)] = P[x_0(\cdot), x(\cdot)] + Q[x_0(\cdot), x(\cdot)].$$

Then

$$\int_C F[x(\cdot)]dx = J_n(F) + \epsilon_n$$

where

$$\epsilon_n = \int_{-\infty}^{\infty}\left\{\int_C Q[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)]\, dx \right.\\ - \frac{1}{2}\int_0^1 (Q[\psi_n(\xi, \cdot), \rho(s, \cdot)/\sqrt{2} - \rho^n(s, \cdot)/\sqrt{2}] \\ \left. + Q[\psi_n(\xi, \cdot), -\rho(s, \cdot)/\sqrt{2} + \rho^n(s, \cdot)/\sqrt{2}])\, ds\right\} d\mu_n.$$

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In particular, if $Q = 0$, $\epsilon_n = 0$, *and if*

$$|Q[x_1(\cdot), x(\cdot)]| \leqslant A \left\{\int_0^1 [x(s)[^2\, ds\right\}^2 \exp\left\{B \int_0^1 [x_0(s)]^2\, ds + B \int_0^1 [x(s)]^2\, ds\right\}$$

with $0 \leqslant B < \pi^2/12$, *then*

$$|\epsilon_n| \leqslant [\sqrt[3]{(15^2)/4}]A \sqrt[6]{[\sec\sqrt{(3B)}]}\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^2 + \frac{1}{4} A\sqrt{[\sec\sqrt{B}]} \int_0^1 \left\{\int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt\right\}^2 \times \exp\left\{\frac{B}{2} \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt\right\} ds.$$

COROLLARY. *If* $F[x(\cdot)]$ *satisfies the conditions of Theorem* 6 *and if*

$$\sum_{i,j=1}^{N} \gamma_{i,j}\, \alpha_i(s)\, \alpha_j(s) \leqslant M$$

for $N = 1, 2, 3, \ldots$ *and all* $s \in [0, 1]$ (this condition holds for all Sturm–Liouville sets of α's) (**4**, 2, p. 272), *then* L_n *exists for* $n = 1, 2, \ldots$, *where*

$$L_n = \lim_{N\to\infty} \int_0^1 \left\{\int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\, \beta_i(t)\right]^2 dt\right\}^2 ds.$$

Moreover,

$$|\epsilon_n| \leqslant [\sqrt[3]{(15^2)/4}]A \sqrt[6]{[\sec\sqrt{(3B)}]}\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^2 + \tfrac{1}{4}A\sqrt{[\sec\sqrt{B}]}\exp\{B(1+M)^2/2\}L_n,$$

and a crude estimate is provided for L_n *by*

$$L_n \leqslant 4M \sum_{i=n+1}^{\infty} \gamma_{i,i}.$$

In particular, if

$$L_n = O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^{1+a} \qquad \textit{where } 0 \leqslant a \leqslant 1,$$

then

$$|\epsilon_n| = O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^{1+a}.$$

Theorem 1 (**2**, p. 118, Theorem 3) and Theorem 3 (which is proved by means of Theorem 2; both Theorems 2 and 3 appear in Cameron's unpublished notes, though the proofs in this paper differ from Cameron's) as well as Theorems 4 and 5 of the present paper are the main results used in the proof of Theorem 6. The corollary to the theorem is proved by means of Lemmas 3.1 and 3.2. These theorems and the lemmas are stated below, after which proofs of Theorem 6 and its corollary follow.

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THEOREM 1. *Suppose that $F[x(\cdot)]$ is, for $x(\cdot) \in C'$, of the form*

$$F[x(\cdot)] = K_0 + \sum_{i=1}^{3} \int_0^1 x(s_1) \ldots x(s_i)\, d_{(i)} K_i(s_1, \ldots, s_i)$$

in which the K_i's are right continuous and of bounded variation (**4**, pp. 345–7) *in any j ($j \leqslant i$) of the variables for the other $i - j$ variables fixed, and where the integrals are understood to be Radon integrals. Then*

$$\int_c F[x(\cdot)]\, dx = \frac{1}{2} \int_0^1 \{F[\rho(s, \cdot)/\sqrt{2}] + F[-\rho(s, \cdot)/\sqrt{2}]\}\, ds.$$

THEOREM 2. *Let $F[x(\cdot)]$ be bounded and continuous in the Hilbert topology in the space C. Then*

$$\lim_{n\to\infty} I_n(F) = \int_c F[x(\cdot)]\, dx.$$

THEOREM 3. *If $F[x(\cdot)] \in L_1(C)$, then*

$$\int_c F[x(\cdot)]\, dx = \int_{-\infty}^{\infty} \int_c F[x(\cdot) - x^n(\cdot) + \psi_n(\xi, \cdot)]\, dx\, d\mu_n.$$

THEOREM 4.

$$\int_c \left\{ \int_0^1 [x(t) - x^n(t)]^2\, dt \right\}^3 dx \leqslant \frac{15}{8} \left(\sum_{i=n+1}^{\infty} \gamma_{i,i} \right)^3.$$

THEOREM 5. *For fixed $i \in \{1, 2, 3\}$, let $H(t_1, \ldots, t_i)$ be right continuous and of bounded variation* (**4**, pp. 345–7) *in any j ($j \leqslant i$) of its variables for the other $i - j$ variables fixed. Then there exists $N(s_1, \ldots, s_i)$ of bounded variation and right continuous such that for all $x(t) \in C'$*

$$\int_0^1 [x(t_1) - x^n(t_1)] \ldots [x(t_i) - x^n(t_i)]\, d_{(i)} H(t_1, \ldots, t_i)$$

is of the form

$$\int_0^1 x(s_1) \ldots x(s_i)\, d_{(i)} N(s_1, \ldots, s_i).$$

(The integrals are to be interpreted as Radon if $x(t) \in C' - C$ and as either Radon or Riemann–Stieltjes if $x(t) \in C$.)

LEMMA 3.1. *Let $\{\alpha_1(s), \alpha_2(s), \alpha_3(s), \ldots\}$ be such that*

$$\int_0^1 \left[\sum_{i=i}^{n} \alpha_i(s)\, \beta_i(t) \right]^2 dt \leqslant M \qquad \textit{for } n = 1, 2, 3, \ldots \textit{ and all } s \in [0, 1].$$

Then

(i) $$\int_0^1 [\rho^n(s, t) - \rho(s, t)]^2\, dt \leqslant (1 + \sqrt{M})^2,$$

(ii) $$\int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\, \beta_i(t) \right]^2 dt \leqslant 4M,$$

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(iii) $$\int_0^1 \left\{ \int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\, \beta_i(t) \right]^2 dt \right\}^2 ds \leqslant 4M \sum_{i=n+1}^{\infty} \gamma_{i,i},$$

(iv) $\lim_{N\to\infty} \int_0^1 \left\{ \int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\, \beta_i(t) \right]^2 dt \right\}^2 ds$ *exists and equals*

$$\int_0^1 \left\{ \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt \right\}^2 ds.$$

LEMMA 3.2. *The condition required in Lemma* 3.1 *is satisfied by all Sturm-Liouville sets of* α'*s*.

The *proof of Theorem* 6 follows. Since

$$\psi_n(\xi, t) \in C, \quad [x(\cdot) - x^n(\cdot)] \in C', \quad \text{and } [\rho(s, \cdot)/\sqrt{2} - \rho^n(s, \cdot)/\sqrt{2}] \in C'$$

for fixed $\xi_1, \ldots, \xi_n$, it follows from (3.2) that the replacements below of F's by sums of P's and Q's are valid. Since $F[x(\cdot)] \in L\ (C)$, Theorem 3 applies to yield

$$\int_c F[x(\cdot)]\, dx = \int_{-\infty}^{\infty} \int_c F[\psi_n(\xi, \cdot) + x(\cdot) - x^n(\cdot)]\, dx\, d\mu_n$$

or

$$\int_c F[x(\cdot)]\, dx = \int_{-\infty}^{\infty} \int_c \{P[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)] + Q[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)]\}\, dx\, d\mu_n. \tag{3.3}$$

Now since $F[x(\cdot)] \in L(C)$ so that

$$\left| \int_c F[x(\cdot)]\, dx \right| < \infty,$$

and since by assumption $J_n(F)$ exists as a finite quantity, i.e.

$$\int_{-\infty}^{\infty} \frac{1}{2} \left\{ \int_0^1 F[\psi_n(\xi, \cdot) + \rho(s, \cdot)/\sqrt{2} - \rho^n(s, \cdot)/\sqrt{2}] + F[\psi_n(\xi, \cdot) - \rho(s, \cdot)/\sqrt{2} + \rho^n(s, \cdot)/\sqrt{2}] \right\} ds\, d\mu_n \tag{3.4}$$

exists as a finite quantity, consider ϵ_n which by definition is given by the second member of the following equations (the third member follows from (3.3)):

$$\begin{aligned} \epsilon_n &= \int_c F[x(\cdot)]\, dx \\ &\quad - \int_{-\infty}^{\infty} \frac{1}{2} \int_0^1 \{F[\psi_n(\xi, \cdot) + \rho(s, \cdot)/\sqrt{2} - \rho^n(s, \cdot)]/\sqrt{2}] \\ &\qquad + F[\psi_n(\xi, \cdot) - \rho(s, \cdot)/\sqrt{2} + \rho^n(s, \cdot)/\sqrt{2}]\}\, ds\, d\mu_n \\ &= \int_{-\infty}^{\infty} \int_c \{P[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)] + Q[\psi_n(\xi, \cdot), x(\cdot) - x^n(\cdot)]\} dx\, d\mu_n \end{aligned} \tag{3.5}$$

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$$-\frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \{P[\psi_n(\xi,\cdot),\rho(s,\cdot)/\sqrt{2}-\rho^n(s,\cdot)/\sqrt{2}]$$
$$+Q[\psi_n(\xi,\cdot),\rho(s,\cdot)/\sqrt{2}-\rho^n(s,\cdot)/\sqrt{2}]$$
$$+P[\psi_n(\xi,\cdot),-\rho(s,\cdot)/\sqrt{2}+\rho^n(s,\cdot)/\sqrt{2}]$$
$$+Q[\psi_n(\xi,\cdot),-\rho(s,\cdot)/\sqrt{2}+\rho^n(s,\cdot)/\sqrt{2}]\}\,ds\,d\mu_n$$
$$=\int_{-\infty}^{\infty}\Big[\int_C\{P[\psi_n(\xi,\cdot),x(\cdot)-x^n(\cdot)]+Q[\psi_n(\xi,\cdot),x(\cdot)-x^n(\cdot)]\}dx$$
$$-\frac{1}{2}\int_0^1\{P[\psi_n(\xi,\cdot),\rho(s,\cdot)/\sqrt{2}-\rho^n(s,\cdot)/\sqrt{2}]$$
$$+Q[\psi_n(\xi,\cdot),\rho(s,\cdot)/\sqrt{2}-\rho^n(s,\cdot)/\sqrt{2}]$$
$$+P[\psi_n(\xi,\cdot),-\rho(s,\cdot)/\sqrt{2}+\rho^n(s,\cdot)/\sqrt{2}]$$
$$+Q[\psi_n(\xi,\cdot)-\rho(s,\cdot)/\sqrt{2}+\rho^n(s,\cdot)/\sqrt{2}]\}ds\Big]d\mu_n.$$

The combining of the two integrals on R^n in passing from the second to the third member of (3.5) can be justified by the finiteness of $\int_C F[x(\cdot)]\,dx$ and of $J_n(F)$. It will now be noticed that the fact that $\int_C F[x(\cdot)]\,dx$ and $J_n(F)$ are each finite implies that in the last member of (3.5) the integral on C and the integral on $[0, 1]$ are each finite for almost all $\xi_1,\ldots,\xi_n \in R^n$. Also, it follows from Theorem 5 that $P[\psi_n(\xi,\cdot), x(\cdot)-x^n(\cdot)]$ is, for fixed $\xi_1,\ldots,\xi_n \in R^n$, a functional satisfying the hypothesis of Theorem 1. Now it can be shown that if $G[x(\cdot)]$ is a functional satisfying the hypothesis of Theorem 1, then $G[A\rho(s,\cdot)]$ is measurable and bounded in s on $[0, 1]$ for each constant value of A (for a proof one can assume without loss of generality that $A > 0$ and K_i is monotonically decreasing in all of its arguments for each i). Thus the integrals (in s) involving

$$P[\psi_n(\xi,\cdot),\mp\rho(s,\cdot)/\sqrt{2}\pm\rho^n(s,\cdot)/\sqrt{2}]$$

in the last member of (3.5) can be combined and then, by Theorem 1, cancelled with the integral (with respect to x) of $P[\psi_n(\xi,\cdot), x(\cdot)-x^n(\cdot)]$. Thus there follows

$$\epsilon_n=\int_{-\infty}^{\infty}\Big\{\int_C Q[\psi_n(\xi,\cdot),x(\cdot)-x^n(\cdot)]\,dx$$
$$-\frac{1}{2}\int_0^1(Q[\psi_n(\xi,\cdot),\rho(s,\cdot)/\sqrt{2}-\rho^n(s,\cdot)/\sqrt{2}]$$
$$+Q[\psi_n(\xi,\cdot)-\rho(s,\cdot)/\sqrt{2}+\rho^n(s,\cdot)/\sqrt{2}])\,ds\Big\}\,d\mu_n.$$

If

$$|Q[x_0(\cdot),x(\cdot)]|\leqslant A\Big\{\int_0^1[x(s)]^2\,\mathrm{ds}\Big\}^2\exp\Big\{B\int_0^1[x_0(s)]^2\,ds+B\int_0^1[x(s)]^2\,ds\Big\},$$

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there follows

$$(3.6)\quad |\epsilon_n| \leqslant A \int_{-\infty}^{\infty}\Big[\int_c \Big\{\int_0^1 [x(s) - x^n(s)]^2\,ds\Big\}^2 \exp\Big\{B\int_0^1 [\psi_n(\xi, s)]^2\,ds + B\int_0^1 [x(s) - x^n(s)]^2\,ds\Big\}\,dx + (2/2)\int_0^1\Big\{\int_0^1 [\rho(s,t)/\sqrt{2} - \rho^n(s,t)/\sqrt{2}]^2\,dt\Big\}^2 \exp\Big\{B\int_0^1 [\psi_n(\xi,t)]^2\,dt + B\int_0^1 [\rho(s,t)/\sqrt{2} - \rho^n(s,t)/\sqrt{2}]^2\,dt\Big\}\,ds\Big]\,d\mu_n.$$

Now

$$(3.7)\quad \int_{-\infty}^{\infty}\Big[\int_c\Big\{\int_0^1 [x(s) - x^n(s)]^2\,ds\Big\}^2 \exp\Big\{B\int_0^1 [\psi_n(\xi, s)]^2\,ds + B\int_0 [x(s) - x^n(s)]^2\,ds\Big\}\,dx\Big]\,d\mu_n$$

(the complete notation for n-fold integration now being used)

$$= \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\int_c\Big\{\sqrt[3]{[e_n(\xi)]^2}\Big[\int_0^1 [x(s) - x^n(s)]\Big]^2 \times \sqrt[3]{[e_n(\xi)]}\exp B\int_0^1\{[\psi_n(\xi,s)]^2 + [x(s) - x^n(s)]^2\}ds\Big\}dx\,d\xi_1\ldots d\xi_n$$

$$\leqslant \sqrt[3]{\Big[\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\int_c e_n(\xi)\Big[\int_0^1 [x(s) - x^n(s)]^2 ds\Big]^3 dx\,d\xi_1\ldots d\xi_n\Big]^2}$$

$$\times \sqrt[3]{\Big[\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\int_c e_n(\xi)\exp\Big\{3B\int_0^1\{[\psi_n(\xi,s)]^2 + [x(s) - x^n(s)]^2\}ds\Big\}dx\,d\xi_1\ldots d\xi_n\Big]}$$

(because of the Hölder inequality)

$$= \sqrt[3]{\Big\{\int_c\Big[\int_0^1 [x(s) - x^n(s)]^2 ds\Big]^3 dx\Big\}^2}$$

$$\times \sqrt[3]{\Big\{\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\int_c \sqrt{[e_n(\xi)]}\exp\Big\{\frac{3B}{2}\int_0^1 [\psi_n(\xi,s) + x(s) - x^n(s)]^2 ds\Big\} \times \sqrt{[e_n(\xi)]}\exp\Big\{\frac{3B}{2}\int_0^1 [x(s) - x^n(s) - \psi_n(\xi,s)[^2 ds\Big\}dx\,d\xi_1\ldots d\xi_n\Big\}}$$

(because $a^2 + b^2 = [(a+b)^2 + (a-b)^2]/2$)

(compressed integration notation again used)

$$\leqslant \sqrt[3]{\Big\{\int_c\Big[\int_0^1 [x(s) - x^n(s)]^2 ds\Big]^3 dx\Big\}^2}$$

$$\times \sqrt[6]{\Big[\int_{-\infty}^{\infty}\int_c \exp\Big\{3B\int_0^1 [x(s) - x^n(s) + \psi_n(\xi,s)]^2\,ds\Big\}\,dx\,d\mu_n\Big]}$$

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$$\times \sqrt[6]{\left[\int_{-\infty}^{\infty} \int_{c} \exp\left\{3B \int_0^1 [-x(s) + x^n(s) + \psi_n(\xi, s)]^2 \, ds\right\} dx \, d\mu_n\right]}$$

$$= \sqrt[3]{\left\{\int_{c}\left[\int_0^1 [x(s) - x^n(s)]^2 \, ds\right]^3 dx\right\}^2}$$

$$\times \sqrt[3]{\left[\int_{-\infty}^{\infty} \int_{c} \exp\left\{3B \int_0^1 [x(s) - x^n(s) + \psi_n(\xi, s)]^2 \, ds\right\} dx \, d\mu_n\right]}$$

(because if $G[x(\cdot)]$ is integrable, so also is $G[-x(\cdot)]$ and the integrals of the two are equal)

$$= \sqrt[3]{\left\{\int_{c}\left[\int_0^1 [x(s) - x^n(s)]^2 \, ds\right]^3 dx\right\}^2} \sqrt[3]{\left[\int_{c} \exp\left\{3B \int_0^1 [x(s)]^2 \, ds\right\} dx\right]}$$

(because of Theorem 3)

$$= \sqrt[6]{[\sec\sqrt{(3B)}]} \sqrt[3]{\left\{\int_{c}\left[\int_0^1 [x(s) - x^n(s)]^2 \, ds\right]^3 dx\right\}^2}$$

(because of formula (4.15)).

Also

$$(3.8) \quad \int_{-\infty}^{\infty} \exp\left\{B \int_0^1 [\psi_n(\xi, t)]^2 \, dt\right\} d\mu_n \int_0^1 \left[\int_0^1 [\rho(s, t)/\sqrt{2} - \rho^n(s, t)/\sqrt{2}]^2 \, dt\right]^2 \exp\left\{B \int_0^1 [\rho(s, t)/\sqrt{2} - \rho^n(s, t)/\sqrt{2}]^2 \, dt\right\} ds$$

$$= \int_{c} \int_{-\infty}^{\infty} \exp\left\{B \int_0^1 [\psi_n(\xi, t)]^2 \, dt\right\} d\mu_n \, dx \times \frac{1}{4} \int_0^1 \left\{\int_0^1 [\rho(s, t) - \rho^n(s, t)]^2 \, dt\right\}^2$$

$$\times \exp\left\{\frac{B}{2} \int_0^1 [\rho(s, t) - \rho^n(s, t)]^2 \, dt\right\} ds \leqslant \frac{1}{4} \int_{c} \int_{-\infty}^{\infty} \exp\left\{B \int_0^1 [\psi_n(\xi, t)]^2 \, dt\right.$$

$$\left. + B \int_0^1 [x(s) - x^n(s)]^2 \, ds\right\} d\mu_n \, dx \times \int_0^1 \left\{\int_0^1 [\rho(s, t) - \rho^n(s, t)]^2 \, dt\right\}^2$$

$$\times \exp\left\{\frac{B}{2} \int_0^1 [\rho(s, t) - \rho^n(s, t)]^2 \, dt\right\} ds$$

(note integration notation again)

$$\leqslant \frac{1}{4} \sqrt{\left[\int_{-\infty}^{\infty} \int_{c} \exp\left\{B \int_0^1 [\psi_n(\xi, s) + x(s) - x^n(s)]^2 \, ds\right\} dx \, d\mu_n\right.}$$

$$\times \sqrt{\left[\int_{-\infty}^{\infty} \int_{c} \exp\left\{B \int_0^1 [-x(s) + x^n(s) + \psi_n(\xi, s)]^2 \, ds\right\} dx \, d\mu_n\right]}$$

$$\times \int_0^1 \left\{\int_0^1 [\rho(s, t) - \rho^n(s, t)]^2 \, dt\right\}^2 \exp\left\{\frac{B}{2} \int_0^1 [\rho(s, t) - \rho^n(s, t)]^2 \, dt\right\} ds$$

$$= \frac{1}{4} \int_{c} \exp\left\{B \int_0^1 [x(s)]^2 \, ds\right\} dx$$

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$$\times \int_0^1 \left\{ \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt \right\}^2 \exp\left\{\frac{B}{2} \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt\right\} ds$$

$$= \frac{1}{4} \sqrt{[\sec\sqrt{B}]} \int_0^1 \left\{ \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt \right\}^2$$

$$\times \exp\left\{\frac{B}{2} \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt\right\} ds$$

(because of formula (4.15)).

From (3.6), (3.7), and (3.8) there now follows

$$(3.9) \quad |\epsilon_n| \leqslant A \sqrt[6]{[\sec\sqrt{(3B)}]^3} \sqrt{\left\{ \int_0^1 \left[\int_0^1 [x(s) - x^n(s)]^2\, ds \right]^3 dx \right\}^2}$$

$$+ \frac{1}{4} A \sqrt{(\sec\sqrt{B})} \int_0^1 \left\{ \int_0^1 [\rho(s,t) - \rho^n(s,t)[^2\, dt \right\}^2$$

$$\times \exp\left\{\frac{B}{2} \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt\right\} ds.$$

Theorem 4 provides the estimate which, when substituted into (3.9), completes the proof of Theorem 6.

The *proof of the corollary* is as follows: From Lemma 3.1 (i),

$$(3.10) \quad \exp\left\{\frac{B}{2} \int_0^1 [\rho(s,t) - \rho^n(s,t)]^2\, dt\right\} \leqslant \exp\left\{\frac{B}{2}(1 + \sqrt{M})^2\right\}.$$

Then, from the estimate for $|\epsilon_n|$ in Theorem 6, from Lemma 3.1 (iv) and from (3.10), the asserted inequality concerning $|\epsilon_n|$ follows. Lemma 3.1 (iii)provides the crude estimate required and Lemma 3.2 proves the parenthetical statement of the corollary.

4. The proofs of Theorems 2 to 5 and of Lemmas 3.1 and 3.2. First a sequence of seven lemmas from which will be obtained Theorems 2 and 3 will be given.

LEMMA 4.1. *Let* $\{f_i(s)\colon i = 1, 2, \ldots, n\}$ *be an orthogonal set of functions of bounded variation on* $[0, 1]$. *Then there exists a normal function* $\theta(s)$ *of bounded variation on* $[0, 1]$ *such that* $f_1(s), \ldots, f_n(s), \theta(s)$ *is an orthogonal set of functions of bounded variation on* $[0, 1]$.

Proof. Let

$$s^j - \sum_{i=1}^n a_{j,i} f_i(s), \qquad j = 1, 2, \ldots, n, n+1,$$

be the component of

$$s^j, \qquad j = 1, 2, \ldots, n, n+1,$$

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which is orthogonal to the set

$$\{f_i(s): i = 1, 2, \ldots, n\}.$$

It is easy to show that the assumption that each of the equations

$$s^j - \sum_{i=1}^{n} a_{j,i} f_i(s) = 0, \qquad j = 1, 2, \ldots, n, n+1,$$

holds for almost all $s \in [0, 1]$ leads to a contradiction of a corollary of the fundamental theorem of algebra. (To do this, one multiplies the jth equation by r_j and notes that r's, not all zero, can be chosen to satisfy

$$\sum_{j=1}^{n+1} a_{j,i} r_j = 0: \qquad i = 1, 2, \ldots, n.)$$

The existence of the required $\theta(s)$ follows at once.

LEMMA 4.2. *For fixed* $(t, t') \in \{(t, t'): 0 \leqslant t < t' \leqslant 1\}$

$$t = \sum_{i=1}^{\infty} \beta_i(t)\, \beta_i(t').$$

In particular, if $0 \leqslant t \leqslant 1$, *then*

$$t = \sum_{i=1}^{\infty} \beta_i^2(t).$$

Proof. Suppose that $\rho(s, t)$ and $\rho(s, t')$ as functions of $s \in [0, 1]$ are given generalized orthonormal expansions in terms of the set of functions $\{\alpha_i(s): i = 1, 2, \ldots\}$. Parseval's equation for the integral of $\rho(s, t)\rho(s, t')$ on $[0, 1]$ yields the result.

LEMMA 4.3. *Let* $\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)$ *be any orthonormal set of functions of bounded variation on* $[0, 1]$. *Let*

$$Q_1(t) = \sqrt{\left(\sum_{i=n+1}^{\infty} \beta_i^2(t)\right)},$$

$$Q_2(t, t') = \sum_{i=n+1}^{\infty} \beta_i(t)\, \beta_i(t'),$$

$$Q_3(t, t') = Q_2(t, t')/Q_1(t)$$

(unless $Q_1(t) = 0$, in which case let $Q_3(t, t') = 0$ also),

$$R_1(t, t') = \sqrt{\{[Q_1(t')]^2 - [Q_3(t, t')]^2\}}$$

(note that $[Q_1(t')]^2 \geqslant [Q_3(t, t')]^2$ follows at once from the Schwarz inequality),

$$R_2(t, t', t'') = \frac{Q_2(t', t'') - Q_3(t, t')Q_3(t, t'')}{R_1(t, t')}$$

(unless $R_1(t, t') = 0$, in which case let $R_2(t, t', t'') = 0$ also),

$$U_1(t, t', t'') = \sqrt{\{[R_1(t, t'')]^2 - [R_2(t, t', t'')]^2\}}$$

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(note that it will follow in the course of the proof of the lemma that

$$[R_1(t, t'')]^2 \geqslant [R_2(t, t', t'')]^2).$$

Then

(i) *There exists a function* $q(s, t)$ *defined on* $[0, 1] \times [0, 1]$, *which is, as a function of* s, *for each fixed* t, *of bounded variation and orthogonal to each of the* α'*s and which is normal. Also, for each fixed* t,

$$(4.1)\quad \rho(s, t) = \sum_{k=1}^{n} \beta_k(t)\, \alpha_k(s) + Q_1(t) q(s, t) \qquad \textit{for almost all } s \in [0, 1].$$

(ii) *For a fixed triple*

$$(t, t', t'') \in \{(t, t', t''): 0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1\}$$

such that

$$[Q_1(t')]^2 [Q_1(t)]^2 \neq [Q_2(t, t')]^2$$

there exist functions $q(s, t)$, $r(s, t, t')$, $u(s, t, t', t'')$ *of bounded variation in* s *on* $[0, 1]$ *such that*

$$\alpha_1(s), \ldots, \alpha_n(s), q(s, t), r(s, t, t'), u(s, t, t', t'')$$

is an orthonormal set of functions of s *on* $[0, 1]$ *and such that for almost all* $s \in [0, 1]$, (4.1) *holds as well as*

$$(4.2)\quad \rho(s, t') = \sum_{i=1}^{n} \beta_i(t') \alpha_i(s) + Q_3(t, t') q(s, t) + R_1(t, t') r(s, t, t'),$$

$$(4.3)\quad \rho(s, t'') = \sum_{i=1}^{n} \beta_i(t'') \alpha_i(s) + Q_3(t, t'') q(s, t) + R_2(t, t', t'') r(s, t, t') + U_1(t, t', t'') u(s, t, t', t'').$$

Proof. (i) To prove (4.1) it is noted that for each fixed t, $\rho(s, t)$ must satisfy exactly one of the following alternatives:

$$(4.4)\qquad \rho(s, t) = \sum_{k=1}^{n} b_k(t) \alpha_k(s)$$

for almost all $s \in [0, 1]$ for an appropriate choice of $\{b_k(t): k = 1, 2, \ldots n\}$,

$$(4.5)\qquad \rho(s, t) = \sum_{k=1}^{n} b_k(t) \alpha_k(s)$$

for almost all $s \in [0, 1]$ is false for all choices of $\{b_k(t): k = 1, 2, \ldots, n\}$.

Multiplication of both sides of (4.4) by $\alpha_j(s)$, followed by integration with respect to s from 0 to 1, yields $b_j(t) = \beta_j(t)$. Hence, if (4.4) holds, then

$$\rho(s, t) = \sum_{k=1}^{n} \beta_k(t) \alpha_k(s)$$

for almost all $s \in [0, 1]$. The squaring of both sides of the last equation followed by integration with respect to s from 0 to 1 and an application of

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Lemma 4.2 yields $Q_1^2(t) = 0$. Hence, if (4.4) holds,

$$\rho(s, t) = \sum_{k=1}^{n} \beta_k(t)\alpha_k(s) + Q_1(t)q(s, t)$$

for almost all $s \in [0, 1]$, where $q(s, t)$ is a normal function of bounded variation orthogonal to each of $\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)$. The existence of $q(s, t)$ is ensured by Lemma 4.1 for each t for the case in which (4.4) holds.

Now suppose (4.5) holds. Then

$$\rho(s, t) - \sum_{k=1}^{n} \beta_k(t)\alpha_k(s) = N(s, t)$$

where $N(s, t)$ is non-zero on a subset of positive measure of $[0, 1]$. Thus

$$\left[\rho(s, t) - \sum_{k=1}^{n} \beta_k(t)\alpha_k(s)\right]^2 = N^2(s, t)$$

where $N^2(s, t)$ is positive on a subset of positive measure of $[0, 1]$. Thus

$$t - \sum_{k=1}^{n} \beta_k^{\,2}(t) = \int_0^1 N^2(s, t)\, ds > 0,$$

so that, by Lemma 4.2, $Q_1^2(\mathrm{t}) > 0$. It follows that

$$q(s, t) = \left[\rho(s, t) - \sum_{k=1}^{n} \beta_k(t)\alpha_k(s)\right] \Big/ Q_1(t)$$

is a normal function of s on $[0, 1]$. It is easy to verify that $q(s, t)$ is orthogonal to each of $\alpha_1(s), \ldots, \alpha_n(s)$, and $q(s, t)$ is clearly of bounded variation in s. Hence, if (4.5) holds, then for all $s \in [0, 1]$

$$\begin{aligned}\rho(s, t) &= \sum_{k=1}^{n} \beta_k(t)\alpha_k(s) + Q_1(t)\left\{\left[\rho(s, t) - \sum_{k=1}^{n} \beta_k(t)\alpha_k(s)\right] \Big/ Q_1(t)\right\} \\ &= \sum_{k=1}^{n} \beta_k(t)\alpha_k(s) + Q_1(t)q(s, t).\end{aligned}$$

Thus, for all $t \in [0, 1]$ for which (4.5) holds, $q(s, t)$ has been defined for all $s \in [0, 1]$. Finally then, for all $t \in [0, 1]$, $q(s, t)$ has been defined for all $s \in [0, 1]$ and $q(s, t)$ satisfies the conclusions of the lemma.

(ii) The proof of the existence of $q(s, t)$ satisfying (4.1) has been completed in part (i) of the proof. The existence of the required $r(s, t, t')$ and $u(s, t, t', t'')$ will now be established. For fixed

$$(t, t', t'') \in \{(t, t', t''): 0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1\},$$

$\rho(s, t')$ must satisfy exactly one of the following:

$$\rho(s, t') = \sum_{i=1}^{n} m_i \alpha_i(s) + mq(s, t) \tag{4.6}$$

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for almost all $s \in [0, 1]$ for an appropriate choice of $m_1, m_2, \ldots, m_n, m$,

$$\rho(s, t') = \sum_{i=1}^{n} m_i \alpha_i(s) + mq(s, t) \tag{4.7}$$

for almost all $s \in [0, 1]$ is false for all choices of $m_1, m_2, \ldots, m_n, m$.

Suppose (4.6) is true. Then multiplication of both sides of (4.6) first by $\alpha_j(s)$ followed by integration from 0 to 1, and then by $q(s, t)$ followed by integration from 0 to 1, yields respectively $m_j = \beta_j(t')$ and

$$m = \int_0^1 \rho(s, t')q(s, t)\, ds.$$

It follows by the definition of $q(s, t)$ (note that $Q_1(t) \neq 0$ by assumption) that $m = Q_3(t, t')$. Thus if (4.6) holds, then

$$\rho(s, t) = \sum_{i=1}^{n} \beta_i(t')\alpha_i(s) + Q_3(t, t')q(s, t) \tag{4.8}$$

for almost all $s \in [0, 1]$, and thus from the squaring of both sides of (4.8) and integrating with respect to s from 0 to 1 there follows $[Q_1(t')]^2 = [Q_3(t, t')]^2$, contrary to the assumption that $[Q_1(t')]^2[Q_1(t)]^2 \neq [Q_2(t, t')]^2$. Thus, (4.7) must hold and so

$$\rho(s, t') - \sum_{i=1}^{n} \beta_i(t')\alpha_i(s) - Q_3(t, t')q(s, t) = R(s, t, t') \tag{4.9}$$

where $R(s, t, t')$ is non-zero on a subset of positive measure of $[0, 1]$.

The squaring of both sides of (4.9) followed by integration with respect to s from 0 to 1 yields, again in view of Lemma 4.2, $[R_1(t, t')]^2 > 0$. Hence

$$r(s, t, t') = \frac{\rho(s, t') - \sum_{i=1}^{n} \beta_i(t')\alpha_i(s) - Q_3(t, t')q(s, t)}{R_1(t, t')}$$

is a normal function of s on $[0, 1]$. Since $\alpha_1(s), \ldots, \alpha_n(s)$ are of bounded variation and so also is $q(s, t)$ (as observed above in the establishment of the existence of $q(s, t)$) it follows that $r(s, t, t')$ is also of bounded variation. It is easy to verify that $r(s, t, t')$ is orthogonal to each of $\alpha_1(s), \ldots, \alpha_n(s)$ and $q(s, t)$. Hence, for almost all $s \in [0, 1]$,

$$\rho(s, t') = \sum_{i=1}^{n} \beta_i(t')\alpha_i(s) + Q_3(t, t')q(s, t) + R_1(t, t')r(s, t, t')$$

and the existence of a normal $r(s, t, t')$ of bounded variation, orthogonal to the α's and to $q(s, t)$, and which satisfies (4.2), is established.

Now the existence of the required $u(s, t, t', t'')$ will be established.

For fixed

$$(t, t', t'') \in \{(t, t', t''): 0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1\},$$

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$\rho(s, t'')$ must satisfy exactly one of the following alternatives:

$$(4.10)\qquad \rho(s, t'') = \sum_{i=1}^{n} m_i \alpha_i(s) + aq(s, t) + br(s, t, t')$$

for almost all $s \in [0, 1]$ for an appropriate choice of $m_1, m_2, \ldots, m_n, a, b$,

$$(4.11)\qquad \rho(s, t'') = \sum_{i=1}^{n} m_i \alpha_i(s) + aq(s, t) + br(s, t, t')$$

for almost all $s \in [0, 1]$ is false for all choices of $m_1, \ldots, m_n, a, b$.

If (4.10) holds, then multiplication of both sides by $\alpha_j(s)$, $q(s, t)$, and $r(s, t, t')$ in turn, followed by integration with respect to s from 0 to 1, yields respectively $m_j = \beta_j(t'')$, $a = Q_3(t, t'')$, and $b = R_2(t, t', t'')$. Hence, if (4.10) holds,

$$\rho(s, t'') = \sum_{i=1}^{n} \beta_i(t'')\alpha_i(s) + Q_3(t, t'')q(s, t) + R_2(t, t', t'')r(s, t, t')$$

for almost all $s \in [0, 1]$ so that

$$t'' = \int_0^1 [\rho(s, t'')]^2\, ds = \sum_{i=1}^{n} \beta_i^2(t'') + [Q_3(t, t'')]^2 + [R_2(t, t', t'')]^2.$$

Thus from Lemma 4.2 follows

$$\begin{aligned} 0 &= [Q_1(t'')]^2 - [Q_3(t, t'')]^2 - [R_2(t, t', t'')]^2 \\ &= [R_1(t, t'')]^2 - [R_2(t, t', t'')]^2 = [U_1(t, t', t'')]^2. \end{aligned}$$

(Note that it has now been shown that for case (4.10)

$$[R_1(t, t'')]^2 - [R_2(t, t', t'')]^2 = 0.)$$

Hence, if (4.10) holds, it follows that for almost all $s \in [0, 1]$,

$$\begin{aligned} \rho(s, t'') = \sum_{i=1}^{n} \beta_i(t'')\alpha_i(s) &+ Q_3(t, t'')q(s, t) \\ &+ R_2(t, t', t'')r(s, t, t') + U_1(t, t', t'')u(s, t, t', t''), \end{aligned}$$

where $u(s, t, t', t'')$ is a normal function of bounded variation orthogonal to each of $\alpha_1(s), \ldots, \alpha_n(s)$, $q(s, t)$, $r(s, t, t')$. The existence of $u(s, t, t', t'')$ is ensured by Lemma 4.1.

Now suppose that (4.11) holds. Then

$$(4.12)\qquad \begin{aligned} \rho(s, t'') - \sum_{i=1}^{n} \beta_i(t'')\alpha_i(s) - Q_3(t, t'')q(s, t) &- R_2(t, t', t'')r(s, t, t') \\ &= U(s, t, t', t'') \end{aligned}$$

where $U(s, t, t', t'')$ is non-zero on a subset of positive measure of the set $[0, 1]$. If each side of (4.12) is squared and integrated with respect to s from 0 to 1

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there results

$$t'' - \sum_{i=1}^{n} \beta_i^2(t'') - [Q_3(t, t'')]^2 - [R_2(t, t', t'')]^2 = \int_0^1 [U(s, t, t', t'')]^2 ds > 0$$

or, using Lemma 4.2 again,

$$[R_1(t, t'')]^2 - [R_2(t, t', t'')]^2 > 0.$$

(Note that it has now been shown that for case (4.11)

$$[R_1(t, t'')]^2 - [R_2(t, t', t'')]^2 > 0.$$

Thus, from the assertion of the previous note, there now follows that for all

$$(t, t', t'') \in \{(t, t', t''): 0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1\},$$
$$[R_1(t, t'')]^2 - [R_2(t, t', t'')]^2 \geqslant 0$$

as asserted in the lemma.) It follows that $U(s, t, t', t'')$ has a normalized form $u(s, t, t', t'')$ and it is easy to verify, using the equations $m_j = \beta_j(t'')$, $a = Q_3(t, t'')$, and $b = R_2(t, t', t'')$ obtained above, that $U(s, t, t', t'')$ and hence $u(s, t, t', t'')$ is orthogonal to each of

$$\alpha_1(s), \ldots, \alpha_n(s), q(s, t), r(s, t, t').$$

Thus if (4.11) holds, then $\rho(s, t'')$ has the form given in the lemma where, as noted above, $u(s, t, t', t'')$ is the normalized form of $U(s, t, t', t'')$. Hence the existence of a normal $u(s, t, t', t'')$ of bounded variation, orthogonal to the α's, to $q(s, t)$, and to $r(s, t, t')$, and which satisfies (4.3), is established and the proof of the lemma is complete.

LEMMA 4.4. *If $\phi(s) = 0$ for almost all $s \in [0, 1]$ and if $\phi(s)$ is of bounded variation and $x(s)$ is continuous on* [0, 1], *then*

$$\int_0^1 \phi(s)\, dx(s) = 0.$$

Proof. Since $\phi(s)$ is of bounded variation, it has at most countably many discontinuities. It is easy to show that $\phi(s) = 0$ at each point of continuity. Also, if $s_1, s_2, \ldots$ is an enumeration of the s-values for which $\phi(s) \neq 0$, then it is easily shown that since $\phi(s)$ is of bounded variation,

$$\sum_{i=1}^{\infty} |\phi(s_i)| < \infty.$$

A consideration of the definition of

$$\int_0^1 \phi(s)\, dx(s),$$

combined with the last inequality, yields the desired result.

LEMMA 4.5. *For all $x \in C$ and any orthonormal set of functions*

$$\{\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)\}$$

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of bounded variation on $[0, 1]$, *there exists* $q(s, t)$ *defined on* $[0, 1] \times [0, 1]$, *of bounded variation in s and orthogonal to the* α*'s and such that for all* $t \in [0, 1]$ *it is possible to express* $x(t)$ *by*

$$x(t) = \sum_{i=1}^{n} \int_0^1 \alpha_i(s)\, dx(s) \beta_i(t) + Q_1(t) \int_0^1 q(s, t)\, dx(s)$$

(where $Q_1(t)$ is as defined in Lemma 4.3) *or*

$$x(t) - \sum_{i=1}^{n} c_i \beta_i(t) = Q_1(t) \int_0^1 q(s, t)\, dx(s). \tag{4.13}$$

Proof. Stieltjes integrations, with respect to $x(s)$, from 0 to 1 of both sides of equation (4.1), followed by an application of Lemma 4.4, yield the desired result.

LEMMA 4.6. *Let n be fixed. Then*

$$\left[x(t) - \sum_{i=1}^{n} c_i \beta_i(t)\right]^2$$

(i) *for fixed* $x \in C$, *is measurable with respect to* $t \in [0, 1]$ *and integrable with respect to t,*

(ii) *for fixed* $t \in [0, 1]$, *is measurable with respect to* $x \in C$ *and integrable with respect to x,*

(iii) *measurable with respect to* $(t, x) \in [0, 1] \times C$ *and integrable with respect to* (t, x).

Proof. (i) follows from the continuity of the expression in t. The statement of measurability in (ii) follows from the facts that $x(t)$ is continuous (in the uniform topology) in x for fixed t and that c_i is integrable (and thus measurable of course). The statement of integrability in (ii) follows from the fact which will be proved below, that

$$\left[x(t) - \sum_{i=1}^{n} c_i \beta_i(t)\right]^2$$

is dominated by a functional in $L(C)$. Now

$$\begin{aligned}
[x(t) - x^n(t)]^2 &= \left[x(t) - \sum_{i=1}^{n} c_i \beta_i(t)\right]^2 \\
&= \left[x(t) - \sum_{i=1}^{n} \int_0^1 x(s) d\alpha_i(s) \cdot \beta_i(t)\right]^2 \\
&\leqslant [x(t)]^2 + 2\sum_{i=1}^{n} \left| x(t) \int_0^1 x(s) d\alpha_i(s) \beta_i(t) \right| + \left[\sum_{i=1}^{n} \int_0^1 x(s) d\alpha_i(s) \cdot \beta_i(t)\right]^2 \\
&\leqslant \left\{\max_{0 \leqslant t \leqslant 1} |x(t)|\right\}^2 + 2\sum_{i=1}^{n} \left\{\max_{0 \leqslant t \leqslant 1} |x(t)|\right\}\left\{\max_{0 \leqslant t \leqslant 1} |\beta_i(t)|\right\} \cdot \left| \int_0^1 x(s)\, d\alpha_i(s) \right| \\
&\quad + \left[\sum_{i=1}^{n} \int_0^1 x(s)\, d\alpha_i(s) \cdot \beta_i(t)\right]^2.
\end{aligned}$$

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But

$$\left| \int_0^1 x(s)\, d\alpha_i(s) \right| \leqslant \max_{0 \leqslant t \leqslant 1} |x(t)| \left\{ \operatorname*{Var}_{0 \leqslant s \leqslant 1} \alpha_i(s) \right\}.$$

Thus

$$[x(t) - x^n(t)]^2 < K \left\{ \max_{0 \leqslant t \leqslant 1} |x(t)| \right\}^2 \in L(C)$$

for appropriate $K > 0$, and the desired dominance of $[x(t) - x^n(t)]^2$ is established.

The statement of measurability in (iii) follows from the following observations. $x(t)$ is continuous in (t, x) (in the topology induced in $[0, 1] \times C$ by the usual topology on $[0, 1]$ and the uniform topology on C) and thus $x(t)$ is measurable in (t, x). Since $\beta_i(t)$ is integrable in t, and c_i is integrable in x, it follows that $c_i\, \beta_i(t)$ is integrable in (t, x) and thus of course is measurable in (t, x). The statement of integrability in (iii) now follows from the dominance of

$$\left[x(t) - \sum_{i=1}^{n} c_i\, \beta_i(t) \right]^2$$

again by $K\{\max_{0 \leqslant t \leqslant 1} |x(t)|\}^2$ now considered as a function on $[0, 1] \times C$.

LEMMA 4.7. *Let* $\{y(j; t) : j = 1, 2, 3, \ldots\}$ *be any subsequence of the sequence* $\{x^n(t) : n = 1, 2, 3, \ldots\}$. *Then for some subsubsequence* $\{y(j_k; t) : k = 1, 2, 3, \ldots\}$

$$\underset{k \to \infty}{\overset{[0,1]}{\text{L.I.M.}}}\; y(j_k; t) = x(t),$$

i.e.

$$\lim_{k \to \infty} \int_0^1 [y(j_k; t) - x(t)]^2\, dt = 0 \qquad \textit{for almost all } x \in C.$$

Proof. The existence of each of the following integrals is ensured by Lemma 4.6. Also note that $y(j; t) = x^{n(j)}(t)$.

$$\int_C \int_0^1 \left[x(t) - \sum_{i=1}^{n} c_i\, \beta_i(t) \right]^2 dt\, dx = \int_0^1 \int_C \left[x(t) - \sum_{i=1}^{n} c_i\, \beta_i(t) \right]^2 dx\, dt.$$

If both sides of (4.13), where n is replaced by $n(j)$, are squared and integrated on $[0, 1] \times C$, there follows

$$\begin{aligned} &\int_0^1 \int_C \left[x(t) - \sum_{i=1}^{n(j)} c_i\, \beta_i(t) \right]^2 dx\, dt \\ &\qquad = \int_0^1 \int_C \left[\sum_{i=n(j)+1}^{\infty} \beta_i^2(t) \right] \left[\int_0^1 q(s, t)\, dx(s) \right]^2 dx\, dt \\ &\qquad = \frac{1}{2} \int_0^1 \left[\sum_{i=n(j)+1}^{\infty} \beta_i^2(t) \right] dt \end{aligned}$$

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(because if f is the Stieltjes integral with respect to $x(s)$, on the interval [0, 1], of a normal function, then $\int_c f^2\,dx = \frac{1}{2}$)

$$= \frac{1}{2} \sum_{i=n(j)+1}^{\infty} \gamma_{i,i}.$$

But

$$\lim_{j\to\infty} \sum_{i=n(j)+1}^{\infty} \gamma_{i,i} = \lim_{j\to\infty} \int_0^1 \sum_{i=n(j)+1}^{\infty} \beta_i^{\,2}(t)\,dt$$
$$= \int_0^1 \lim_{j\to\infty} \sum_{i=n(j)+1}^{\infty} \beta_i^{\,2}(t)\,dt = 0$$

by Lemma 4.2. Hence there exists a subsubsequence $\{y(j_k;t): k = 1, 2, 3, \ldots\}$ such that

$$\lim_{k\to\infty} \int_0^1 [x(t) - y(j_k;t)]^2\,dt = 0$$

for almost all $x \in C$ and the proof is complete.

The theorems mentioned prior to the seven lemmas will now be stated and proved.

THEOREM 2. *Let $F[x(\cdot)]$ be continuous in the Hilbert topology in the space C and let $F[x(\cdot)]$ be bounded. Then*

$$\lim_{n\to\infty} I_n(F) = \int_c F[x(\cdot)]\,dx.$$

Proof. Note first that

$$I_n(F) = \int_c F[x^n(\cdot)]\,dx.$$

Let $\{y(j;t): j = 1, 2, 3, \ldots\}$ be any subsequence of the sequence

$$\{x^n(t): n = 1, 2, 3, \ldots\}.$$

Then by Lemma 4.7, there exists a subsubsequence $\{y(j_k;t): k = 1, 2, 3, \ldots\}$ such that

$$\underset{k\to\infty}{\overset{[0,1]}{\text{L.I.M.}}}\, y(j_k;t) = x(t).$$

Since $F[x(\cdot)]$ is continuous in the Hilbert topology, it follows that

$$\lim_{k\to\infty} F[y(j_k;\cdot)] = F[x(\cdot)].$$

Since $F[x(\cdot)]$ is bounded, Lebesgue's bounded convergence theorem applies to yield

$$\int_c F[x(\cdot)]\,dx = \lim_{k\to\infty} \int_c F[y(j_k;\cdot)]\,dx,$$

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that is, the limit of the sequence of I's based on this subsequence of x's is equal to

$$\int_C F[x(\cdot)]\,dx.$$

The conclusion of the theorem follows at once.

If the α's are chosen to be the odd harmonic cosine functions

$$(\alpha_i(s) = \sqrt{2}\cos\{[i - 1/2]\pi s\}; \qquad i = 1, 2, 3, \ldots)$$

then, as shown in (**2**, p. 112) (it should be noted that the $\beta_j(t)$ of (0.2) in (**2**, p. 112) is called $\alpha_j(t)$ in the present paper) the conclusion of Theorem 2 (of the present paper) holds, provided $F[x(\cdot)]$ is dominated by a suitable (unbounded) integrable functional. In the following example one does not even need to know of such a dominating functional.

Example. Let

$$F[x(\cdot)] = \exp\left(B\int_0^1 [x(t)]^2 dt\right)$$

where $0 \leqslant B < \frac{1}{4}\pi^2$. Then

$$\lim_{n\to\infty} I_n(F) = \int_C F[x(\cdot)]\,dx.$$

Proof. As shown in (**2**, p. 112, (0.5))

$$x^n(t) = \frac{2}{\pi}\sum_{j=1}^{n} c_j\, g_j(t)/(2j-1),$$

where $g_j(t)$: $j = 1, 2, 3, \ldots$ is an orthonormal set of functions. (Note that $x^n(t)$ is a partial sum in a Fourier expansion of $x(t)$). Thus

$$F[x^n(\cdot)] = \exp\left[\frac{4B}{\pi^2}\sum_{j=1}^{n} c_j^2/(2j-1)^2\right].$$

Again it is noted that

$$I_n(F) = \int_C F[x^n(\cdot)]\,dx$$

and that $F[x^n(\cdot)]$: $n = 1, 2, 3, \ldots$ is an increasing sequence of non-negative functionals such that $\lim_{n\to\infty} F[x^n(\cdot)] = F[x(\cdot)]$. The assertion of the example is thus established.

$F[x^n(\cdot)]$ can be easily computed by Wiener's formula for functions of n linear functionals, viz.

$$(4.14) \qquad \int_C f(c_1, \ldots, c_n)\,dx = \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty} e_n(\xi) f(\xi_1, \ldots, \xi_n)\,d\xi_1 \ldots d\xi_n.$$

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The infinite product in $\lim_{n\to\infty} I_n(F)$ can be found in (**5**, p. 114) and the result is

$$(4.15)\qquad \int_C \exp\left(B\int_0^1 [x(t)]^2 dt\right) dx = \sqrt{(\sec\sqrt{B})}, \qquad 0 \leqslant B \leqslant \pi^2/4.$$

THEOREM 3. *If* $F[x(\cdot)] \in L_1(C)$, *then*

$$\int_C F[x(\cdot)]dx = \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty} e_n(\xi)\int_C F[x(\cdot) - x^n(\cdot) + \psi_n(\xi,\cdot)]\,dx\,d\xi_1\ldots d\xi_n.$$

Proof. Assume first that $F[x(\cdot)]$ is bounded and continuous in the Hilbert topology. By Theorem 2

$$\begin{aligned}\int_C F[x(\cdot)]\,dx &= \lim_{n\to\infty} I_n(F)\\ &= \lim_{n\to\infty}\int_{-\infty}^{\infty}(\nu)\int_{-\infty}^{\infty}(1/\sqrt{\pi^\nu})\exp(-\xi_1^2\ldots-\xi_\nu^2)\\ &\times\left\{\int_{-\infty}^{\infty}(n-\nu)\int_{-\infty}^{\infty}(1/\sqrt{\pi^{n-\nu}})\exp(-\xi_{\nu+1}^2\ldots-\xi_n^2)\right.\\ &\times\left. F\left[\psi_\nu(\xi,\cdot)+\sum_{j=\nu+1}^{n}\xi_j\beta_j(\cdot)\right]d\xi_{\nu+1}\ldots d\xi_n\right\}d\xi_1\ldots d\xi_\nu.\end{aligned}$$

Thus, by the use of the formula for functions of n linear functionals, viz.

$$\int_C f\left\{\int_0^1\alpha_1(t)\,dx(t),\ldots,\int_0^1\alpha_n(t)\,dx(t)\right\}dx = \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}e_n(\xi)f(\xi_1,\ldots,\xi_n)\,d\xi_1\ldots d\xi_n,$$

and from the definition of $x^n(t)$, there follows

$$\begin{aligned}(4.16)\qquad &\int_C F[x(\cdot)]\,dx\\ &= \lim_{n\to\infty}\int_{-\infty}^{\infty}(\nu)\int_{-\infty}^{\infty}(1/\sqrt{\pi^\nu})\exp(-\xi_1^2-\ldots-\xi_\nu^2)\\ &\qquad\times\int_C F[\psi_\nu(\xi,\cdot)+x^n(\cdot)-x^\nu(\cdot)]\,dx\,d\xi_1\ldots d\xi_\nu\\ &= \lim_{n\to\infty}\int_{-\infty}^{\infty}(\nu)\int_{-\infty}^{\infty}e_\nu(\xi)\int_C F[\psi_\nu(\xi,\cdot)+x^n(\cdot)-x^\nu(\cdot)]\,dx\,d\xi_1\ldots d\xi_\nu.\end{aligned}$$

By Lemma 4.7 there exists a subsequence $\{y(j;t): j = 1, 2, 3, \ldots\}$ of $\{x^n(t): n = 1, 2, 3, \ldots\}$ such that

$$\underset{j\to\infty}{\overset{[0,1]}{\text{L.I.M.}}}\; y(j;t) = x(t).$$

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By Lebesgue's bounded convergence theorem

$$(4.17)\quad \lim_{j\to\infty} \int_C F[\psi_\nu(\xi, \cdot) + y(j; \cdot) - x^\nu(\cdot)]\, dx = \int_C F[\psi_\nu(\xi, \cdot) + x(\cdot) - x^\nu(\cdot)]\, dx.$$

From (4.16) it follows that

$$(4.18)\quad \int_C F[x(\cdot)]\, dx = \lim_{j\to\infty} \int_{-\infty}^{\infty} (\nu) \int_{-\infty}^{\infty} \int_C e_\nu(\xi) F[\psi_\nu(\xi, \cdot) + y(j; \cdot) - x^\nu(\cdot)]\, dx\, d\xi_1 \ldots d\xi_\nu.$$

Since $F[x(\cdot)]$ is bounded,

$$\int_C F[\psi_\nu(\xi, \cdot) + y(j; \cdot) - x^\nu(\cdot)]\, dx$$

is also bounded. Thus, for fixed ν, Lebesgue's bounded convergence theorem can be applied to the right side of (4.18) to yield

$$\begin{aligned}\int_C F[x(\cdot)]\, dx &= \int_{-\infty}^{\infty} (\nu) \int_{-\infty}^{\infty} e_\nu(\xi) \lim_{j\to\infty} \int_C F[\psi_\nu(\xi, \cdot) + y(j; \cdot) - x^\nu(\cdot)]\, dx\, d\xi_1 \ldots d\xi_\nu \\ &= \int_{-\infty}^{\infty} (\nu) \int_{-\infty}^{\infty} e_\nu(\xi) \int_C F[\psi_\nu(\xi, \cdot) + x(\cdot) - x^\nu(\cdot)]\, dx\, d\xi_1 \ldots d\xi_\nu\end{aligned}$$

because of (4.17). Thus the theorem is established in the case for which $F[x(\cdot)]$ is bounded and continuous in the Hilbert topology.

Next, as in Cameron's paper **(2**, p. 120**)**, assume that $F[x(\cdot)] = \chi_Q[x(\cdot)]$ is the characteristic functional of the quasi-interval

$$Q\colon \lambda_j < x(t_j) \leqslant \mu_j, \qquad j = 1, 2, \ldots, p; 0 < t_1 < t_2 < \ldots < t_p \leqslant 1,$$

and let

$$F_{\epsilon,\delta}[x(\cdot)] = \prod_{j=1}^{p} \phi_{j,\epsilon}\left[\frac{1}{\delta} \int_{t_j-\delta}^{t_j} x(s)\, ds\right],$$

where $\phi_{j,\epsilon}(u)$ is the continuous "trapezoidal" function that is zero outside the interval $\lambda_j < u < \mu_j + \epsilon$, equals unity in the interval $\lambda_j + \epsilon < u < \mu_j$, and is linear in the remaining intervals. It is clear that $F_{\epsilon,\delta}$ is continuous in the Hilbert topology and bounded, so it comes under the case of Theorem 3, which has already been proved, and so the conclusion of Theorem 3 holds for it. But it is also clear that for all $x \in C$,

$$\lim_{\delta\to 0^+} F_{\epsilon,\delta}[x(\cdot)] = F_\epsilon[x(\cdot)]$$

where

$$F_\epsilon[x(\cdot)] = \prod_{j=1}^{p} \phi_{j,\epsilon}[x(t_j)],$$

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and by the principle of bounded convergence it follows that the conclusion of Theorem 3 also holds for $F_\epsilon[x(\cdot)]$. Similarly for all $x \in C$, there follows

$$\lim_{\epsilon\to 0+} F_\epsilon[x(\cdot)] = F[x(\cdot)],$$

and again by bounded convergence it follows that the conclusion of Theorem 3 holds for $F[x(\cdot)]$, i.e. for the characteristic function of a quasi-interval. The theorem can be established for simple functionals, positive functionals, and then integrable functionals by standard procedures.

Another sequence of four lemmas will now be given to prove Theorem 4.

LEMMA 4.8. *If* $\{\alpha_i(t): i = 1, 2, 3, \ldots\}$ *is a complete orthonormal set of functions of bounded variation on* $[0, 1]$, *then it is impossible for a specific function* $f(t)$ *and constants* c_i $(i = n + 1, n + 2, \ldots)$, n *being any fixed positive integer, that*

$$\alpha_i(t) = c_i f(t): i = n+1, n+2, \ldots \tag{4.19}$$

on a subset E *of* $[0, 1]$ *where* E *has positive measure.*

Proof. Assume that

$$\alpha_i(t) = c_i f(t), \qquad i = n+1, n+2, \ldots,$$

on a subset E of $[0, 1]$ where E has positive measure, $f(t)$ is a specific function, and c_i are constants; $i = n + 1, n + 2, \ldots$. Let

$$f_j(t) = \begin{cases} t^{j+1} & \text{for } t \in E, \\ 0 & \text{elsewhere,} \end{cases} \qquad j = 0, 1, \ldots n, n+1,$$

$$g_0(t) = \begin{cases} f(t) & \text{for } t \in E, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$g_j(t) = \begin{cases} \alpha_j(t) & \text{for } t \in E, \\ 0 & \text{elsewhere,} \end{cases} \qquad j = 1, 2, \ldots, n.$$

The components of $f_j(t): j = 0, 1, \ldots, n, n + 1$ which are orthogonal to all members of $g_0(t), g_1(t), \ldots, g_n(t)$ on the set $[0, 1]$ (or equivalently on the set E, of course) are

$$f_i(t) - \sum_{j=0}^{n} a_{i,j}\, g_j(t), \qquad i = 0, 1, 2, \ldots, n, n+1,$$

for appropriate constants $a_{i,j}$ $(i = 0, 1, 2, \ldots, n + 1; j = 0, 1, 2, \ldots, n)$. Now it is impossible that

$$f_i(t) - \sum_{j=0}^{n} a_{i,j}\, g_j(t) = 0, \qquad i = 0, 1, 2, \ldots, n+1,$$

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should hold for almost all $t \in E$. For otherwise let the ith equation be multiplied by r_i ($i = 0, 1, 2, \ldots, n+1$). Since the system of equations

$$\sum_{i=0}^{n+1} a_{i,j} r_i = 0, \qquad j = 0, 1, 2, \ldots, n,$$

has a non-trivial solution (for the r_i's), it follows that

$$r_0 t + r_1 t^2 + \ldots + r_n t^{n+1} + r_{n+1} t^{n+2} = 0$$

for almost all $t \in E$ with at least one of the r's being non-zero. This contradicts a corollary of the fundamental theorem of algebra. Thus

$$f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t) \not\equiv 0$$

for some $I \in \{0, 1, 2, \ldots, n, n+1\}$ on a subset $F \subset E$ where F has positive measure. Now

$$(4.20) \qquad \int_0^1 \left[f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t)\right]\alpha_k(t)\,dt = \int_E [f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t)]\alpha_k(t)\,dt, \qquad k = 1, 2, 3, \ldots,$$

since

$$f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t) = 0$$

save for $t \in E$. Also, $\alpha_k(t) = g_k(t)$ for $t \in E$ and $k \in \{1, 2, \ldots, n\}$ and

$$f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t)$$

was constructed to be orthogonal on E to each function of the set

$$\{g_i(t) \colon i = 1, 2, \ldots, n\}.$$

Thus

$$(4.21) \qquad \int_E \left[f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t)\right]\alpha_k(t)\,dt = \int_E \left[f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t)\right]g_k(t)\,dt = 0, \qquad k = 1, 2, 3, \ldots, n.$$

Hence it is seen from (4.20) and (4.21) that

$$f_I(t) - \sum_{j=0}^{n} a_{I,j} g_j(t)$$

is orthogonal on $[0, 1]$ to the set of functions $\{\alpha_i(t) \colon i = 1, 2, \ldots n\}$. Furthermore, from (4.19) there follows, because of the definition of $g_0(t)$ and the fact that

$$f_I(t) - \sum_{j=0}^{n} \alpha_{I,j} g_j(t)$$

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was constructed orthogonal to $g_0(t)$ on E, that

$$(4.22)\qquad \int_E \left[f_I(t) - \sum_{j=0}^{n} a_{I,j}\, g_j(t)\right]\alpha_k(t)\, dt$$
$$= c_k \int_E \left[f_I(t) - \sum_{j=0}^{n} a_{I,j}\, g_j(t)\right] f(t)\, dt$$
$$= c_k \int_E \left[f_I(t) - \sum_{j=0}^{n} a_{I,j}\, g_j(t)\right] g_0(t)\, dt = 0,$$
$$k = n+1, n+2, \ldots .$$

From (4.20) and (4.22) it now follows that

$$f_I(t) - \sum_{j=0}^{n} a_{I,j}\, g_j(t)$$

is orthogonal on [0, 1] also to the set of functions $\{\alpha_i(t) \colon i = n+1, n+2, \ldots\}$. Thus

$$f_I(t) - \sum_{j=0}^{n} a_{I,j}\, g_j(t)$$

is orthogonal on [0, 1] to the set of functions $\{\alpha_i(t) \colon i = 1, 2, 3, \ldots\}$, but is non-zero on a subset of positive measure of [0, 1]. Hence a contradiction has been obtained and the proof of Lemma 4.8 is complete.

LEMMA 4.9. *It is impossible that, for any positive integer* n,

$$\sum_{i=n+1}^{\infty} \beta_i^2(t) = 0$$

should hold on an interval of [0, 1].

Proof. Assume that

$$\sum_{i=n+1}^{\infty} \beta_i^2(t) = 0$$

on an interval. Then $\beta_i(t) = 0 \colon i = n+1, n+2, \ldots$ on an interval, say $\{t \colon 0 \leqslant a < t < b \leqslant 1\}$. Since

$$\beta_i(t) = \int_0^t \alpha_i(s)\, ds, \qquad i = 1, 2, \ldots, n+1, n+2, \ldots$$

it then follows that $\alpha_i(t) = 0 \colon i = n+1, n+2, \ldots$ for almost all

$$t \in \{t \colon 0 \leqslant a < t < b \leqslant 1\}.$$

This is impossible because of Lemma 4.8 and the proof of Lemma 4.9 is complete.

LEMMA 4.10. *It is impossible that, for any positive integer* n,

$$(4.23)\qquad \left[\sum_{i=n+1}^{\infty} \beta_i^2(t)\right]\left[\sum_{j=n+1}^{\infty} \beta_j^2(t')\right] = \left[\sum_{i=n+1}^{\infty} \beta_i(t)\beta_i(t')\right]^2$$

should hold on a subrectangle of $\{t \colon 0 \leqslant t \leqslant 1\} \times \{t' \colon 0 \leqslant t' \leqslant 1\}$.

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Proof. Assume the contrary. Without loss of generality assume that (4.23) holds on $\{t{:}a < t < b\} \times \{t'{:}c < t' < d\}$ where $0 \leqslant a < b \leqslant c \leqslant 1, 0 \leqslant c < d \leqslant 1$. By Lemma 4.9 it is impossible that $\beta_i(t') = 0$, $i = n+1, n+2, \ldots$ for all $t' \in \{t'{:}c < t' < d\}$. Thus, there exists $t'_0 \in \{t'{:}c < t' < d\}$ such that $\beta_i(t'_0) = 0{:}i = n+1, n+2, \ldots$ is false. From the last observation and the fact that (4.23) is the equality part of a Schwarz inequality, it follows, for the t'_0 mentioned above, that

$$\beta_i(t) = k(t)\beta_i(t_0'), \qquad i = n+1, n+2, \ldots \tag{4.24}$$

for $t \in \{t{:}a < t < b\}$ where $k(t)$ is a fixed function. Differentiation of (4.24) with respect to t yields

$$\alpha_i(t) = k'(t)\beta_i(t_0') \quad \text{or } \alpha_i(t) = c_i k'(t), \qquad i = n+1, n+2, \ldots,$$

for almost all $t \in \{t{:}a < t < b\}$. This is impossible according to Lemma 4.8 and the proof of Lemma 4.10 is complete.

LEMMA 4.11. *Let $\alpha_1(s), \ldots, \alpha_n(s)$ be any orthonormal set of functions of bounded variation on* $[0, 1]$ *and let $Q_1(t)$, $Q_2(t, t')$, $Q_3(t, t')$, $R_1(t, t')$, $R_2(t, t', t'')$, and $U_1(t, t', t'')$ be defined as in Lemma* 4.3. *Then for fixed* (t, t', t'') *such that*

$$(t, t', t'') \in \{(t, t', t''){:}\ 0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1\}$$

and

$$[Q_1(t')]^2[Q_1(t)]^2 \neq [Q_2(t, t')]^2,$$

there exist functions $q(s, t)$, $r(s, t, t')$, $u(s, t, t', t'')$ of bounded variation in s on $[0, 1]$ *such that*

$$\alpha_1(s), \ldots, \alpha_n(s), q(s, t), r(s, t, t'), u(s, t, t', t'')$$

are orthonormal in s on $[0, 1]$, *and such that for $x(\cdot) \in C$ it is possible to express $x(t)$, $x(t')$, and $x(t'')$ as follows:*

$$x(t) = \sum_{i=1}^{n} \beta_i(t) \int_0^1 \alpha_i(s)\,dx(s) + Q_1(t) \int_0^1 q(s, t)\,dx(s),$$

$$x(t') = \sum_{i=1}^{n} \beta_i(t') \int_0^1 \alpha_i(s)\,dx(s) + Q_3(t, t') \int_0^1 q(s, t)\,dx(s) + R_1(t, t') \int_0^1 r(s, t, t')\,dx(s),$$

$$x(t'') = \sum_{i=1}^{n} \beta_i(t'') \int_0^1 \alpha_i(s)\,dx(s) + Q_3(t, t'') \int_0^1 q(s, t)\,dx(s) + R_2(t, t', t'') \int_0^1 r(s, t, t')\,dx(s) + U_1(t, t', t'') \int_0^1 u(s, t, t', t'')\,dx(s).$$

Proof. Integration (Stieltjes) of equations (4.2), (4.3), and (4.4) from Lemma 4.3 in conjunction with Lemma 4.4 establishes the result.

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LEMMA 4.12. *For fixed* $(t, t', t'') \in \{t, t', t''):0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1\}$ *such that*

$$[Q_1(t')]^2[Q_1(t)]^2 \neq [Q_2(t, t')]^2,$$

the following equation holds:

$$(4.25)\quad \int_C [x(t) - x^n(t)]^2[x(t') - x^n(t')]^2[x(t'') - x^n(t'')]^2\, dx$$
$$= \tfrac{1}{8}[Q_1(t)]^2[Q_1(t')]^2[Q_1(t'')]^2$$
$$+ \tfrac{1}{4}\{[Q_2(t, t')]^2[Q_1(t'')]^2 + [Q_2(t, t'')]^2[Q_1(t')]^2 + [Q_2(t', t'')]^2[Q_1(t)]^2\}$$
$$+ Q_2(t', t'')Q_2(t, t')Q_2(t', t'').$$

Proof. It will now be noted that if f is the Stieltjes integral with respect to $x(s)$, on the interval [0, 1], of a normal function, then

$$\int_C f^2\, dx = 1/2, \qquad \int_C f^4\, dx = 3/4, \qquad \int_C f^6\, dx = 15/8.$$

Lemma 4.11 along with a routine computation using the formula for functions of n linear functionals (as in the proof of Theorem 3) completes the proof of the lemma.

LEMMA 4.13. *For all triples* (t, t', t'') *such that* $0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1$ *the Wiener integration formula, mentioned in Lemma* 4.12, *holds.*

Proof. The result has been established in Lemma 4.12 for all (t, t', t'') such that $0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1$ and such that

$$[Q_1(t')]^2[Q_1(t)]^2 \neq [Q_2(t, t')]^2.$$

For arbitrary $t'' \in [0, 1]$, it follows from Lemma 4.10 that the only possible pairs (t, t') satisfying $0 \leqslant t \leqslant t' \leqslant t'' \leqslant 1$ for which the result perhaps does not hold are limit points of (t, t') where the result does hold. But the right side of (4.25) is continuous in (t, t') since, by Lemma 4.2,

$$\sum_{i=n+1}^{\infty} \beta_i^{\,2}(t) = t - \sum_{i=1}^{n} \beta_i^{\,2}(t)$$

and

$$\sum_{i=n+1}^{\infty} \beta_i(t)\beta_i(t') = t - \sum_{i=1}^{n} \beta_i(t)\beta_i(t')$$

(for $0 \leqslant t \leqslant t' \leqslant 1$) and since each $\beta_i(t)$ is continuous. It will be shown that the left side of (4.25) is also continuous in (t, t'). It then follows that (4.25) must hold for all (t, t') pairs such that $0 \leqslant t \leqslant t' \leqslant t'' \leqslant t$.

To show that the left side of (4.25) is continuous in (t, t'), note is made of the inequality

$$[x(t) - x^n(t)]^2 \leqslant K\{\max_{0\leqslant t\leqslant 1}|x(t)|\}^2 \in L(C),$$

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for appropriate $K > 0$, which was obtained in the proof of Lemma 4.6. Then it follows that

$$[x(t) - x^n(t)]^2[x(t') - x^n(t')]^2[x(t'') - x^n(t'')]^2 \leqslant K^3\{\max_{0\leqslant t\leqslant 1}|x(t)|\}^6,$$

so that by Lebesgue's dominated convergence theorem

$$\lim_{\substack{\epsilon\to 0\\ \nu\to 0}} \int_c [x(t+\epsilon) - x^n(t+\epsilon)]^2[x(t'+\nu) - x^n(t'+\nu)]^2[x(t'') - x^n(t'')]^2\,dx$$
$$= \int_c \lim_{\substack{\epsilon\to 0\\ \nu\to 0}} [x(t+\epsilon) - x^n(t+\epsilon]^2[x(t'+\nu) - x^n(t'+\nu)]^2[x(t'') - x^n(t'')]^2\,dx.$$

The proof of the lemma is complete.

The theorem that provides the estimate for

$$\sqrt[3]{\left\{\int_c \left[\int_0^1 [x(s) - x^n(s)]^2\,ds\right]^3 dx\right\}^2}$$

will now be given.

THEOREM 4.

$$\int_c \left\{\int_0^1 [x(t) - x^n(t)]^2 dt\right\}^3 dx \leqslant \frac{15}{8}\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^3.$$

Proof.

$$\int_c \left\{\int_0^1 [x(t) - x^n(t)]^2\,dt\right\}^3 dx$$
$$= 6\int_0^1 \int_0^{t''} \int_0^{t'} \int_c [x(t) - x^n(t)]^2[x(t') - x^n(t')]^2[x(t'') - x^n(t'')]^2\,dx\,dt\,dt'\,dt''.$$

Thus, by Lemma 4.13,

$$\int_c \left\{\int_0^1 [x(t) - x^n(t)]^2\,dt\right\}^3 dx$$
$$= 6\int_0^1 \int_0^{t''} \int_0^{t'} \Bigg\{1/8\left[\sum_{i=n+1}^{\infty} \beta_i^2(t)\right]\left[\sum_{j=n+1}^{\infty} \beta_j^2(t')\right]\left[\sum_{k=n+1}^{\infty} \beta_k^2(t'')\right]$$
$$+\frac{1}{4}\left[\sum_{i=n+1}^{\infty} \beta_i(t)\beta_i(t')\right]^2\left[\sum_{j=n+1}^{\infty} \beta_j^2(t'')\right]$$
$$+\frac{1}{4}\left[\sum_{i=n+1}^{\infty} \beta_i(t)\beta_i(t'')\right]^2\left[\sum_{j=n+1}^{\infty} \beta_j^2(t')\right]$$
$$+\frac{1}{4}\left[\sum_{i=n+1}^{\infty} \beta_i(t')\beta_i(t'')\right]^2\left[\sum_{j=n+1}^{\infty} \beta_j^2(t)\right]$$

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$$+\left[\sum_{i=n+1}^{\infty}\beta_i(t)\beta_i(t')\right]\left[\sum_{j=n+1}^{\infty}\beta_j(t)\beta_j(t'')\right]\left[\sum_{k=n+1}^{\infty}\beta_k(t')\beta_k(t'')\right]\Bigg\}\,dt\,dt'\,dt''$$

$$\leqslant \int_0^1\int_0^1\int_0^1\Bigg\{1/8\left[\sum_{i=n+1}^{\infty}\beta_i^2(t)\right]\left[\sum_{j=n+1}^{\infty}\beta_j^2(t')\right]\left[\sum_{k=n+1}^{\infty}\beta_k^2(t'')\right]$$

$$+\frac{3}{4}\left[\sum_{i=n+1}^{\infty}\beta_i^2(t)\right]\left[\sum_{k=n+1}^{\infty}\beta_k^2(t'')\right]\left[\sum_{j=n+1}^{\infty}\beta_j^2(t')\right]$$

$$+\left[\sum_{i=n+1}^{\infty}\beta_i^2(t)\right]\left[\sum_{j=n+1}^{\infty}\beta_j^2(t')\right]\left[\sum_{k=n+1}^{\infty}\beta_k^2(t'')\right]\Bigg\}\,dt\,dt'\,dt''$$

$$=\frac{15}{8}\left(\sum_{i=n+1}^{\infty}\gamma_{i,i}\right)^3$$

and the proof of the theorem is complete.

Seven more lemmas are required in the proof of Theorem 5.

LEMMA 4.14. *Suppose $f(s)$ is continuous on* $[0, 1]$. *Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition of norm less than δ and for every $g(s)$ such that*

$$\operatorname{Var}_{0\leqslant s\leqslant 1} g(s) < M$$

the following inequality holds:

$$\left|\sum_{i=1}^{m} f(s_i)[g(s_i) - g(s_{i-1})] - \int_0^1 f(s)dg(s)\right| < \epsilon.$$

Proof. Hille (**3**, pp. 292-4) shows that if $f(s)$ is continuous and $g(s)$ is of bounded variation, then for any partition of norm less than δ

$$\left|\sum_{i=1}^{m} f(s_i)[g(s_i) - g(s_{i-1})] - \int_0^1 f(s)dg(s)\right|$$

$$\leqslant 2\max\{|f(u) - f(v)| : u, v \in [0, 1], |u - v| \leqslant \delta\} \operatorname*{Var}_{0\leqslant s\leqslant 1} g(s).$$

The proof is obvious from this inequality.

LEMMA 4.15. *If $f(s, t)$ is of bounded variation in the sense that it is of bounded variation in (s, t) on* $[0, 1] \times [0, 1]$, *and is of bounded variation in s for a particular t and in t for a particular s and if $g(s)$ is bounded and Riemann–Stieltjes integrable with respect to $f(s, t)$ for each $t \in$* $[0, 1]$, *then*

$$\int_0^1 g(s)\,df(s, t)$$

is of bounded variation in t on $[0, 1]$.

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Proof. Suppose $g(s)$ is not identically zero (otherwise the proof is trivial). Then

$$\sum_{j=1}^{n}\left|\int_0^1 g(s)\,df(s,t_j) - \int_0^1 g(s)\,df(s,t_{j-1})\right|$$
$$= \sum_{j=1}^{n}\left|\int_0^1 g(s)\,d[f(s,t_j) - f(s,t_{j-1})]\right|$$
$$\leqslant \sum_{j=1}^{n}\int_0^1 |g(s)|\,|d[f(s,t_j) - f(s,t_{j-1})]|$$
$$\leqslant \sup_{0\leqslant s\leqslant 1}|g(s)|\sum_{j=1}^{n}\operatorname*{Var}_{0\leqslant s\leqslant 1}[f(s,t_j) - f(s,t_{j-1})].$$

Now there exists an s-partition such that

$$\operatorname*{Var}_{0\leqslant s\leqslant 1}[f(s,t_j) - f(s,t_{j-1})]$$
$$\leqslant \sum_{i=1}^{m}|f(s_i,t_j) - f(s_i,t_{j-1}) - f(s_{i-1},t_j) + f(s_{i-1},t_{j-1})| + \epsilon\Big/\Big(n\sup_{0\leqslant s\leqslant 1}|g(s)|\Big)$$

for $j = 1, 2, \ldots, n$.

Thus

$$\sum_{j=1}^{n}\left|\int_0^1 g(s)\,df(s,t_j) - \int_0^1 g(s)\,df(s,t_{j-1})\right|$$
$$\leqslant \sup_{0\leqslant s\leqslant 1}|g(s)|\sum_{j=1}^{n}\sum_{i=1}^{m}|f(s_i,t_j) - f(s_{i-1},t_j) - f(s_i,t_{j-1}) + f(s_{i-1},t_{j-1})| + \epsilon$$
$$\leqslant \sup_{0\leqslant s\leqslant 1}|g(s)|\operatorname*{Var}_{[0,1]\times[0,1]} f(s,t) + \epsilon.$$

LEMMA 4.16. *If $h(s,t)$ is of bounded variation on $[0,1]\times[0,1]$ (as in Lemma 4.15) and is right continuous in s for each t, and if $f(t)$ is bounded and Riemann–Stieltjes integrable with respect to $h(s,t)$ for each s, then*

$$\int_0^1 f(t)\,dh(s,t)$$

is right continuous in s.

Proof. Suppose $f(t)$ is not identically zero (otherwise the proof is trivial). It will be shown that $\operatorname{Var}_{0\leqslant t\leqslant 1}[h(s+\delta,t) - h(s,t)] \to 0$ as $\delta\to 0$ and then the lemma follows from the observation that

$$\left|\int_0^1 f(t)\,dh(s+\delta,t) - \int_0^1 f(t)\,dh(s,t)\right| \leqslant \sup_{0\leqslant t\leqslant 1}|f(t)|\operatorname*{Var}_{0\leqslant t\leqslant 1}[h(s+\delta,t) - h(s,t)].$$

The proof of the above-mentioned property follows: Since $h(s,t)$ is of bounded variation on $[0,1]\times[0,1]$, and hence on $[s_0,1]\times[0,1]$: $0\leqslant s_0\leqslant 1$,

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for any $\epsilon > 0$, there exists a partition of $[s_0, 1] \times [0, 1]$ by the lines

$$s = s_i, \quad i = 0, 1, 2, \ldots m, \qquad \text{and } t = t_j, \quad j = 0, 1, 2, \ldots n,$$

such that

$$(4.26) \quad \operatorname*{Var}_{[s_0,1]\times[0,1]} h(s,t) - \sum_{i=1}^{m} \sum_{j=1}^{n} |g(s_i, t_j) - g(s_{i-1}, t_j) - g(s_i, t_{j-1}) + g(s_{i-1}, t_{j-1})| < \epsilon/3.$$

Furthermore, because of right continuity of $g(s, t)$ in s it can be assumed (by introducing one new column of points if necessary and using a new m) that

$$(4.27) \quad |g(s_0, t_j) - g(s_1, t_j)| < \epsilon/[3(n+1)], \qquad j = 0, 1, 2, \ldots, n,$$

holds as well as (4.26). But

$$(4.28) \quad \operatorname*{Var}_{[s_0,1]\times[0,1]} h(s,t) \geqslant \operatorname*{Var}_{[s_1,1]\times[0,1]} h(s,t) \geqslant \sum_{i=2}^{m} \sum_{j=1}^{n} |g(s_i, t_j) - g(s_{i-1}, t_j) - g(s_i, t_{j-1}) + g(s_{i-1}, t_{j-1})|$$

and thus from (4.26) and (4.28) it follows that

$$\begin{aligned}
\operatorname*{Var}_{[s_0,s_1]\times[0,1]} h(s,t) &= \operatorname*{Var}_{[s_0,1]\times[0,1]} h(s,t) - \operatorname*{Var}_{[s_1,1]\times[0,1]} h(s,t) \\
&\leqslant \sum_{j=1}^{n} |g(s_1, t_j) - g(s_0, t_j) - g(s_1, t_{j-1}) + g(s_0, t_{j-1})| + \epsilon/3 \\
&\leqslant \sum_{j=1}^{n} |g(s_1, t_j) - g(s_0, t_j)| + \sum_{j=1}^{n} |g(s_1, t_{j-1}) - g(s_0, t_{j-1})| + \epsilon/3 \\
&\leqslant 2n\epsilon/[3(n+1)] + \epsilon/3 < \epsilon
\end{aligned}$$

(because of (4.27)). Thus $\operatorname{Var}_{[s,1]\times[0,1]} h(s, t)$ is right continuous in s. Now

$$\operatorname*{Var}_{0\leqslant t\leqslant 1} [h(s+\delta, t) - h(s,t)] = \sup \sum_{j=1}^{n} |h(s+\delta, t_j) - h(s, t_j) - h(s+\delta, t_{j-1}) + h(s, t_{j-1})|$$

(the supremum taken for sums over all partitions of interval $[0, 1]$ of t) and since the set of such sums is a subset of all sums over which the supremum is taken in obtaining $\operatorname{Var}_{[s,s+\delta]\times[0,1]} h(s, t)$, it follows that

$$\operatorname{Var}_{0\leqslant t\leqslant 1}[h(s+\delta, t) - h(s,t)] \to 0 \text{ as } \delta \to 0$$

to complete the proof.

LEMMA 4.17. *If $f(s, t)$ is right continuous in s and is of bounded variation in the sense described in Lemma 4.15, and $g(s) \in C'$ and $h(t)$ is continuous and of bounded variation on $[0, 1]$, then*

$$\int_0^1 \int_0^1 g(s)h(t)\, d_{s,t} f(s,t) = \int_0^1 g(s)\, d\left\{ \int_0^1 h(t) df(s,t) \right\}.$$

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Also, as proved in Lemmas 4.15 *and* 4.16,

$$\left\{ \int_0^1 h(t)\, df(s,t) \right\}$$

is of bounded variation and is right continuous in s on [0, 1].

Proof. The result is obviously true if $g(s) \equiv 0$ on $[0, 1]$; so assume that $g(s)$ is not identically zero on $[0, 1]$. First assume that $g(s) \in C$, so the integral can be interpreted as Riemann-Stieltjes. Let any partition of $\{s: 0 \leqslant s \leqslant 1\}$ by the points $\{s_0, s_1, s_2, \ldots, s_m\}$ such that $0 = s_0 < s_1 < s_2 < \ldots < s_m = 1$ be denoted by π_s. For any π_s and π_t let the partition of

$$\{s: 0 \leqslant s \leqslant 1\} \times \{t: 0 \leqslant t \leqslant 1\}$$

by the lines

$$s = s_i, \quad i = 0, 1, \ldots, m, \qquad \text{and } t = t_j, \quad j = 0, 1, \ldots, n,$$

be denoted by $\pi_s \times \pi_t$ and name $\pi_s \times \pi_t$ the partition of $[0, 1] \times [0, 1]$ induced by the partitions π_s and π_t of $[0, 1]$. For arbitrary positive ϵ select a specific partition π_s of sufficiently small norm that for all partitions π_t of sufficiently small norm the induced partition $\pi_s \times \pi_t$ is such that

$$(4.29) \quad \Bigg| \sum_{i=1}^{m} \sum_{j=1}^{n} g(s_i)h(t_j)[f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})] \\ - \text{R.S.} \int_0^1 \int_0^1 g(s)h(t)\, d_{s,t}\, f(s,t) \Bigg| < \epsilon/3$$

(the definition of

$$\text{R.S.} \int_0^1 \int_0^1 g(s)h(t)\, d_{s,t} f(s,t)$$

ensures this possibility), and also is such that

$$(4.30) \quad \Bigg| \sum_{i=1}^{m} g(s_i)\left\{ \int_0^1 h(t) df(s_i, t) - \int_0^1 h(t) df(s_{i-1}, t) \right\} \\ - \text{R.S.} \int_0^1 g(s)\, d\left\{ \int_0^1 h(t) df(s,t) \right\} \Bigg| < \epsilon/3$$

(the fact that

$$\int_0^1 h(t)\, df(s,t)$$

is of bounded variation, according to Lemma 4.15, ensures this possibility). Now select a specific partition π_t of sufficiently small norm to satisfy (4.29) (also (4.30) is satisfied, of course) and also to satisfy

$$(4.31) \quad \left| \sum_{j=1}^{n} h(t_j)[f(s,t_j) - f(s, t_{j-1})] - \int_0^1 h(t) df(s,t) \right| < \epsilon \Big/ 6m\left\{ \max_{0 \leqslant s \leqslant 1} |g(s)| \right\}$$

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for all $s \in [0, 1]$ (this is possible according to Lemma 4.14 because

$$\mathrm{Var}_{0\leqslant t\leqslant 1} f(s, t)$$

is bounded as a function of s on $[0, 1]$: cf. (**4**, 346)). Now for the particular resulting $\pi_s \times \pi_t$

$$\begin{aligned}(4.32)\quad &\Bigg| \sum_{i=1}^{m} \sum_{j=1}^{n} g(s_i)h(t_j)[f(s_i, t_j) - f(s_{i-1}, t_j) - f(s_i, t_{j-1}) + f(s_{i-1}, t_{j-1})] \\ &\qquad - \sum_{i=1}^{m} g(s_i)\left\{\int_0^1 h(t)\, df(s_i, t) - \int_0^1 h(t)\, df(s_{i-1}, t)\right\}\Bigg| \\ &\leqslant \left| \sum_{i=1}^{m} g(s_i)\left\{ \sum_{j=1}^{n} h(t_j)[f(s_i, t_j) - f(s_i, t_{j-1})] - \int_0^1 h(t)\, df(s_i, t)\right\}\right| \\ &+ \left| \sum_{i=1}^{m} g(s_i)\left\{ \sum_{j=1}^{n} h(t_j)\, f(s_{i-1}, t_j) - f(s_{i-1}, t_{j-1})] - \int_0^1 h(t)\, df(s_{i-1}, t)\right\}\right| \\ &\leqslant 2m\left\{\max_{0\leqslant s\leqslant 1} |g(s)|\right\}\epsilon \Big/ 6m\left\{\max_{0\leqslant s\leqslant 1} |g(s)|\right\} = \epsilon/3\end{aligned}$$

because of (4.31). The inequalities (4.29), (4.30), and (4.32) imply that

$$\left|\text{R.S.} \int_0^1 \int_0^1 g(s)h(t)\, d_{s,t}f(s, t) - \int_0^1 g(s)\, d\int_0^1 h(t)df(s, t)\right| < \epsilon$$

and the fact that ϵ is an arbitrary positive number completes the proof for the case in which $g(s) \in C$. For the case $g(s) \in C'$, let σ be the s-value for which $g(s)$ takes its jump and let $g^*(s)$ be $g(s)$ modified to be right continuous at σ. Following the pattern used in obtaining the expression for $\rho^n(s, t)$ at the end of §2, one obtains

$$\begin{aligned}&\int_0^1 \int_0^1 g(s)h(t)\, d_{s,t}f(s, t) \\ &\quad = \int_0^\sigma \int_0^1 g(s)h(t)\, d_{s,t}f(s, t) + \int_\sigma^1 \int_0^1 g(s)h(t)\, d_{s,t}f(s, t) \\ &\quad = \text{R.S.} \int_0^\sigma \int_0^1 g(s)h(t)\, d_{s,t}\, g(s, t) + \text{R.S.} \int_\sigma^1 \int_0^1 g^*(s)h(t)\, d_{s,t}f(s, t).\end{aligned}$$

The conclusion of the lemma (with the 1 for the upper limit on s replaced by σ for application to the first integral and a similar replacement for the second) is applicable to each of the last two R.S. integrals and it follows that

$$\begin{aligned}&\int_0^1 \int_0^1 g(s)h(t)\, d_{s,t}f(s, t) \\ &= \text{R.S.} \int_0^\sigma g(s)\, d\left\{\int_0^1 h(t)\, df(s, t)\right\} + \text{R.S.} \int_\sigma^1 g^*(s)\, d\left\{\int_0^1 h(t)\, df(s, t)\right\} \\ &= \int_0^\sigma g(s)\, d\left\{\int_0^1 h(t)\, df(s, t)\right\} + \int_\sigma^1 g(s)\, d\left\{\int_0^1 h(t)\, df(s, t)\right\}\end{aligned}$$

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(because, as proved in Lemma 4.16,

$$\int_0^1 h(t)\, df(s, t)$$

is right continuous). The proof of the lemma for the case $g(s) \in C'$ follows by adding the last two integrals.

The proof of Lemma 4.18 is similar to that of Lemma 4.15 and those of Lemmas 4.19 and 4.20 are similar to that of Lemma 4.17. Thus, only statements of these lemmas will be given.

LEMMA 4.18. *Suppose that* $f(r) \in C'$ *and* $g(s)$ *and* $h(t)$ *are continuous and of bounded variation on* $[0, 1]$. *Suppose also that* $H(r, s, t)$ *is right continuous and is of bounded variation in the sense that*

(i) *it is of bounded variation in* (r, s, t) *on* $[0, 1] \times [0, 1] \times [0, 1]$,

(ii) *it is of bounded variation in any pair of* r, s, t *for the third member fixed*,

(iii) *it is of bounded variation in any one of* r, s, t *for the other pair fixed*.

Then

$$\int_0^1 \int_0^1 \int_0^1 f(r)g(s)h(t)\, d_{r,s,t}H(r, s, t) = \int_0^1 f(r)\, d\left\{\int_0^1 \int_0^1 g(s)h(t)\, d_{s,t}H(r, s, t)\right\}$$

and

$$\left\{\int_0^1 \int_0^1 g(s)h(t)\, d_{s,t}H(r, s, t)\right\}$$

is of bounded variation and is right continuous in r.

LEMMA 4.19. *Suppose that* $f(r)$ *and* $g(s)$ *are in* C' *and that* $h(t)$ *is continuous and of bounded variation on* $[0, 1]$. *Suppose also that* $H(r, s, t)$ *is right continuous and of bounded variation in the sense described in Lemma* 4.18. *Then*

$$\int_0^1 \int_0^1 \int_0^1 f(r)g(s)h(t)\, d_{r,s,t}H(r, s, t) = \int_0^1 \int_0^1 f(r)g(s)\, d_{r,s}\left\{\int_0^1 h(t)\, dH(r, s, t)\right\}$$

and

$$\left\{\int_0^1 h(t)\, dH(r, s, t)\right\}$$

is right continuous and of bounded variation in (r, s).

LEMMA 4.20. *Suppose that* $f(r)$ *and* $g(s)$ *are in* C' *and* $h(t) \in C$. *Suppose furthermore that* $N(r, s)$ *is right continuous and of bounded variation on* $[0, 1] \times [0, 1]$ *and that* $M(t)$ *is of bounded variation. Then*

$$\int_0^1 \int_0^1 \int_0^1 f(r)g(s)h(t)\, d_{r,s,t}[N(r, s)M(t)] = \int_0^1 \int_0^1 f(r)g(s)\, d_{r,s}N(r, s) \cdot \int_0^1 h(t)\, dM(t).$$

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The *proof of Theorem* 5 now follows. It is first noted that for fixed n

$$\int_0^1 [x(t) - x^n(t)]dH(t) < \infty.$$

Because of the definition of $x^n(t)$,

$$\begin{aligned}
&\int_0^1 [x(t) - x^n(t)]\, dH(t) \\
&\quad = \int_0^1 \left[x(t) - \sum_{j=1}^n \int_0^1 x(s) d\alpha_j(s) \cdot \beta_j(t) \right] dH(t) \\
&\quad = \int_0^1 [x(t)]\, dH(t) - \sum_{j=1}^n \left[\int_0^1 \beta_j(t)\, dH(t) \int_0^1 x(s)\, d\alpha_j(s) \right],
\end{aligned}$$

where the equality of the second member to the third member of the last equation is justified by the finiteness of

$$\int_0^1 x(t)\, dH(t)$$

and of

$$\int_0^1 [x(t) - x^n(t)]\, dH(t).$$

Since

$$\int_0^1 \beta_j(t)\, dH(t), \qquad j = 1, 2, \ldots, n,$$

are finite constants, the proof of the theorem, for $i = 1$, is complete.

Now for fixed n

$$\int_0^1 \int_0^1 [x(s) - x^n(s)][x(t) - x^n(t)]\, d_{s,t} H(s, t) < \infty.$$

Also, because of the definition of $x^n(t)$,

$$\begin{aligned}
&\int_0^1 \int_0^1 [x(s) - x^n(s)][x(t) - x^n(t)]\, d_{s,t} H(s, t) \\
&\quad = \int_0^1 \int_0^1 \left[x(s) - \sum_{i=1}^n \int_0^1 x(u)\, d\alpha_i(u) \cdot \beta_i(s) \right] \\
&\qquad\qquad \times \left[x(t) - \sum_{j=1}^n \int_0^1 x(v)\, d\alpha_j(v) \cdot \beta_j(t) \right] d_{s,t} H(s, t) \\
&\quad = \int_0^1 \int_0^1 x(s) x(t) d_{s,t} H(s, t) \\
&\qquad\qquad - \sum_{j=1}^n \int_0^1 x(v)\, d\alpha_j(v) \int_0^1 \int_0^1 x(s) \beta_j(t)\, d_{s,t} H(s, t) \\
&\quad - \{\text{one similar term}\} \\
&\quad + \sum_{i=1}^n \sum_{j=1}^n \int_0^1 \int_0^1 \beta_i(s) \beta_j(t)\, d_{s,t} H(s, t) \cdot \int_0^1 x(u)\, d\alpha_i(u) \cdot \int_0^1 x(v)\, d\alpha_j(v).
\end{aligned}$$

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An application of Lemma 4.17 and Fubini's theorem completes the proof of the theorem for the case $i = 2$.

In the case for which $i = 3$, it is noted that for fixed n

$$\int_0^1 \int_0^1 \int_0^1 [x(r) - x^n(r)][x(s) - x^n(s)][x(t) - x^n(t)]\, d_{r,s,t}H(r, s, t)$$

$$= \int_0^1 \int_0^1 \int_0^1 \left[x(r) - \sum_{i=1}^{n} \int_0^1 x(u)d\alpha_i(u)\cdot\beta_i(r)\right]$$

$$\times\left[x(s) - \sum_{j=1}^{n} \int_0^1 x(v)d\alpha_j(v)\cdot\beta_j(s)\right]$$

$$\times\left[x(t) - \sum_{k=1}^{n} \int_0^1 x(w)d\alpha_k(w)\cdot\beta_k(t)\right] d_{r,s,t}H(r, s, t)$$

$$= \int_0^1 \int_0^1 \int_0^1 x(r)x(s)x(t)\, d_{r,s,t}H(r, s, t)$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{n} \int_0^1 x(v)\, d\alpha_j(v)\cdot \int_0^1 x(w)\, d\alpha_k(w)$$

$$\times \int_0^1 \int_0^1 \int_0^1 x(r)\beta_j(s)\beta_k(t)\, d_{r,s,t}H(r, s, t)$$

$+$ {two similar terms}

$$-\sum_{k=1}^{n} \int_0^1 x(w)\, d\alpha_k(w) \int_0^1 \int_0^1 \int_0^1 x(r)x(s)\beta_k(t)\, d_{r,s,t}H(r, s, t)$$

$-$ {two similar terms}

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \int_0^1 x(u)\, d\alpha_i(u) \int_0^1 x(v)\, d\alpha_j(v) \int_0^1 x(w)\, d\alpha_k(w)$$

$$\times \int_0^1 \int_0^1 \int_0^1 \beta_i(r)\beta_j(s)\beta_k(t)\, d_{r,s,t}H(r, s, t).$$

An application of Lemmas 4.16 to 4.20 and Fubini's theorem completes the proof of the theorem.

Now the proof of Lemma 3.1 will be given. Let $\alpha_1(s), \alpha_2(s), \ldots$ be such that

$$\int_0^1 \left[\sum_{i=1}^{n} \alpha_i(s)\beta_i(t)\right]^2 dt \leqslant M$$

for $n = 1, 2, 3, \ldots$ and all $s \in [0, 1]$. Then

$$\int_0^1 [\rho^n(s, t) - \rho(s, t)]^2\, dt$$

$$\leqslant \int_0^1 [\rho^n(s, t)]^2\, dt + 2\int_0^1 |\rho^n(s, t)|\rho(s, t)\, dt + \int_0^1 [\rho(s, t)]^2\, dt$$

$$\leqslant M + 2\sqrt{\left[\int_0^1 [\rho^n(s, t)]^2\, dt \cdot \int_0^1 [\rho(s, t)]^2\, dt\right]} + 1$$

$$\leqslant M + 2\sqrt{M} + 1 = (1 + \sqrt{M})^2,$$

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and the proof of part (i) is complete. The proof of (ii) is as follows:

$$\int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\beta_i(t) \right]^2 dt = \int_0^1 \left[\sum_{i=1}^{N} \alpha_i(s)\beta_i(t) - \sum_{i=1}^{n} \alpha_i(s)\beta_i(t) \right]^2 dt$$

$$\leqslant 2 \int_0^1 \left[\sum_{i=1}^{N} \alpha_i(s)\beta_i(t) \right]^2 dt + 2 \int_0^1 \left[\sum_{i=1}^{n} \alpha_i(s)\beta_i(t) \right]^2 dt \leqslant 4M.$$

To prove part (iii) one need only note that

$$\int_0^1 \left\{ \int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\beta_i(t) \right]^2 dt \right\}^2 ds \leqslant 4M \int_0^1 \int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\beta_i(t) \right]^2 dt\, ds$$

(because of (11))

$$= 4M \int_0^1 \int_0^1 \left[\sum_{i=n+1}^{N} \alpha_i(s)\beta_i(t) \right]^2 ds\, dt = 4M \int_0^1 \left[\sum_{i=n+1}^{N} \beta_i^2(t) \right] dt$$

$$= 4M \sum_{i=n+1}^{N} \gamma_{i,i} \; (\leqslant 2M).$$

The proof of part (iv) follows. Let ϵ be an arbitrary positive number and let ϵ_1 be chosen to satisfy the following conditions:

(a) $0 < \epsilon_1 < 1$,
(b) $\epsilon_1 < \epsilon/\sqrt{3}$,
(c) $\sqrt{\epsilon_1} < \epsilon/12(1 + \sqrt{M})^3$,
(d) $\epsilon_1 < \epsilon/3(1 + \sqrt{M})^4$,
(e) $\sqrt{\epsilon_1} < \epsilon/2[2(1 + \sqrt{M})^2 + 1][2(1 + \sqrt{M}) + 1]$,
(f) $\epsilon_1 < \epsilon/32M^2$.

Since

$$\lim_{m\to\infty} \int_0^1 \int_0^1 [\rho(s,t) - \rho^m(s,t)]^2 dt\, ds = \lim_{m\to\infty} \int_0^1 \int_0^1 [\rho(s,t) - \rho^m(s,t)]^2 ds\, dt$$

$$= \lim_{m\to\infty} \int_0^1 \sum_{i=m+1}^{\infty} \beta_i^2(t)\, dt = \lim_{m\to\infty} \sum_{i=m+1}^{\infty} \gamma_{i,i} = 0,$$

it follows that there exists m_0 such that for all $m > m_0$

$$\int_0^1 [\rho(s,t) - \rho^m(s,t)]^2 dt < \epsilon_1,$$

save for $s \in s(m) \subset [0, 1]$ where $s(m)$ is of measure less than or equal to ϵ_1. Now the Minkowski inequality yields

$$\sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2 dt \right)}$$

$$\leqslant \sqrt{\left(\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2 dt \right)} + \sqrt{\left(\int_0^1 [\rho^m(s,t) - \rho(s,t)]^2 dt \right)}$$

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and

$$\sqrt{\left(\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right)}$$
$$\leqslant \sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)} + \sqrt{\left(\int_0^1 [\rho^m(s,t) - \rho(s,t)]^2\,dt\right)}$$

so that

$$\left|\sqrt{\left(\int_0 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right)} - \sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)}\right|$$
$$\leqslant \sqrt{\left(\int_0^1 [\rho^m(s,t) - \rho(s,t)]^2\,dt\right)}.$$

Hence, for $m > m_0$ and $s \notin s(m)$,

$$\left|\sqrt{\left(\int_0 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right)} - \sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)}\right| \leqslant \epsilon_1.$$

Equivalently, for $m > m_0$ and $s \notin s(m)$,

$$\text{(4.33)}\quad \left[\sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)} - \sqrt{\epsilon_1}\right]^4 - \epsilon_1^2$$
$$\leqslant \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right]^2$$
$$\leqslant \left[\sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)} + \sqrt{\epsilon_1}\right]^4.$$

It will now be noted that for $0 \leqslant b \leqslant 1$

$$[a+b]^2 = a^2 + 2ab + b^2 \leqslant a^2 + 2ab + b = a^2 + (2a+1)b.$$

Thus for $0 \leqslant b \leqslant 1$ and $0 \leqslant (2a+1)b \leqslant 1$,

$$\text{(4.34)}\qquad [a+b]^4 \leqslant a^4 + (2a^2+1)(2a+1)b.$$

Also, by part (i) of the lemma

$$\text{(4.35)}\qquad \int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt \leqslant (1+\sqrt{M})^2$$

for $n = 1, 2, 3, \ldots$ and all $s \in [0, 1]$. Because of condition (a) on ϵ_1, (4.34) in conjunction with (4.35) yields

$$\text{(4.36)}\quad \left[\sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)} + \sqrt{\epsilon_1}\right]^4$$
$$\leqslant \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2$$
$$+ [2(1+\sqrt{M})^2 + 1][2(1+\sqrt{M}) + 1]\sqrt{\epsilon_1}.$$

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It will also be noted that

$$[a - b]^2 = a^2 - 2ab + b^2 \geqslant a^2 - 2ab$$

and thus

$$[a - b]^4 \geqslant a^4 - 4a^3b.$$

Thus

$$\left[\sqrt{\left(\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right)} - \sqrt{\epsilon_1}\right]^4 \tag{4.37}$$
$$\geqslant \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2 - 4(1+\sqrt{M})^3\sqrt{\epsilon_1}.$$

Therefore for $m > m_0$ and $s \notin s(m)$ it follows from (4.33), (4.36), and (4.37) that

$$\left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2 - 4(1+\sqrt{M})^3\sqrt{\epsilon_1} - \epsilon_1^2$$
$$\leqslant \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right]^2$$
$$\leqslant \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2$$
$$+ [2(1+\sqrt{M})^2 + 1][2(1+\sqrt{M}) + 1]\sqrt{\epsilon_1},$$

and thus, for $m > m_0$,

$$\int_{[0,1]-s(m)} \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2 ds - 4(1+\sqrt{M})^3\sqrt{\epsilon_1} - \epsilon_1^2 \tag{4.38}$$
$$\leqslant \int_{[0,1]-s(m)} \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right]^2 ds$$
$$\leqslant \int_{[0,1]-s(m)} \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2 ds$$
$$+ [2(1+\sqrt{M})^2 + 1][2(1+\sqrt{M}) + 1]\sqrt{\epsilon_1}.$$

Also, because of part (ii) of the lemma and the fact that the measure of $s(m)$ is no greater than ϵ_1,

$$0 \leqslant \int_{s(m)} \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right]^2 ds \leqslant 16M^2\epsilon_1. \tag{4.39}$$

Addition of inequalities (4.38) and (4.39) yields, for $m > m_0$,

$$\int_{[0,1]-s(m)} \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2 ds - 4(1+\sqrt{M})^3\sqrt{\epsilon_1} - \epsilon_1^2$$
$$\leqslant \int_0^1 \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2\,dt\right]^2 ds$$
$$\leqslant \int_{[0,1]-s(m)} \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2\,dt\right]^2 ds$$
$$+ [2(1+\sqrt{M})^2 + 1][2(1+\sqrt{M}) + 1]\sqrt{\epsilon_1} + 16M^2\epsilon_1.$$

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From (4.35) and the fact that $s(m)$ has measure no greater than ϵ_1 it now follows, for $m > m_0$, that

$$\int_0^1 \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2 \, dt \right]^2 ds - 4(1+\sqrt{M})^3 \sqrt{\epsilon_1} - (1+\sqrt{M})^4 \epsilon_1 - \epsilon_1^2$$

$$\leqslant \int_0^1 \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2 \, dt \right]^2 ds$$

$$\leqslant \int_0^0 \left[\int_0^0 [\rho^n(s,t) - \rho(s,t)]^2 \, dt \right]^2 ds$$

$$+ [2(1+\sqrt{M})^2 + 1][2(1+\sqrt{M}) + 1]\sqrt{\epsilon_1} + 16M^2\epsilon_1.$$

Now, in view of conditions (b), (c), (d), (e), (f) on ϵ_1, it follows that, for $m > m_0$,

$$\int_0^1 \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2 \, dt \right]^2 ds - \epsilon$$

$$\leqslant \int_0^1 \left[\int_0^1 [\rho^m(s,t) - \rho^n(s,t)]^2 \, dt \right]^2 ds$$

$$\leqslant \int_0^1 \left[\int_0^1 [\rho^n(s,t) - \rho(s,t)]^2 \, dt \right]^2 ds + \epsilon,$$

which completes the proof of Lemma 3.1.

Finally the proof of Lemma 3.2 now follows. The required boundedness for all Sturm-Liouville sets of α's follows from the boundedness for the Fourier cosine functions and the asymptotic approach of the functions of any Sturm-Liouville set to the functions of the Fourier cosine set. More precisely, by (**4**, vol. 2, p. 722) (it will be convenient here to suppose the indices on the α's begin at 0 rather than 1), the sum

$$\sum_{i=0}^{n} \alpha_i(u)\alpha_i(s) - \left[1 + 2\sum_{i=1}^{n} \cos i\pi u \cos i\pi s \right]$$

is bounded for all u and s on $[0, 1]$ and for all positive integers n. Thus

$$\int_0^1 \left\{ \sum_{i=0}^{n} \alpha_i(u)\alpha_i(s) - \left[1 + 2\sum_{i=1}^{n} \cos i\pi u \cos i\pi s \right] \right\} \rho(u,t) \, du$$

is bounded for all s and t on $[0, 1]$ and all n. Thus, to show that

$$\sum_{i=0}^{n} \beta_i(t)\alpha_i(s)$$

is bounded and hence that

$$\sum_{i,j=0}^{n} \gamma_{i,j}\alpha_i(s)\alpha_i(s) \left(= \int_0^1 \left[\sum_{i=0}^{n} \beta_i(t)\alpha_i(s) \right]^2 dt \right)$$

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is bounded for all s on $[0, 1]$ and all positive integers n, it suffices to notice that

$$(4.40)\quad \int_0^1 \left[1 + 2\sum_{i=1}^{n} \cos i\pi u \cos i\pi s\right] \rho(u, t)\, du$$

$$= t + \frac{2}{\pi}\sum_{i=1}^{n} [\sin i\pi t \cos i\pi s]/i$$

$$= t + \frac{1}{\pi}\sum_{i=1}^{n} [\sin i\pi(t + s)/i + \sin i\pi(t - s)/i]$$

and that the boundedness of the last expression in t, s and n follows from the boundedness of

$$\sum_{i=1}^{n} \sin i\pi u/i$$

(**4**, vol. 2, pp. 493–8) in u and n. It is noted that the left member of (4.40) is just

$$\sum_{i=0}^{n} \beta_i(t)\alpha_i(s)$$

for the Fourier cosine functions and that no use was made of a possible averaging effect from the integration on t in establishing the boundedness of

$$\int_0^1 \left[\sum_{i=0}^{n} \beta_i(t)\alpha_i(s)\right]^2 dt\left(= \sum_{i,j=0}^{n} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s)\right).$$

5. In the present section the boundedness condition on

$$\sum_{i,j=1}^{n} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s),$$

required in the corollary to Theorem 6 (which has been shown to hold for all Sturm–Liouville sets of α's) is also verified for the sine and Haar functions. The author does not know whether it holds for all sets of α's. Order estimates are found for

$$\lim_{n\to\infty} \int_0^1 \left[\sum_{i,j=n+1}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s)\right]^2 ds$$

for the Fourier cosine and the Fourier sine functions as well as for the Haar functions. For the Fourier cosine and the Haar functions this order is

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^2,$$

whereas for the Fourier sine functions it is

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^1$$

and not

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^{1+\epsilon} \qquad \text{for any } \epsilon > 0.$$

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This shows that the best bound on $|\epsilon_n|$ provided by the corollary for all admissible sets of α's is

$$O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^{1}.$$

It should be noted that, now that specific sets of α's are being dealt with, the α's will not always be indexed by the indices $1, 2, 3, \ldots$. Any change in the indexing of the α's will of course give rise to corresponding changes in the indexing of the β's and γ's.

Though a specific bound is found in this paper for the sums of products of γ's and α's for the Haar functions, it is of interest to note again that, just as for Sturm–Liouville sets of α's, this boundedness really follows from the boundedness of the sums of products of the β's and α's. This latter remark is justified by a result given in (**1**, pp. 47–49) for which, by adopting the notation there used and letting $f(t) = \rho(t, u)$, one obtains that the sums of products $\beta(u)$'s and $\alpha(x)$'s are bounded by 1. Again, the author does not know whether, for all sets of α's the sums of products of β's and α's is bounded.

Finally now the specific considerations are made for the three above-mentioned sets of α's.

(i) The Fourier cosine functions:

$$\alpha_0(s) = \begin{cases} 1 & \text{if } s \in [0, 1), \\ 0 & \text{if } s = 1, \end{cases}$$

$$\alpha_i(s) = \begin{cases} \sqrt{2}\cos i\pi s & \text{if } s \in [0, 1), \\ 0 & \text{if } s = 1, \end{cases} \qquad i = 1, 2, 3, \ldots,$$

$$\beta_0(t) = t,$$

$$\beta_i(t) = (\sqrt{2}/i\pi)\sin i\pi t, \qquad i = 1, 2, 3, \ldots,$$

$$\gamma_{0,0} = 1/3,$$

$$\gamma_{0,1} = \sqrt{2}(-1)^{i+1}/i^2\pi^2, \qquad i = 1, 2, 3, \ldots,$$

$$\gamma_{i,j} = 0 \text{ if } i \neq j, \qquad i, j = 1, 2, 3, \ldots,$$

$$\gamma_{i,i} = 1/i^2\pi^2, \qquad i = 1, 2, 3, \ldots.$$

(a)
$$\sum_{i,j=0}^{N} \gamma_{i,j}\,\alpha_i(s)\alpha_j(s) \leqslant 4/3$$

for $N = 0, 1, 2, \ldots$ and all $s \in [0, 1]$. The sum

$$\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6$$

yields the estimate at once.

(b)

$$(5.1) \qquad \lim_{N\to\infty} \int_0^1 \left(\sum_{i,j=n}^{N} \gamma_{i,j}\,\alpha_i(s)\alpha_j(s)\right)^2 ds \leqslant 16/\pi^4 n^2, \qquad n \geqslant 1.$$

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To establish this inequality, it need only be noted that

$$\sum_{i,j=n}^{N} \gamma_{i,j}\,\alpha_i(s)\alpha_j(s) \leqslant (2/\pi^2)\sum_{i=n}^{\infty}(1/i^2).$$

$$(5.2) \qquad \left(\sum_{i=n}^{\infty}\gamma_{i,i}\right)^2 \geqslant 1/\pi^4 n^2, \qquad n \geqslant 1,$$

is easy to verify from the definition of the γ's. From (5.1) and (5.2) it follows that

$$\lim_{N\to\infty}\int_0^1\left[\sum_{i,j=n}^{N}\gamma_{i,j}\,\alpha_i(s)\alpha_j(s)\right]^2 ds = O\left(\sum_{i=n}^{\infty}\gamma_{i,i}\right)^2.$$

(ii) The Haar functions:

$$\alpha_0^{(0)}(s) = \begin{cases} 1 & \text{for } s \in [0,1), \\ 0 & \text{for } s = 0, \end{cases}$$

$$\alpha_0^{(1)}(s) = \begin{cases} 1 & \text{for } s \in [0,1/2), \\ -1 & \text{for } s \in [1/2,1), \\ 0 & \text{for } s = 0, \end{cases}$$

$$\alpha_1^{(1)}(s) = \begin{cases} \sqrt{2} & \text{for } s \in [0,1/4), \\ -\sqrt{2} & \text{for } s \in [1/4,1/2), \\ 0 & \text{elsewhere}, \end{cases}$$

$$\alpha_1^{(2)}(s) = \begin{cases} \sqrt{2} & \text{for } s \in [1/2,3/4), \\ -\sqrt{2} & \text{for } s \in [3/4,1), \\ 0 & \text{elsewhere}, \end{cases}$$

and in general

$$\alpha_n^{(1)}(0) = 2^{n/2}, \qquad \alpha_n^{(2n)}(1) = -2^{n/2},$$

$$\alpha_n^{(k)}(s) = \begin{cases} 2^{n/2} & \text{for } s \in [(2k-2)/2^{n+1}, (2k-1)/2^{n+1}), \\ -2^{n/2} & \text{for } s \in ((2k-1)/2^{n+1}, 2k/2^{n+1}], \\ 0 & \text{elsewhere}, \end{cases}$$

where n ranges over $1, 2, 3, \ldots$ and k ranges over $1, 2, 3, \ldots, 2^n$.

The two-indices notation is used also for the β's as follows:

$$\beta_n^{(k)}(t) = \int_0^t \alpha_n^{(k)}(s)\,ds.$$

It can be shown that

$$\beta_0^{(0)}(t) = t,\ \beta_0^{(1)}(t) = \begin{cases} t & \text{for } t \in [0,1/2], \\ 1-t & \text{for } t \in [1/2,1], \end{cases}$$

and for $n \geqslant 1$, $k = 1, 2, \ldots, 2^n$,

$$\beta_n^{(k)}(t) = \begin{cases} 0 & \text{for } t \in [0, (2k-2)/2^{n+1}], \\ \sqrt{2^n}(t-(2k-2)/2^{n+1}) & \text{for } t \in [(2k-2)/2^{n+1}, (2k-1)/2^{n+1}], \\ \sqrt{2^n}(-t+2k/2^{n+1}) & \text{for } t \in [(2k-1)/2^{n+1}, 2k/2^{n+1}], \\ 0 & \text{for } t \in [2k/2^{n+1}, 1]. \end{cases}$$

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The two-indices notation on the β's gives rise naturally to a four-indices notation on the γ's. Thus

$$\gamma_{n,m}{}^{(k),(w)} = \int_0^1 \beta_n{}^{(k)}(t)\beta_m{}^{(w)}(t)\,dt.$$

It can be shown that the values for the various γ's are those given below:

$$\gamma_{0,0}{}^{(0),(0)} = 1/3,$$
$$\gamma_{0,0}{}^{(0),(1)} = 1/8,$$
$$\gamma_{0,n}{}^{(0),(k)} = \sqrt{2^n}(2k-1)/2^{3n+3} \qquad \text{for } n \geqslant 1, k = 1, 2, \ldots, 2^n,$$
$$\gamma_{0,n}{}^{(1),(1)} = 1/12,$$
$$\gamma_{0,n}{}^{(1),(k)} = \begin{cases} \sqrt{2^n}(2k-1)/2^{3n+3} & \text{for } k \leqslant 2^{n-1}, n \geqslant 1, \\ \sqrt{(2^n/2^{3n+3})}[2^{n+1} - (2k-1)] & \text{for } k = 2^{n-1}+1, \\ & \qquad 2^{n-1}+2, \ldots, 2^n, \end{cases}$$
$$\gamma_{n,n}{}^{(k),(k)} = 1/(12\cdot 2^{2n}), \qquad n \geqslant 1, k = 1, 2, \ldots, 2^n,$$
$$\gamma_{n,n}{}^{(k),(j)} = 0, \qquad n \geqslant 1, k \neq j,$$
$$\gamma_{n,n+p}{}^{(k),(w)} = \begin{cases} (1/2^{3n+3p+3})[(2w-1) - 2^p(2k-1)]2^n\sqrt{2^p} \\ \qquad\qquad \text{if } (2k-2)2^{p-1}+1 \leqslant w \leqslant (2k-1)2^{p-1}, \\ (1/2^{3n+3p+3})[-(2w-1) + 2(2k)]2^n\sqrt{2^p} \\ \qquad\qquad \text{if } (2k-1)2^{p-1}+1 \leqslant w \leqslant (2k)2^{p-1}, \\ 0 \qquad \text{otherwise}, \end{cases}$$
$$n \geqslant 1, p \geqslant 1.$$

(a) The expression

$$\sum_{i,j=1}^{N} \gamma_{i,j}\,\alpha_i(s)\alpha_j(s),$$

for fixed $n \geqslant 2$, used where the α's each have one index and the indices begin at 1, is here replaced by

$$\begin{aligned} g_{n,r}{}^{N,p}(s) = &\sum_{k=r}^{2^n}\sum_{w=r}^{2^n} \gamma_{n,n}{}^{(k),(w)}\alpha_n{}^{(k)}(s)\alpha_n{}^{(w)}(s) \\ &+ \sum_{i,j=n+1}^{N}\sum_{k=1}^{2^i}\sum_{w=1}^{2^j} \gamma_{i,j}{}^{(k),(w)}\alpha_i{}^{(k)}(s)\alpha_j{}^{(w)}(s) \\ &+ \sum_{k=1}^{p}\sum_{w=1}^{p} \gamma_{N+1,N+1}{}^{(k),(w)}\alpha_{N+1}{}^{(k)}(s)\alpha_{N+1}{}^{(w)}(s) \\ &+ 2\sum_{i=n+1}^{N}\sum_{w=1}^{2^i}\sum_{k=r}^{2^n} \gamma_{i,n}{}^{(w),(k)}\alpha_i{}^{(w)}(s)\alpha_n{}^{(k)}(s) \\ &+ 2\sum_{k=r}^{2n}\sum_{w=1}^{p} \gamma_{n,N+1}{}^{(k),(w)}\alpha_n{}^{(k)}(s)\alpha_{N+1}{}^{(w)}(s) \\ &+ 2\sum_{i=n+1}^{N}\sum_{k=1}^{2^i}\sum_{w=1}^{p} \gamma_{i,N+1}{}^{(k),(w)}\alpha_i{}^{(k)}(s)\alpha_{N+1}{}^{(w)}(s), \end{aligned}$$

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where $1 \leqslant r \leqslant 2^n$ and $1 \leqslant p \leqslant 2^{N+1}$. It can now be shown that this latter sum is bounded independently of N, p, and s by a decreasing function of the variables n and r. (To show that the expression corresponding to

$$\sum_{i,j=n}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s)$$

is bounded is not more difficult. However, as seen from the corollary to Theorem 6, it is the bound, as a decreasing function of n and r, which is useful in estimating the error in Theorem 6.)

A routine but long calculation yields

$$\limsup_{\substack{N,p\\(1\leqslant p\leqslant 2^n)}} \int_0^1 [g_{n,r}{}^{N,p}(s)]^2\, ds \leqslant [13/(3.2^n)]^2.$$

This result is based on the inequalities

$$\sum_{k=1}^{2^n} |\alpha_n{}^{(k)}(s)| \leqslant 2\sqrt{2^n}, \quad \sum_{k=1}^{2^n} [\alpha_n{}^{(k)}(s)]^2 \leqslant 2.2^n, \qquad n = 1, 2, \ldots ; s \in [0, 1].$$

Now the expression

$$\sum_{i=n+1}^{\infty} \gamma_{i,i},$$

used when the α's each have one index and the indices begin at 1, is here replaced by

$$\begin{aligned}\sum_{i=r}^{2^n} \gamma_{n,n}{}^{(i),(i)} - \sum_{i=n+1}^{\infty} \sum_{j=1}^{2^i} \gamma_{i,i}{}^{(j),(j)}\\ &= (2^n - r)/(12.2^n) + \sum_{i=n+1}^{\infty} 2^i/(12.2^{2i}) = (2^n - r)/(12.2^n) + 1/(12.2^n)\\ &\geqslant 1/(12.2^n).\end{aligned}$$

Thus

$$\limsup_{\substack{N,p\to\infty\\(1\leqslant p\leqslant 2^N)}} \int_0^1 [g_{n,r}{}^{N,p}(s)]^2\, ds \leqslant 2704\left(\sum_{i=r}^{2^n} \gamma_{n,n}{}^{(i),(i)} + \sum_{i=n+1}^{\infty} \sum_{j=1}^{2^i} \gamma_{i,i}{}^{(j),(j)}\right)^2.$$

Thus the Haar functions, when ordered in a single sequence, satisfy the condition

$$|\epsilon_n| = O\left(\sum_{i=n+1}^{\infty} \gamma_{i,i}\right)^2.$$

(iii) The Fourier sine functions:

$$\begin{aligned} \alpha_i(s) &= \sqrt{2}\sin i\pi s &&\text{for } s \in [0, 1], i = 1, 2, 3, \ldots,\\ \beta_i(t) &= \sqrt{2}(1 - \cos i\pi t)/i\pi, && i = 1, 2, 3, \ldots,\\ \gamma_{i,j} &= 2[1 + \delta_{i,j}/2]/ij\pi^2, && i, j = 1, 2, 3, \ldots. \end{aligned}$$

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In this case, it is false for each ϵ, $0 < \epsilon < 1$, that

$$(5.3)\qquad \lim_{N\to\infty} \int_0^1 \Big[\sum_{i,j=n+1}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s) \Big]^2 ds = O\Big(\sum_{i=n+1}^{\infty} \gamma_{i,i} \Big)^{1+\epsilon}.$$

The proof of this follows:

$$\sum_{i,j=1}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s) = 4\Big[\Big(\sum_{i=1}^{N} (\sin i\pi s)/i\Big)^2 + \frac{1}{2}\Big(\sum_{i=1}^{N} \sin^2 i\pi s/i^2\Big)\Big] \Big/ \pi^2$$

and thus

$$(5.4)\qquad \int_0^1 \Big[\sum_{i,j=n+1}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s) \Big]^2 ds \geqslant \frac{16}{\pi^4} \int_0^1 \Big[\sum_{i=n+1}^{N} (\sin i\pi s)/i \Big]^4 ds.$$

Hence

$$\lim_{N\to\infty} \int_0^1 \Big[\sum_{i,j=n+1}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s) \Big]^2 ds \geqslant \frac{16}{\pi^4} \lim_{N\to\infty} \int_0^1 \Big[\sum_{i=n+1}^{N} (\sin i\pi s)/i \Big]^4 ds$$

(the existence of the limit on the left follows from Lemma 3.1(iv) and that of the right follows below). Now, as noted just after (4.37), there exists M such that

$$\Big| \sum_{i=n+1}^{N} (\sin i\pi s)/i \Big| \leqslant M$$

for all n, N ($\geqslant n+1$) and $s \in [0, 1]$. Also

$$\lim_{N\to\infty} \sum_{i=n+1}^{N} (\sin i\pi s)/i = \pi(1-s)/2 - \sum_{i=1}^{n} (\sin i\pi s)/i \quad \text{for almost all } s \in [0,1].$$

Thus, Lebesgue's bounded convergence theorem yields

$$(5.5)\qquad \lim_{N\to\infty} \int_0^1 \Big[\sum_{i=n+1}^{N} (\sin i\pi s)/i \Big]^4 ds = \int_0^1 \Big[\pi(1-s)/2 - \sum_{i=1}^{n} (\sin i\pi s)/i \Big]^4 ds.$$

It will be shown below that

$$(5.6)\qquad \int_0^1 \Big[\pi(1-s)/2 - \sum_{i=1}^{n} (\sin i\pi s)/i \Big]^4 ds \geqslant K/n \qquad \text{for some } K > 0$$

for all sufficiently large n.

From (5.5) and (5.6) follows

$$\lim_{N\to\infty} \int_0^1 \Big[\sum_{i=n+1}^{N} (\sin i\pi s)/i \Big]^4 ds \geqslant K/n$$

and thus from (5.4)

$$(5.7)\qquad \lim_{N\to\infty} \int_0^1 \Big[\sum_{i,j=n+1}^{N} \gamma_{i,j}\, \alpha_i(s)\alpha_j(s) \Big]^2 ds \geqslant K/n.$$

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Now

$$\sum_{i=n+1}^{\infty} \gamma_{i,i} = \frac{3}{2} \sum_{i=n+1}^{\infty} (1/i^2) \leqslant 3/(2n). \tag{5.8}$$

(5.7) and (5.8) clearly established (5.3).

To establish (5.7), note that $(\sin u)/u \geqslant 1 - u^2/6 \geqslant 5/6$ for $0 < u \leqslant 1$. Thus for fixed $n > 0$ and $0 < s \leqslant 2/[(2n+1)\pi]$,

$$\int_0^{(n+1/2)\pi s} (\sin u)/u \, du \geqslant 5(n+1/2)\pi s/6. \tag{5.9}$$

Now Hobson (**4**, vol. 2, p. 495) shows that

$$\sum_{i=1}^{n} (\sin i\pi s)/i - \pi(1-s)/2 \tag{5.10}$$
$$= \int_0^{(n+1/2)\pi s} (\sin u)/u \, du - \pi/2 + \theta A/(n+1/2)$$

for $0 \leqslant s \leqslant 1$ and $n = 1, 2, 3, \ldots$, where A is a constant independent of n and s and $-1 \leqslant \theta \leqslant 1$. From (5.9) it can be seen that for sufficiently large n

$$\int_0^{(n+1/2)\pi s} (\sin u)/u \, du - \pi/2 + \theta A/(n+1/2) \geqslant 2(2n+1)\pi s/6 \tag{5.11}$$

for $1/[(2n+1)\pi] \leqslant s \leqslant 2/[(2n+1)\pi]$. It follows from (5.9) and (5.10) that

$$\int_0^1 \left[\sum_{i=1}^{n} (\sin i\pi s)/i - \pi(1-s)/2 \right]^4 ds$$
$$\geqslant \int_{1/\pi(2n+1)}^{2/\pi(2n+1)} [(2n+1)\pi s/3]^4 \, ds \geqslant K/n$$

for some $K > 0$ and all sufficiently large n. Thus (5.6) is established and the proof is complete.

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References

1. G. Alexits, *Convergence problems of orthogonal series* (New York, 1961).
2. R. H. Cameron, *A "Simpson's Rule" for the numerical evaluation of Wiener's integrals in function space*, Duke Math. J., *18* (1951), 111–130.
3. Einar Hille, *Analytic function theory*, vol. 1 (New York, 1959).
4. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, 3rd ed. (Cambridge, 1957).
5. E. C. Titchmarsh, *The theory of functions*, 2nd ed. (Oxford, 1950).

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