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ABSOLUTE PURITY

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1. Introduction. Throughout this paper we use the Bourbaki [**1**] conventions for rings and modules: all rings are associative but not necessarily commutative and have a 1; all modules are unital.

Our purpose is to extend and simplify some recent results of Maddox [**7**], Megibben [**8**], Enochs [**3**], and the author [**5**] on absolutely pure modules by introducing several new dimensions, and using the absolutely pure dimension introduced by the author in [**6**]. This completes some work on character modules and dimension in [**5**] and [**6**].

An A-module will be called an *FFR-module* if and only if it has a resolution by finitely generated free A-modules. A cyclic module A/I, where I is a one-sided ideal, is called fp-*cyclic* if and only if I is finitely generated. We use "fp" for finitely presented (modules), i.e. a module of the form F/K where F and K are both finitely generated modules and F is free.

2. The dimensions. *Flat dimensions.* For any left A-module M we define fl $M = \inf(n|\mathrm{Tor}^{n+1}(X, M) = 0$ for all right A-modules $X)$. In [**6**] we have shown that fl $M = \inf(n|\mathrm{Tor}^{n+1}(X, M) = 0$ for all fp cyclic $X)$. This is the usual weak dimension of Cartan-Eilenberg [**2**]. We define also fl′ $M = \inf(n|\mathrm{Tor}^{n+1}(X, M) = 0$ for all FFR-modules $X)$. If these infs have no finite value we write fl $M = \infty$, etc.

Absolutely pure dimension. We define

$$\text{ap } M = \inf(n|\mathrm{Ext}^{n+1}(X, M) = 0 \text{ for all fp modules } X).$$

Some basic properties of this dimension were established in [**6**].

Injective dimensions. We define:

(1) inj $M = \inf(n|\mathrm{Ext}^{n+1}(X, M) = 0$ for all $X)$, which is also equal to $\inf(n|\mathrm{Ext}^{n+1}(X, M) = 0$ for all cyclic $X)$.

(2) inj′ $M = \inf(n|\mathrm{Ext}^{n+1}(X, M) = 0$ for all FFR-modules $X)$.

(3) winj $M = \inf(n|\mathrm{Ext}^{n+1}(X, M) = 0$ for all fp cyclics $X)$, which is called the weak injective dimension.

Projective dimension. we define

$$\text{pr } M = \inf(n|\mathrm{Ext}^{n+1}(M, X) = 0 \text{ for all } X).$$

We use capitals to denote the corresponding global dimensions: thus

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FL(A) = weak or flat global dimensions of the ring A = sup fl M, with the sup taken over all modules M. When only one ring A is under consideration we sometimes just write FL in place of FL(A), etc. Some of these dimensions are of course one-sided, in which case they are denoted as l.INJ′, r.AP, etc.

If d denotes any one of these dimensions then clearly $d' \leqq d$, i.e. $d'M \leqq dM$ for all M, and hence $D'(A) \leqq D(A)$. We have also: inj′ $\leqq$ ap $\leqq$ inj and winj $\leqq$ ap $\leqq$ inj with corresponding results for the global dimension case. It is also well-known that INJ = PR = the global dimension in the sense of Cartan-Eilenberg [**2**].

3. Relationship between the dimensions. Character modules. For any A-module M we define its character module:

$$M^* = \mathrm{Hom}_Z(M, Q/Z).$$

Further details may be found in [**5**]. In particular we shall frequently use the fact that $M = 0$ if and only if $M^* = 0$.

The following theorem extends earlier results of the author in [**5**] and [**6**], and completely describes the relationship between dimension and character modules.

THEOREM 3.1. *For any module M we have*
(1) $\mathrm{fl}'\ M = \mathrm{inj}'\ M^*$;
(2) $\mathrm{inj}'\ M = \mathrm{fl}'\ M^*$;
(3) $\mathrm{fl}\ M = \mathrm{winj}\ M^* = \mathrm{ap}\ M^* = \mathrm{inj}\ M^*$.

Proof. We use two basic isomorphisms:

$$\text{(i)} \qquad \mathrm{Ext}^n(N, M^*) \simeq (\mathrm{Tor}^n(N, M))^*$$

which is valid for all right A-modules N, left A-modules M, and all $n \geqq 1$, and

$$\text{(ii)} \qquad (\mathrm{Ext}^n(N, M))^* \simeq \mathrm{Tor}^n(M^*, N)$$

which is valid for all left A-modules M, all left FFR A-modules N, and all $n \geqq 1$.

We have used (i) in [**5**] and remark that (ii) is a slight extension of our results in [**6**]. We now show (1):

$$\begin{aligned} \mathrm{fl}'\ M \leqq n &\Leftrightarrow \mathrm{Tor}^{n+1}(N, M) = 0 \text{ for all FFR } N \\ &\Leftrightarrow \mathrm{Ext}^{n+1}(N, M^*) = 0 \text{ for all FFR } N \\ &\Leftrightarrow \mathrm{inj}'\ M \leqq n. \end{aligned}$$

(2) is shown in exactly the same way. For (3), we have

$$\begin{aligned} \text{inj } M^* \leqq n &\Leftrightarrow \text{Ext}^{n+1}(N, M^*) = 0 \text{ for all } N \\ &\Leftrightarrow \text{Tor}^{n+1}(N, M) = 0 \text{ for all } N \\ &\Leftrightarrow \text{fl } M \leqq n \\ &\Leftrightarrow \text{Tor}^{n+1}(N, M) = 0 \text{ for all fp cyclic } N \\ &\Leftrightarrow \text{Ext}^{n+1}(N, M^*) = 0 \text{ for all fp cyclic } N \\ &\Leftrightarrow \text{winj } M^* \leqq n. \end{aligned}$$

Since winj $\leqq$ ap $\leqq$ inj we have the desired result.

COROLLARY 1. *For any ring A we have:*

$$\text{r.INJ}' = \text{l.FL}' \leqq \text{FL} \leqq \text{l.WINJ} \leqq \text{l.AP} \leqq \text{l.INJ} = \text{l.PR}$$

and l.INJ$'$ = r.FL$'$. *Also:* r. *could replace* l. *in last four terms.*

Proof. r.INJ$'$ = sup inj$'$ M = sup fl$'$ M^* $\leqq$ sup fl$'$ N = l.FL$'$, for M right A-module, and N left A-module. Similarly l.FL$'$ $\leqq$ r.INJ$'$. The other statements are clear.

COROLLARY 2. *We have that* fl$'$ M = fl$'$ M^{**} *and* inj$'$ M = inj$'$ M^{**} *for all* modules M.

4. Coherent rings. We recall that a ring A is called *left coherent* if and only if every finitely generated left ideal is finitely presented. (See Bourbaki [**1**]).

THEOREM 4.1. *If A is a left coherent ring then*
(1) *every finitely presented left A-module is an* FFR-*module.*
(2) ap M = inj$'M$ = winj$'M$ *for all left A-modules M.*
(3) fl M = fl$'M$ *for all left A-modules M.*

Proof. (1) is given in [**5**]. To show (2) note that by (1) we have ap M = inj$'M$ for all M. Now

$$\begin{aligned} \text{ap } M \leqq n &\Leftrightarrow \text{Ext}^{n+1}(N, M) = 0 \text{ for all fp } N \\ &\Leftrightarrow \text{Tor}^{n+1}(M^*, N) = 0 \text{ for all fp } N \\ &\qquad \text{using the second isomorphism and (1)} \\ &\Leftrightarrow \text{Tor}^{n+1}(M^*, N) = 0 \text{ for all fp cyclic } N \\ &\Leftrightarrow \text{Ext}^{n+1}(N, M) = 0 \text{ for all fp cyclic } N \\ &\Leftrightarrow \text{winj } M \leqq n. \end{aligned}$$

(3) follows from (1) and the definitions.

COROLLARY. *If A is left coherent then* l.AP = l.WINJ = l.INJ$'$ = r.FL$'$ = FL

5. Global dimensions. We can now give alternative characterizations of the global dimensions, which will be used in the next section.

THEOREM 5.1. *For any ring A we have*
(1) PR = INJ = sup(pr$M$|M *cyclic*)
(2) AP = sup(pr M|M *finitely presented*)
(3) WINJ = sup(pr M|M fp *cyclic*)
(4) r.INJ′ = sup(pr M|M *is right* FFR)
= l.FL′ = sup (fl M| *is left* FFR)
(5) FL = sup (fl M|M *is* fp *cyclic*)

Proof. (1) is well-known and is given in Cartan-Eilenberg [**2**] For (2) we have $AP(A) \leqq n \Leftrightarrow$ ap$N \leqq n$ for all N

$\Leftrightarrow$ $\text{Ext}^{n+1}(M, N) = 0$ for all N and for all fp modules M
$\Leftrightarrow$ pr $M \leqq n$ for all fp M
$\Leftrightarrow$ sup pr $M \leqq n$.

The proof of the other parts is similar.

COROLLARY. *If any one of these dimensions is* $\geqq 1$, *we have, for example,*

$$\text{AP} = 1 + \sup \text{pr } K,$$

with the sups *taken over all finitely generated submodules of (finitely generated) free (or projective) modules.*

For WINJ *and* INJ *we need only look at the dimension of the one-sided ideals in this case.*

6. Special cases.

THEOREM 6.1. *The following statements are equivalent for any ring A:*
(1) AP(A) = 0;
(2) WINJ(A) = 0;
(3) FL(A) = 0;
(4) *A is von Neuman regular; where in* (1) *and* (2) *either* l. *or* r. *may be used.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (3) $\Leftrightarrow$ (4) is well-known. But if A is regular then all submodules are pure (see [**4**]). Hence (4) $\Rightarrow$ (1). It is also well-known that (3) and (4) are left-right symmetric.

THEOREM 6.2. *For any ring A the following conditions are equivalent:*
(1) *A is left semihereditary, but not regular.*
(2) l.AP (A) = 1.
(3) l.WINJ(A) = 1.
(4) FL(A) = 1 *and A is left coherent.*

Proof. (1) $\Rightarrow$ (2). In Cartan-Eilenberg [**2**] it is shown that finitely generated submodules of projective left modules are projective if and only if the ring is left semihereditary. Hence by Theorem 5.1 we have l.AP(A) $\leqq$ 1. But by Theorem 6.1 AP(A) $\neq$ 0.

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(2) $\Rightarrow$ (1). By Theorem 5.1 all finitely generated left ideals are projective and hence A is left semihereditary. By Theorem 6.1 A is not regular.

(2) $\Rightarrow$ (3). Clearly $\text{WINJ}(A) \leqq 1$ and cannot be zero by Theorem 6.1.

(3) $\Rightarrow$ (4). Again $\text{FL}(A) = 1$. By Theorem 5.1 every finitely generated left ideal is projective, hence finitely presented, and A is left coherent.

(4) $\Rightarrow$ (2). Since A is left coherent we can use the corollary of Theorem 4.1.

Remark 1. Using Theorem 6.1 (as in the proof) we could delete "regular" from statement (1) of the Theorem and write $\leqq 1$ in place of $= 1$ everywhere.

Remark 2. These results extend and simplify some of the results of Enochs [**3**].

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