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LAMÉ EQUATION, QUANTUM EULER TOP AND ELLIPTIC BERNOULLI POLYNOMIALS

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Abstract A generalization of the odd Bernoulli polynomials related to the quantum Euler top is introduced and investigated. This is applied in order to compute the coefficients of the spectral polynomials for the classical Lamé operator.

Keywords: Lamé equation; quantum top; Bernoulli polynomials

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1. Introduction

The classical Bernoulli polynomials can be defined through the generating function

$$\frac{z\mathrm{e}^{zx}}{\mathrm{e}^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k,$$

$$B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x^3 - \tfrac{1}{2}3x^2 + \tfrac{1}{2}x, \quad \dots$$

(see, for example, [**1**, **5**]). They appear naturally in the calculation of sums of powers of the natural numbers $S_{k-1}(n) = 1^{k-1} + 2^{k-1} + \dots + n^{k-1}$ in a simple way:

$$S_{k-1}(n) = \frac{B_k(n+1) - B_k}{k},$$

where $B_k = B_k(0)$ are the *Bernoulli numbers*. All odd Bernoulli numbers except $B_1 = -\frac{1}{2}$ are known to be zero, so the odd Bernoulli polynomials (up to a multiple k) can be thought of as an 'analytic continuation' of the sums of powers from natural argument n to real (or complex) x.

In this paper we introduce a new class of polynomials, which can be considered as an elliptic generalization of the *odd* Bernoulli polynomials $B_{2k+1}(x)$. They are related to the quantum top and to the classical *Lamé operator*

$$L_s = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + s(s+1)\wp(z).$$

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where $\wp$ is the Weierstrass elliptic function [**5**], satisfying the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

It is well known (see [**10**]) that the Lamé operator (considered on the real line shifted by the imaginary half-period) for integer s has a remarkable property: its spectrum has exactly s gaps. The ends of the spectrum E_j correspond to the doubly periodic solutions of the Lamé equation (so-called *Lamé functions*). The corresponding polynomials

$$R_{2s+1}(E) = \prod_{j=0}^{2s}(E - E_j(s))$$

will be called *Lamé spectral polynomials*. The computation of the polynomials $R_{2s+1}(E)$ for given $s = 1, 2, 3, \dots$ goes back to Hermite and Halphen [**30**]. In more recent times this has been investigated within the finite-gap theory initiated by Novikov [**20**] (see [**2**,**24**,**25**] for the latest results in this direction).

Here we consider a related but different problem: we would like to express the coefficient b_k of the spectral polynomial $R_{2s+1}(E) = E^{2s+1} + b_1E^{2s} + b_2E^{2s-1} + \cdots + b_{2s+1}$ as a *function of* s (and thus for *all* values of parameter s). We will show that in this relation there naturally appear some new polynomials which generalize the odd Bernoulli polynomials.

The following remarkable relation between the Lamé equation and the quantum Euler top, going back to Kramers and Ittmann [**12**, **13**], will be crucial for us. Consider the quantum mechanical Hamiltonian of the Euler top (see, for example, [**16**]):

$$\hat{H} = a_1\hat{M}_1^2 + a_2\hat{M}_2^2 + a_3\hat{M}_3^2,$$

where the angular momentum operators $\hat{M}_j$ satisfy the standard commutation relations $[\hat{M}_1, \hat{M}_2] = \mathrm{i}\hat{M}_3$, $[\hat{M}_2, \hat{M}_3] = \mathrm{i}\hat{M}_1$, $[\hat{M}_3, \hat{M}_1] = \mathrm{i}\hat{M}_2$ (we assume that $\hbar = 1$ for simplicity).

The operator $\hat{H}$ naturally acts in any representation of the Lie algebra so(3). In particular, it acts in the representation space with spin s of dimension $2s + 1$ as a finite-dimensional operator $\hat{H}_s$. The claim is that if the parameters $a_i = e_i$ are the roots e_1, e_2, e_3 of the equation $4\wp^3 - g_2\wp - g_3 = 0$, then the characteristic polynomial of the operator $\hat{H}_s$ coincides with the spectral Lamé polynomial:

$$\det(\lambda I - \hat{H}_s) = R_{2s+1}(\lambda). \tag{1.1}$$

We discuss this in more detail in the next section.

The Weierstrass condition $e_1 + e_2 + e_3 = 0$ is unnatural from this point of view (and, moreover, contradicts the 'physical' condition of positivity of a_i), so we consider the case when the parameters a_i are arbitrary. Let us introduce new parameters g_1, g_2 and g_3, which are symmetric functions of a_1, a_2 and a_3 defined by the relation

$$4(z - a_1)(z - a_2)(z - a_3) = 4z^3 - g_1z^2 - g_2z - g_3. \tag{1.2}$$

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We define the *elliptic Bernoulli polynomials* $\mathcal{B}_{2k+1}$ as the coefficients in the expansion of the trace of the resolvent of $\hat{H}_s$ at infinity,

$$\operatorname{tr}(\lambda I - \hat{H}_s)^{-1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}_{2k+1}(s)}{\lambda^{k+1}}, \tag{1.3}$$

or, equivalently, by the relation

$$\mathcal{B}_{2k+1}(s) = \operatorname{tr} \hat{H}_s^k.$$

$\mathcal{B}_{2k+1}$ is a polynomial in s of degree $2k+1$ with coefficients, which are themselves polynomials in g_1, g_2, g_3 with rational coefficients. Strictly speaking we should write $\mathcal{B}_{2k+1}(s; g_1, g_2, g_3)$ rather than $\mathcal{B}_{2k+1}(s)$, but we will use both notations depending on the context. When $g_2 = g_3 = 0$, these polynomials reduce, up to a factor, to the classical odd Bernoulli polynomials:

$$\mathcal{B}_{2k+1}(s; g_1, 0, 0) = \frac{g_1^k}{(2k+1)2^{2k-1}} B_{2k+1}(s+1).$$

The corresponding elliptic curve Γ given by the equation

$$y^2 = 4x^3 - g_1 x^2 - g_2 x - g_3$$

degenerates to a rational curve in this case. If $g_1 = 0$, we have the standard Weierstrass form of an elliptic curve. The polynomials $\mathcal{B}_{2k+1}(s; 0, g_2, g_3)$ are called *reduced elliptic Bernoulli polynomials* and denoted as $\mathcal{B}^{\mathrm{W}}_{2k+1}(s; g_2, g_3)$:

$$\begin{aligned}
\mathcal{B}^{\mathrm{W}}_1 &= 2s+1,\\
\mathcal{B}^{\mathrm{W}}_3 &= 0,\\
\mathcal{B}^{\mathrm{W}}_5 &= \tfrac{1}{60} g_2 s(s+1)(2s-1)(2s+1)(2s+3),\\
\mathcal{B}^{\mathrm{W}}_7 &= \tfrac{1}{280} g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),\\
\mathcal{B}^{\mathrm{W}}_9 &= \tfrac{1}{1680} g_2^2 s(s+1)(2s-1)(2s+1)(2s+3)(4s^4+8s^3-11s^2-15s+21).
\end{aligned}$$

The coefficients of $\mathcal{B}^{\mathrm{W}}_{2k+1}$ are homogeneous polynomials in g_2, g_3 of weight $2k$ if we assume as usual that the weights of g_2 and g_3 are 4 and 6, respectively (in other words, they are modular forms of weight $2k$; see, for example, [**17**]). Two interesting special cases, $g_2 = 0$ and $g_3 = 0$, are called *lemniscatic* and *equianharmonic*, respectively, and correspond to elliptic curves with additional symmetries.

We will present some effective ways to compute the elliptic Bernoulli polynomials, investigate their properties and then apply them to the calculation of the coefficients of the Lamé spectral polynomials. In particular, we prove that the coefficient $b_k = b_k(s)$ of the Lamé spectral polynomial

$$R_{2s+1}(E) = \prod_{j=0}^{2s} (E - E_j(s)) = E^{2s+1} + b_1 E^{2s} + b_2 E^{2s-1} + \cdots + b_{2s+1}$$

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is a polynomial in s, g_2, g_3 with rational coefficients. It can be computed using the reduced elliptic Bernoulli polynomials by the following recurrence relation with $b_0 = 1$:

$$b_k = -\frac{1}{k}\sum_{j=1}^{k} \mathcal{B}^{\mathrm{W}}_{2j+1}(s) b_{k-j}.$$

The first coefficients are

$$\begin{aligned}
b_1 &= 0,\\
b_2 &= -\frac{g_2}{120} s(s+1)(2s-1)(2s+1)(2s+3),\\
b_3 &= -\frac{g_3}{840} s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),\\
b_4 &= \frac{g_2^2}{201\,600} s(s-1)(s+1)(2s-1)(2s+1)(2s+3)(56s^4+76s^3-94s^2+201s+630)
\end{aligned}$$

(more of them are given in § 5, below).

Note that once the coefficients $b_k(s)$ are known for $k = 0, 1, \dots, 2s$ one can find the eigenvalues of the quantum Euler top in the representation with spin s (integer or half-integer) by solving the corresponding algebraic equation $R_{2s+1}(E) = 0$.

We conclude with the discussion of possible relations and further developments.

2. The Lamé equation and the quantum Euler top

The observation that the Lamé equation is closely related to the quantum top was made by Kramers and Ittmann in the early age of quantum mechanics [**12**,**13**] (see also [**28**,**29**]). They showed that the corresponding Schrödinger equation is separable in the elliptic coordinate system and that the resulting differential equations are of Lamé form. We are going to re-derive this result here and reformulate it in modern terms.

Consider the Hamiltonian

$$\hat{H} = a_1\hat{M}_1^2 + a_2\hat{M}_2^2 + a_3\hat{M}_3^2$$

acting in the space of functions on the unit sphere

$$q_1^2 + q_2^2 + q_3^2 = 1, \tag{2.1}$$

using the standard representation of the angular momenta as the first-order differential operators:

$$\begin{aligned}
\hat{M}_1 &= -\imath(q_2\partial_3 - q_3\partial_2),\\
\hat{M}_2 &= -\imath(q_3\partial_1 - q_1\partial_3),\\
\hat{M}_3 &= -\imath(q_1\partial_2 - q_2\partial_1).
\end{aligned}$$

Let us introduce the *elliptic* (or *sphero-conical*) coordinates u_1, u_2 on this sphere as the roots of the quadratic equation

$$\frac{q_1^2}{a_1 - u} + \frac{q_2^2}{a_2 - u} + \frac{q_3^2}{a_3 - u} = 0, \tag{2.2}$$

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where the parameters a_1, a_2, a_3 are the same as in the top's Hamiltonian. We then have the following expressions for the Cartesian coordinates in terms of u_1, u_2:

$$\left.\begin{aligned} q_1^2 &= \frac{(a_1-u_1)(a_1-u_2)}{(a_1-a_2)(a_1-a_3)}, \\ q_2^2 &= \frac{(a_2-u_1)(a_2-u_2)}{(a_2-a_1)(a_2-a_3)}, \\ q_3^2 &= \frac{(a_3-u_1)(a_3-u_2)}{(a_3-a_1)(a_3-a_2)}. \end{aligned}\right\} \tag{2.3}$$

The system has an obvious quantum integral $\hat{M}^2 = \sum \hat{M}_i^2$, which is the Casimir operator:

$$[\hat{M}^2, \hat{M}_i] = 0, \quad i = 1, 2, 3.$$

One can check that in the elliptic coordinate system the operators $\hat{H}$ and $\hat{M}^2$ have the form

$$\hat{M}^2 = -\frac{4}{u_1-u_2}\left[\sqrt{-P(u_1)}\frac{\partial}{\partial u_1}\left(\sqrt{-P(u_1)}\frac{\partial}{\partial u_1}\right) + \sqrt{P(u_2)}\frac{\partial}{\partial u_2}\left(\sqrt{P(u_2)}\frac{\partial}{\partial u_2}\right)\right], \tag{2.4}$$

$$\hat{H} = -\frac{4}{u_1-u_2}\left[u_2\sqrt{-P(u_1)}\frac{\partial}{\partial u_1}\left(\sqrt{-P(u_1)}\frac{\partial}{\partial u_1}\right) + u_1\sqrt{P(u_2)}\frac{\partial}{\partial u_2}\left(\sqrt{P(u_2)}\frac{\partial}{\partial u_2}\right)\right], \tag{2.5}$$

where $P(u) = (u-a_1)(u-a_2)(u-a_3)$. Note that the operator $\hat{M}^2$ corresponds to the standard Laplacian $-\Delta$ on the unit sphere.

Since $\hat{M}^2$ and $\hat{H}$ commute, one can look for joint eigenfunctions. The spectral problem $\hat{M}^2\psi = \mu\psi$ is well known in the theory of spherical harmonics (see, for example, [**19**]). It is known that the spectrum has the form $\mu = s(s+1)$ for non-negative integer values of s. The dimension of the corresponding eigenspace V_s is $2s+1$ and V_s is an irreducible representation of dimension $2s+1$ of the rotation group SO_3 called *representation with spin s*.

It turns out that the joint eigenvalue problem

$$\begin{aligned} \hat{M}^2\phi &= s(s+1)\phi, \\ \hat{H}\phi &= E\phi \end{aligned}$$

is separable in the elliptic coordinates u_1, u_2 (see [**12**,**13**,**28**,**29**]). Namely, if we substitute $\phi(u_1,u_2) = \phi_1(u_1)\phi_2(u_2)$ into this system, we find that each of the functions $\phi_1(u_1)$, $\phi_2(u_2)$ satisfies the same differential equation:

$$\left(4[P(u)]^{1/2}\frac{\mathrm{d}}{\mathrm{d}u}\left([P(u)]^{1/2}\frac{\mathrm{d}}{\mathrm{d}u}\right) - s(s+1)u + E\right)\psi = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\psi + \frac{1}{2}\left[\frac{1}{u-a_1} + \frac{1}{u-a_2} + \frac{1}{u-a_3}\right]\frac{\mathrm{d}}{\mathrm{d}u}\psi = \frac{1}{4}\frac{s(s+1)u-E}{(u-a_1)(u-a_2)(u-a_3)}\psi. \tag{2.6}$$

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A remarkable fact is that this is an algebraic form of the following slightly generalized version of the Lamé differential equation:

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi + s(s+1)\wp_*(z)\psi = E\psi, \tag{2.7}$$

where $\wp_*(z)$ is a solution of the differential equation

$$(\wp_*')^2 = 4(\wp_* - a_1)(\wp_* - a_2)(\wp_* - a_3). \tag{2.8}$$

Indeed, after the change of variables $u = \wp_*(z)$, equation (2.7) coincides with (2.6) (see [**30**]). When the sum $a_1 + a_2 + a_3 = 0$, equation (2.8) determines the Weierstrass elliptic function $\wp(z)$; otherwise, it differs from it by adding a constant.

It is well known (see, for example, [**5**]) that for ϕ to be a regular solution on the sphere the corresponding ψ must be doubly periodic, which implies that s is integer and E must have one of the $2s+1$ characteristic values $E_m(s)$. For each $E_m(s)$ there exists exactly one (up to a factor) doubly periodic solution to the Lamé equation $\mathcal{E}_s^m(u)$, which is called the *Lamé function*. Therefore, the basis of the eigenfunctions of the operator $\hat{H}$ in the invariant subspace V_s consists of $2s+1$ solutions $\phi(u_1, u_2)$ of the form $\mathcal{E}_s^m(u_1)\mathcal{E}_s^m(u_2)$. These are sometimes called *ellipsoidal harmonics* (see [**30**]).

Thus, we come to the following result (cf. [**12, 13, 28, 29**]).

Theorem 2.1. *The characteristic polynomial of the quantum top Hamiltonian $\hat{H}_s$ in the representation space with integer spin s coincides with the spectral polynomial $R_{2s+1}(\lambda) = \prod_{j=0}^{2s}(\lambda - E_j(s))$ of the generalized Lamé operator (2.7).*

Remark 2.2. Turbiner [**26**] has discovered a similar but different relation between the Lamé equation and certain quadratic elements of the universal enveloping algebra of sl(2). The Lamé spectral polynomials are known to be factorizable and Turbiner's result gives an interesting interpretation for the factors in these terms.

Remark 2.3. The simple relationship between the quantum Euler top and the Lamé equation mentioned above is a little misleading. Indeed, there are several spectral problems related to the Lamé equation. We have considered only the smooth real periodic version related to real x shifted by the imaginary half-period. If we considered real x, we would have a singular version (since $\wp$ has poles on the real line), whose spectrum has nothing to do with the quantum top. In turn, the quantum Euler top in the representation with half-integer spin s has eigenvalues which are just some special double eigenvalues of the periodic Lamé operator, which in this case has infinitely many gaps.

3. Elliptic Bernoulli polynomials

Now we define the *elliptic Bernoulli polynomials* $\mathcal{B}_{2k+1}(s)$ as the traces of the powers of $\hat{H}_s$, where $\hat{H}_s$ is, as before, the quantum top operator $\hat{H}$ in the representation with spin s:

$$\mathcal{B}_{2k+1}(s; g_1, g_2, g_3) = \operatorname{tr} \hat{H}_s^k, \quad k = 0, 1, 2, \dots. \tag{3.1}$$

Here the parameters $g_1 = 4(a_1 + a_2 + a_3)$, $g_2 = -4(a_1a_2 + a_2a_3 + a_1a_3)$ and $g_3 = 4a_1a_2a_3$ are defined by the relation (1.2).

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Theorem 3.1. *The trace* $\operatorname{tr}\hat{H}_s^k$ *is a polynomial in* s *of degree* $2k+1$ *antisymmetric with respect to* $s=-\frac{1}{2}$*, whose coefficients are polynomials in* g_1*,* g_2*,* g_3 *with rational coefficients. When* $g_2=g_3=0$*, it reduces (up to a factor and shift) to the corresponding classical odd Bernoulli polynomial:*

$$\mathcal{B}_{2k+1}(s;g_1,0,0)=\frac{g_1^k}{(2k+1)2^{2k-1}}B_{2k+1}(s+1). \tag{3.2}$$

The first part essentially follows from the Harish-Chandra general results [**9**] (see also [**4**, p. 268]), but here we give a direct proof.

Proof. Consider the standard basis in V_s consisting of the eigenvectors $|j\rangle$ of $\hat{M}_3$: $\hat{M}_3|j\rangle=j|j\rangle$, $j=-s,-s+1,\dots,s-1,s$. In this basis, the Hamiltonian $\hat{H}$ is a tri-diagonal symmetric matrix $H=H_s$ with the following elements (see, for example, [**16**, p. 417]):

$$\left.\begin{aligned}\langle j|H|j\rangle&=\tfrac{1}{2}(a_1+a_2)[s(s+1)-j^2]+a_3j^2,\\ \langle j|H|j+2\rangle=\langle j+2|H|j\rangle&=\tfrac{1}{4}(a_1-a_2)\sqrt{(s-j)(s-j-1)(s+j+1)(s+j+2)}.\end{aligned}\right\} \tag{3.3}$$

Note that both expressions are symmetric with respect to $s=-\frac{1}{2}$; they are also homogeneous polynomials of degree 1 in a_1,a_2,a_3. Now, consider any diagonal element of H^k; it has the form

$$\langle j|H^k|j\rangle=\sum_{i_1,i_2,\dots,i_{k-1}}\langle j|H|i_1\rangle\langle i_1|H|i_2\rangle\cdots\langle i_{k-1}|H|j\rangle,$$

where the distance between two consecutive indices i_l, i_{l+1} is either 0 or ±2. Since the starting point and the ending point coincide, if the matrix element $\langle i_l|H|i_l+2\rangle$ appears along the path, the element $\langle i_l+2|H|i_l\rangle$ also appears along the path. This proves that the diagonal matrix elements of H^k are polynomials of degree $2k$ in both s and j. From (3.3) they are symmetric with respect to $s=-\frac{1}{2}$ and homogeneous symmetric polynomials of degree k in a_1, a_2, a_3. Now summing over $j=-s,-s+1,\dots,s-1,s$ and taking into account the fact that the sums of the odd powers of j are zero while the sums of even powers $2l$ are the odd Bernoulli polynomials $B_{2l+1}(s+1)$ (multiplied by $2/(2l+1)$), we have the first statement of the theorem. The antisymmetry of $\mathcal{B}_{2k+1}(s)$ with respect to $s=-\frac{1}{2}$ follows from the well-known property of the Bernoulli polynomials: $B_m(1-s)=(-1)^mB_m(s)$. The symmetry in a_1, a_2, a_3 is clear from the definition of $\mathcal{B}_{2k+1}$.

In the case when $a_1=a_2=0$, we have $g_2=g_3=0$, $g_1=4a_3$ and $\hat{H}=a_3\hat{M}_3^2$. The spectrum of H_s is then very simple: $\lambda_j=a_3j^2$ for $j=-s,-s+1,\dots,s-1,s$. Since the sum

$$\sum_{j=1}^{s}j^{2k}=\frac{1}{2k+1}B_{2k+1}(s+1),$$

we thus obtain (3.2). This completes the proof of Theorem 2.1. □

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Note that from the point of view of the elliptic curve Γ given by the equation

$$y^2 = 4x^3 - g_1x^2 - g_2x - g_3,$$

the case $g_2 = g_3 = 0$ corresponds to the limit when one of the periods goes to infinity (the 'trigonometric limit'). There are two more interesting special cases: the *lemniscatic* case when $g_1 = g_3 = 0$ and the *equianharmonic* case when $g_1 = g_2 = 0$, corresponding to the elliptic curves with additional symmetries.

It is natural also to consider the Weierstrass reduction $g_1 = 0$; we will call the corresponding polynomials $\mathcal{B}^{\mathrm{W}}_{2k+1}(s; g_2, g_3) = \mathcal{B}_{2k+1}(s; 0, g_2, g_3)$ the *reduced elliptic Bernoulli polynomials*.

Theorem 3.2. *The elliptic Bernoulli polynomial $\mathcal{B}_{2k+1}$ has the following properties:*

(i) *as a polynomial in g_1, g_2, g_3, $\mathcal{B}_{2k+1}$ is homogeneous of weight $2k$, where the weights of g_1, g_2 and g_3 are assumed to be 2, 4 and 6, respectively;*

(ii) *$\mathcal{B}_{2k+1}$ for $k \geqslant 1$ is divisible by $s(s+1)(2s+1)$;*

(iii) *in the reduced case $\mathcal{B}^{\mathrm{W}}_{2k+1}$ is divisible by $s(s+1)(2s-1)(2s+1)(2s+3)$ for all k and by $s(s+1)(2s-1)(2s+1)(2s+3)(2s-3)(2s+5)$ for odd k;*

(iv) *in the lemniscatic case $\mathcal{B}_{2k+1}(s; 0, g_2, 0) = 0$ for odd integer k;*

(v) *in the equianharmonic case $\mathcal{B}_{2k+1}(s; 0, 0, g_3) = 0$ if k is not divisible by 3;*

(vi) *in the isotropic case $a_1 = a_2 = a_3 = a$, i.e. $g_1 = 12a$, $g_2 = -12a^2$, $g_3 = 4a^3$, we have $\mathcal{B}_{2k+1}(s) = a^k(2s+1)s^k(s+1)^k$.*

Proof. The proof of the first two claims follows from the definition and the antisymmetry property. To prove the third claim, consider the representation with spin $s = \frac{1}{2}$. It is easy to check that $\hat{H}$ acts as the 2×2 scalar matrix $\frac{1}{4}(a_1 + a_2 + a_3)\,\mathrm{Id}$, which is zero in the reduced case. Therefore, $\mathcal{B}^{\mathrm{W}}_{2k+1}(\frac{1}{2}) = 0$ for all k. By antisymmetry with respect to $-\frac{1}{2}$ we also have $\mathcal{B}^{\mathrm{W}}_{2k+1}(-\frac{3}{2}) = 0$. For half-integer s, we know from Kramers's theorem (see [**16**, Paragraph 60]) that the eigenvalues are no longer distinct but are double roots. For the particular case in which $s = \frac{3}{2}$, these eigenvalues take the values $\pm\sqrt{[3(a_1^2 + a_2^2 + a_3^2)/2]}$ (see [**16**, p. 419]). Therefore, for odd k, $\mathcal{B}^{\mathrm{W}}_{2k+1}(\frac{3}{2}) = 0$ and, again by antisymmetry, $\mathcal{B}^{\mathrm{W}}_{2k+1}(-\frac{5}{2}) = 0$. The lemniscatic and equianharmonic cases follow from the first claim. In the isotropic case $\hat{H}_s = as(s+1)\,\mathrm{Id}$, which implies the last statement. □

In the general case the elliptic Bernoulli polynomials are not zero and their highest coefficients are described by the following theorem.

Theorem 3.3. *The leading term of the elliptic Bernoulli polynomial $\mathcal{B}_{2k+1}(s) = A_0s^{2k+1} + A_1s^{2k} + \cdots + A_{2s}$ can be written*

$$A_0s^{2k+1} = 2\int_0^s \operatorname{Res} \xi^{-1}[\gamma(s^2 - j^2)\xi + (\alpha s^2 + \beta j^2) + \gamma(s^2 - j^2)\xi^{-1}]^k \,\mathrm{d}j, \tag{3.4}$$

where $\alpha = \frac{1}{2}(a_1 + a_2)$, $\beta = \frac{1}{2}(2a_3 - a_1 - a_2)$, $\gamma = \frac{1}{4}(a_1 - a_2)$.

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Indeed, for large s and j, the leading behaviour of the matrix elements of $\hat{H}$ is

$$\langle j|\hat{H}|j\rangle = \tfrac{1}{2}(a_1+a_2)[s^2-j^2]+a_3j^2 = \alpha s^2+\beta j^2,$$
$$\langle j|\hat{H}|j+2\rangle = \langle j+2|\hat{H}|j\rangle = \tfrac{1}{4}(a_1-a_2)(s^2-j^2) = \gamma(s^2-j^2).$$

Therefore, the leading term of the diagonal element $\langle j|\hat{H}^k|j\rangle$ coincides with the constant term of the Laurent polynomial $[\gamma(s^2-j^2)\xi+(\alpha s^2+\beta j^2)+\gamma(s^2-j^2)\xi^{-1}]^k$ in auxiliary variable ξ. Replacing the summation over j by the integration, which is fine in the leading order, we come to our formula.

Note that the fact that the final result is a symmetric function of a_1, a_2, a_3 (and thus is a polynomial in g_1, g_2, g_3) is not at all obvious from this formula.

Remark 3.4. From the quasi-classical arguments we can write the highest coefficient A_0 as the following integral over the unit sphere:

$$A_0 = \frac{1}{2\pi}\int_{|M|^2=1} H^k \,\mathrm{d}\Omega = \frac{1}{2\pi}\int_{|M|^2=1}(a_1M_1^2+a_2M_2^2+a_3M_3^2)^k \,\mathrm{d}\Omega, \tag{3.5}$$

where $\mathrm{d}\Omega$ is the area element on the unit sphere. Thus, formula (3.4) gives an expression for this integral. It would be interesting to compare it with the calculation of this integral using elliptic coordinates.

4. An effective way to compute the elliptic Bernoulli polynomials

Although the definition of the elliptic Bernoulli polynomials themselves gives a way to compute them as traces of powers of the given matrices H_s, it does not seem to be as effective as the following procedure, which is based on the fact that the matrix H_s is tri-diagonal.

Indeed, in the standard basis $|j\rangle$ of the space V_s the eigenvalue problem $\hat{H}\psi=\lambda\psi$ leads to the following difference equation:

$$c_{n-2}\psi_{n-2}+v_n\psi_n+c_n\psi_{n+2}=\lambda\psi_n, \tag{4.1}$$

where

$$c_n = \tfrac{1}{4}(a_1-a_2)\sqrt{(s-n)(s-n-1)(s+n+1)(s+n+2)},$$
$$v_n = \tfrac{1}{2}(a_1+a_2)^2[s(s+1)-n^2]+a_3n^2.$$

For such an equation one can use the standard procedure (see, for example, [**6**]) from the theory of solitons to find the local spectral densities, which are difference analogues of the famous Korteweg–de Vries densities [**21**]. In our case it works as follows.

Let $\chi_n = c_n\psi_{n+2}/\psi_n$. Then equation (4.1) becomes

$$c_{n-2}^2+(v_n-\lambda)\chi_{n-2}+\chi_n\chi_{n-2}=0. \tag{4.2}$$

We look for a solution in the form $\chi_n=\lambda-\sum_{k=0}^{\infty}\chi_{n,k}\lambda^{-k}$. Substitution of this expression into equation (4.2) gives $\chi_{n,0}=v_n$, $\chi_{n,1}=c_{n-2}^2$, $\chi_{n,2}=c_{n-2}^2v_{n-2}$ and, for general

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$k \geqslant 1$, the recurrence relation

$$\chi_{n,k+1} = \sum_{i=1}^{k} \chi_{n,i}\chi_{n-2,k-i}. \tag{4.3}$$

Let $X = \sum_{k=0}^{\infty} \chi_{n,k}\lambda^{-(k+1)}$ so that $\chi_n = \lambda(1-X)$ and $\log\chi_n = \log\lambda - \sum_{i=1}^{\infty} X^i/i$. Thus, we have

$$\log\chi_n - \log\lambda = -\sum_{i=1}^{\infty} \frac{\mathcal{I}_{n,i}}{\lambda^i}, \tag{4.4}$$

where $\mathcal{I}_{n,1} = v_n$, $\mathcal{I}_{n,2} = c_{n-2}^2 + v_n^2/2$, $\mathcal{I}_{n,3} = c_{n-2}^2 v_{n-2} + v_n c_{n-2}^2 + v_n^3/3, \dots$.

On the other hand, one can check that

$$\prod_n \frac{\chi_n}{\lambda} = \prod_m \biggl(1 - \frac{E_m(s)}{\lambda}\biggr),$$

where the $E_m(s)$ are the eigenvalues of $\hat{H}_s$. Thus,

$$\sum_n (\log\chi_n - \log\lambda) = -\sum_n \sum_{i=1}^{\infty} \frac{\lambda_n^i}{i\lambda^i} = \sum_{i=1}^{\infty} \frac{\operatorname{tr}\hat{H}_s^i}{i\lambda^i}.$$

Comparing this with (4.4), we obtain

$$\operatorname{tr}\hat{H}_s^k = k\sum_n \mathcal{I}_{n,k} = k\sum_{n=-s}^{s} \mathcal{I}_{n,k}.$$

Theorem 4.1. *The elliptic Bernoulli polynomials $\mathcal{B}_{2k+1}$ can be computed as*

$$\mathcal{B}_{2k+1} = k\sum_{n=-s}^{s} \mathcal{I}_{n,k}, \tag{4.5}$$

where $\mathcal{I}_{n,k}$ are the local densities determined by the relations (4.3), (4.4).

This gives a very effective way to compute the elliptic Bernoulli polynomials, since the local densities are polynomials in c_n^2 and v_n (and hence in n) and thus the summation over n can be done with the use of the standard Bernoulli polynomials. We have applied this procedure to find the first ten elliptic Bernoulli polynomials using Mathematica (see eight of them in the appendix).

5. Application: coefficients of the Lamé spectral polynomials

We will again consider the generalized version of the Lamé operator (2.7). The coefficients $b_k = b_k(s)$ of the corresponding spectral polynomial

$$R_{2s+1}(E) = \prod_{i=0}^{2s}(E - E_i) = E^{2s+1} + b_1E^{2s} + b_2E^{2s-1} + \dots + b_kE^{2s-k+1} + \dots + b_{2s+1}$$

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up to a sign are the elementary symmetric functions of the eigenvalues: $b_k = (-1)^k e_k$, where

$$e_1 = \sum E_i, \quad e_2 = \sum_{i<j} E_i E_j, \quad e_3 = \sum_{i<j<k} E_i E_j E_k, \quad \dots .$$

The elementary symmetric functions are related to power sums $\mathcal{B}_{2k+1}(s) = \sum E_i^k$ by the following well-known relations:

$$ke_k = \sum_{j=1}^{k} (-1)^{j-1} \mathcal{B}_{2j+1} e_{k-j}$$

with $e_0 = b_0 = 1$ (see, for example, [**18**]). This implies the following.

Theorem 5.1. *The coefficients b_k of the Lamé spectral polynomial $R_{2s+1}(E)$ are related to the elliptic Bernoulli polynomials $\mathcal{B}_{2j+1}(s)$ by the recurrent relations*

$$b_k = -\frac{1}{k} \sum_{j=1}^{k} \mathcal{B}_{2j+1}(s) b_{k-j}.$$

The coefficient b_k is a polynomial in s, g_1, g_2, g_3 with rational coefficients. As a polynomial in s it has degree $3k$ and is divisible by $(s+1)s(s-1)\cdots(s-[(k-2)/2])$.

One can apply this result also to the case of half-integer spin s: in this case all the roots of the polynomial $R_{2s+1}(E)$ are double and correspond to the doubly periodic solutions of the Lamé equation.

In the reduced case ($g_1 = 0$) the degree of b_k drops to $[5k/2]$ (for $k > 1$). Using the explicit form of the elliptic Bernoulli polynomials given in the appendix, one can find the first seven coefficients b_k, which in the reduced case are

$$\begin{aligned}
b_1 &= 0,\\
b_2 &= -\frac{g_2}{120} s(s+1)(2s-1)(2s+1)(2s+3),\\
b_3 &= -\frac{g_3}{840} s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),\\
b_4 &= \frac{g_2^2}{201\,600} s(s-1)(s+1)(2s-1)(2s+1)(2s+3)(56s^4+76s^3-94s^2+201s+630),\\
b_5 &= +\frac{g_2 g_3}{1\,108\,800} s(s-1)(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),\\
&\qquad \times (88s^4+68s^3-302s^2+663s+1890),\\
b_6 &= \frac{g_3^2}{201\,801\,600}(s-2)(s-1)s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\qquad \times (4576s^5+12\,944s^4-20\,720s^3+48\,312s^2+597\,150s+779\,625)\\
&\quad - \frac{g_2^3}{10\,378\,368\,000}(s-2)(s-1)s(s+1)(2s-5)(2s-3)(2s-1)(2s+1)(2s+3)\\
&\qquad \times (16\,016s^6+89\,232s^5+197\,160s^4+544\,280s^3\\
&\qquad\qquad +2\,033\,829s^2+385\,8813s+2\,619\,540),
\end{aligned}$$

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$$
\begin{aligned}
b_7 = -\frac{g_2^2 g_3}{24\,216\,192\,000}&(s-3)(s-2)(s-1)s(s+1)(2s-5)(2s-3) \\
&\times (2s-1)(2s+1)(2s+3)(2s+5) \\
&\quad\times (32\,032s^6 + 189\,072s^5 + 463\,440s^4 + 1\,682\,920s^3 \\
&\qquad + 7\,301\,418s^2 + 15\,249\,213s + 11\,351\,340).
\end{aligned}
$$

6. Concluding remarks

We have shown that for any given k the coefficient $b_k(s)$ of the spectral Lamé polynomial R_{2s+1} can be computed effectively for all values of parameter s. In particular, for fixed s it gives an alternative way to compute the whole polynomial. It would be interesting to compare this approach with the classical one going back to Halphen and Hermite [**30**] and recently developed further by Belokolos and Enolski [**2**] and Takemura [**24, 25**] following the work of Krichever [**14**].

However, we believe that the elliptic Bernoulli polynomials are of interest in themselves. In particular, one can expect interesting relations with the arithmetic of the corresponding elliptic curves and the representation theory. In this regard we mention the elliptic generalization of the Bernoulli numbers: the so-called *Bernoulli–Hurwitz numbers* BH_{2k}, whose arithmetic was investigated in [**11, 22**].

Another interesting possible relation is with the zeta function $\zeta_H(z) = \operatorname{tr} \hat{H}^{-z}$ of the quantum top and its special values. A lemniscatic case $a_3 = \frac{1}{2}(a_1 + a_2)$ could be particularly interesting from the arithmetic point of view.

Recall that the parameter s was originally integer or half-integer (spin). A natural question is the role of these values in the theory of elliptic Bernoulli polynomials. We conjecture that, as in the case of the usual Bernoulli polynomials (see, for example, [**27**]), these values are the asymptotic positions of the real roots of the polynomials $\mathcal{B}_{2k+1}$ for large k. More precisely, we conjecture that, for real s in the bounded interval, the ratio

$$
\frac{\mathcal{B}_{2k+1}(s)}{\mathcal{B}'_{2k+1}(0)} \to \frac{\sin 2\pi s}{2\pi} \quad \text{as } k \to \infty.
$$

Actually, we believe that this is true for each component of $\mathcal{B}_{2k+1}$, which is a coefficient at monomial $g_1^p g_2^q g_3^r$.

It is interesting to look at the graphs. In Figure 1 we show the graphs of the coefficients of the polynomial $\mathcal{B}_{15}(s)$ at g_1^7, $g_1^3 g_2^2$, $g_1^2 g_2 g_3$ and $g_2^2 g_3$, respectively. We normalize each polynomial by dividing it by its first derivative at zero and then multiplying it by 2π. The sinusoidal behaviour for small s looks quite plausible.

We note that the even Bernoulli polynomials (or, more precisely, closely related Faulhaber polynomials) also have elliptic versions related to the Lamé operator. They were introduced in [**8**] motivated by [**7**] as certain complete elliptic integrals of the second kind and have quite different properties. The fact that the theory of the Lamé equation leads to two different classes of polynomials, both related to Bernoulli polynomials (one to odd, another to even) seems to be remarkable. To make the picture even more intriguing we note that the integrals in the definition of the elliptic Faulhaber polynomials come from the formal expansion of the trace of the resolvent of the Lamé operator (cf. (1.3)).

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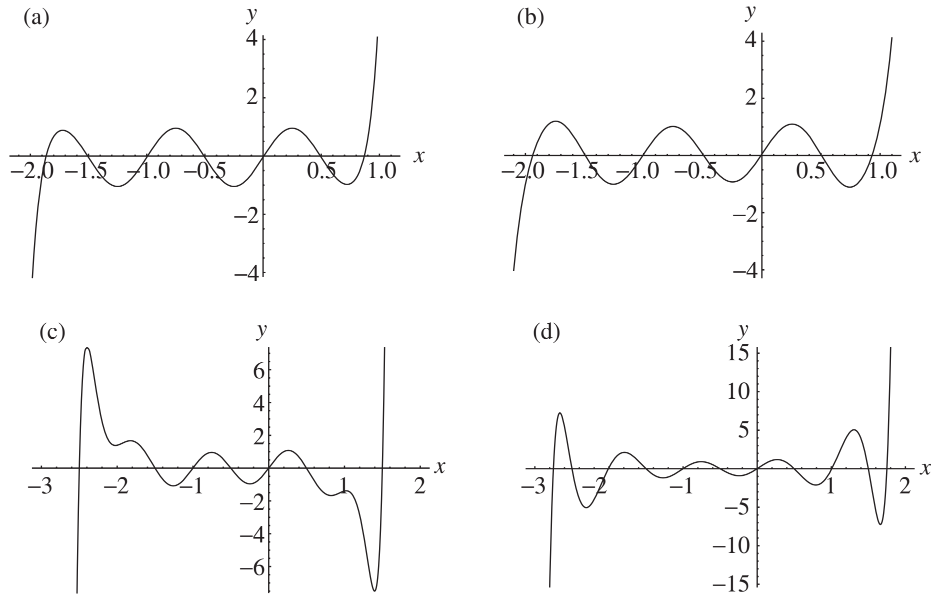

Figure 1. Coefficients of the polynomial $\mathcal{B}_{15}(s)$ at (a) g_1^7, (b) $g_1^3 g_2^2$, (c) $g_1^2 g_2 g_3$ and (d) $g_2^2 g_3$.

Another interesting problem is to investigate the analogues of elliptic Bernoulli polynomials related to the Sklyanin algebra. From [**15**] it is known that Sklyanin's representation [**23**] gives a certain difference analogue of the Lamé equation, so one can consider the traces of powers of the generator S_0 as functions of the corresponding spin. We would like to mention the very interesting paper [**3**], where traces on the Sklyanin algebra are discussed. In particular, the formulae (2.20), (2.21) from [**3**] give an explicit expression of the traces of S_0 and S_0^2 in terms of elliptic functions, which show that they are no longer polynomials.

Finally, one can consider our results from the general point of view of the quantization of integrable systems. One can usually find the spectrum in a closed form only if the classical system is integrable in elementary functions. The Euler top is probably the most natural classical problem integrable in elliptic functions. The question of the nature of its integrability in the quantum case seems not to be as easy as it may look. We hope that our paper adds something in this direction as well.

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Appendix A. The first eight elliptic Bernoulli polynomials

$$\mathcal{B}_1 = 2s + 1,$$
$$\mathcal{B}_3 = \tfrac{1}{12} g_1 s(s+1)(2s+1),$$

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$$\begin{aligned}
\mathcal{B}_5 &= \tfrac{1}{240}g_1^2\tfrac{1}{60}g_2 s(s+1)(2s-1)(2s+1)(2s+3),\\
\mathcal{B}_7 &= \tfrac{1}{1344}g_1^3 s(s+1)(2s+1)(3s^4+6s^3-3s+1)\\
&\quad+\tfrac{1}{1120}g_1g_2s(s+1)(2s-1)(2s+1)(2s+3)(6s^2+6s-5)\\
&\quad+\tfrac{1}{280}g_3s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5),\\
\mathcal{B}_9 &= \tfrac{1}{11\,520}g_1^4 s(s+1)(1+2s)(5s^6+15s^5+5s^4-15s^3-s^2+9s-3)\\
&\quad+\tfrac{1}{3360}g_1^2g_2s(s+1)(2s-1)(2s+1)(2s+3)(5s^4+10s^3-5s^2-10s+7)\\
&\quad+\tfrac{1}{840}g_1g_3s^2(s+1)^2(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\quad+\tfrac{1}{1680}g_2^2s(s+1)(2s-1)(2s+1)(2s+3)(4s^4+8s^3-11s^2-15s+21),\\
\mathcal{B}_{11} &= \tfrac{1}{33\,792}g_1^5 s(s+1)(2s+1)(s^2+s-1)(3s^6+9s^5+2s^4-11s^3+3s^2+10s-5)\\
&\quad+\tfrac{1}{50\,688}g_1^3g_2s(s+1)(2s-1)(2s+1)(2s+3)\\
&\qquad\times(20s^6+60s^5-10s^4-120s^3+44s^2+114s-75)\\
&\quad+\tfrac{1}{29\,568}g_1^2g_3s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\qquad\times(10s^4+20s^3-4s^2-14s+21)\\
&\quad+\tfrac{1}{29\,568}g_1g_2^2s(s+1)(2s-1)(2s+1)(2s+3)\\
&\qquad\times(40s^6+120s^5-86s^4-372s^3+242s^2+448s-315)\\
&\quad+\tfrac{1}{7392}g_2g_3s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\qquad\times(8s^4+16s^3-34s^2-42s+63),\\
\mathcal{B}_{13} &= \tfrac{1}{5\,591\,040}g_1^6 s(s+1)(2s+1)(105s^{10}+525s^9+525s^8-1050s^7-1190s^6+2310s^5\\
&\qquad+1420s^4-3285s^3-287s^2+2073s-691)\\
&\quad+\tfrac{1}{5\,125\,120}g_1^4g_2s(s+1)(2s-1)(2s+1)(2s+3)\\
&\qquad\times(525s^8+2100s^7+350s^6-6300s^5-70s^4+12\,810s^3\\
&\qquad-4105s^2-11\,910s+7601)\\
&\quad+\tfrac{1}{2\,842\,840}g_1^3g_3s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\qquad\times(350s^6+1050s^5-100s^4-1950s^3+1433s^2+2583s-1650)\\
&\quad+\tfrac{1}{2\,562\,560}g_1^2g_2^2s(s+1)(2s-1)(2s+1)(2s+3)\\
&\qquad\times(1400s^8+5600s^7-1450s^6-23\,950s^5+5438s^4+57\,326s^3\\
&\qquad-24\,627s^2-58\,215s+41\,481)\\
&\quad+\tfrac{1}{320\,320}g_1g_2g_3s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\qquad\times(200s6+600s5-670s^4-2340s^3+1922s^2+3192s-2475)\\
&\quad+\tfrac{1}{960\,960}g_2^3s(s+1)(2s-1)(2s+1)(2s+3)\\
&\qquad\times(400s^8+1600s^7-1640s^6-10\,520s^5+8193s^4+35\,786s^3\\
&\qquad-28\,282s^2-48\,195s+43\,659)\\
&\quad+\tfrac{1}{160\,160}g_3^2s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)\\
&\qquad\times(80s^6+240s^5-840s^4-2080s^3+4401s^2+5481s-7425),
\end{aligned}$$

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$$
\begin{aligned}
\mathcal{B}_{15} = {} & \tfrac{1}{737\,280} g_1^7 s(s+1)(2s+1) \\
& \qquad \times (3s^1 2 + 18s^1 1 + 24s^{10} - 45s^9 - 81s^8 + 144s^7 + 182s^6 \\
& \qquad\qquad - 345s^5 - 217s^4 + 498s^3 + 44s^2 - 315s + 105) \\
& + \tfrac{1}{1\,597\,440} g_1^5 \tilde{g}_2 s(s+1)(2s-1)(2s+1)(2s+3) \\
& \qquad \times (42s^{10} + 210s^9 + 105s^8 - 840s^7 - 364s^6 + 2730s^5 + 205s^4 \\
& \qquad\qquad - 5540s^3 + 1650s^2 + 5078s - 3185) \\
& + \tfrac{1}{13\,178\,880} g_1^4 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \qquad \times (315s^8 + 1260s^7 + 140s^6 - 3990s^5 + 1265s^4 \\
& \qquad\qquad + 10\,650s^3 - 5152s^2 - 11\,352s + 9009) \\
& + \tfrac{1}{13\,178\,880} g_1^3 g_2^2 s(s+1)(2s-1)(2s+1) \\
& \qquad \times (2s + 3(2520s^1 0 + 12\,600s^9 + 1750s^8 - 68\,600s^7) \\
& \qquad\qquad - 13\,130s^6 + 253\,630s^5 - 14\,558s^4 - 557\,066s^3 \\
& \qquad\qquad + 206\,601s^2 + 542\,619s - 360\,360) \\
& + \tfrac{1}{1\,098\,240} g_1^2 g_2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \qquad \times (280s^8 + 1120s^7 - 670s^6 - 5930s^5 + 3047s^4 + 17\,284s^3 \\
& \qquad\qquad - 11\,237s^2 - 21\,054s + 18\,018) \\
& + \tfrac{1}{3\,294\,720} g_1 g_2^3 s(s+1)(2s-1)(2s+1)(2s+3) \\
& \qquad \times (1120s^{10} + 5600s^9 - 2400s^8 - 43\,200s^7 - 8814s^6 + 201\,162s^5 \\
& \qquad\qquad - 60\,127s^4 - 517\,124s^3 + 256\,797s^2 + 557\,766s - 405\,405) \\
& + \tfrac{1}{274\,560} g_1 g_3^2 s^2 (s+1)^2 (2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \qquad \times (80s^6 + 240s^5 - 840s^4 - 2080s^3 + 4401s^2 + 5481s - 7425) \\
& + \tfrac{1}{274\,560} g_2^2 g_3 s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
& \qquad \times (80s^8 + 320s^7 - 600s^6 - 2920s^5 + 4037s^4 + 13\,314s^3 \\
& \qquad\qquad - 16\,959s^2 - 24\,156s + 27\,027).
\end{aligned}
$$

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