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*Fourth Meeting, 12th February 1896.*

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**The Steady Motion of a Spherical Vortex.**

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The possibility of the steady motion of a spherical vortex of constant vorticity in an infinite homogeneous liquid was first pointed out by Hill in the *Phil. Trans.*, 1894, pp. 213–245. He had already discussed a case of motion which had for the surfaces always containing the same particles those given by the equation

$$\omega^2 \left( \frac{\omega^2}{a^2} + \frac{(z - Z)^2}{c^2} - 1 \right) = \text{constant},$$

a particular surface being

$$\omega^2(\omega^2 + (z - Z)^2 - a^2) = 0.$$

Since these surfaces are of invariable form it is possible to imagine the fluid limited by any one of them, provided a rigid frictionless boundary having the shape of the limiting surface be supplied and supposed to move parallel to the axis of  $z$  with velocity  $\dot{Z}$ . His previous result gave the velocity components of a possible rotational motion inside this boundary. Further he showed that for the particular case of the surface being a sphere the rotational motion inside is continuous as regards velocity normal and pressure with a certain irrotational motion in all space outside. This irrotational motion is that produced by the sphere moved in the same direction with the same velocity.

Thus in this case the boundary may be removed and the possibility of the state of motion known as Hill's Spherical Vortex is established.

The object of this paper is to show that by using the ordinary hydrodynamical equations, this and other allied types of steady motion not yet noted may be quickly demonstrated. As the problems deal with spheres and spherical shells the equations are taken in spherical coordinates.

§ 1.

Consider the possibility of a spherical vortex of constant vorticity and density  $\rho_1$  moving with velocity  $V$  in an infinite homogeneous liquid of density  $\rho_2$ . Impressing on the whole an equal and opposite velocity we have the case of the vortex at rest and fluid streaming from infinity with velocity  $V$ . This requires the following equations to be satisfied by the current function :

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1 - \mu^2}{r^2} - \frac{\partial^2}{\partial \mu^2}\right)\psi_1 = M r^2 \sin^2 \theta, \quad \dots \quad *(1)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1 - \mu^2}{r^2} - \frac{\partial^2}{\partial \mu^2}\right)\psi_1 = 0, \quad \dots \quad (2)$$

where 
$$\omega = -\frac{M}{2} r \sin \theta,$$

$$\psi_2 = -\frac{1}{2} V r^2 \sin^2 \theta \text{ at infinity, } \dots \quad (3)$$

$$\psi_1 \text{ and } \psi_2 \text{ constant at } r = a; \quad \dots \quad (4)$$

also the pressure must be continuous at  $r = a$ .  $\dots \quad (5)$

As in Hill's paper we have 
$$\psi_1 = \frac{M}{10} r^2 (r^2 - a^2) \sin^2 \theta, \quad \dots \quad (6)$$

$$\psi_2 = -\frac{1}{2} V \left(r^2 - \frac{a^3}{r}\right) \sin^2 \theta. \quad \dots \quad (7)$$

We have still to examine the pressure equations.

These may be written 
$$\frac{p}{\rho_1} + \frac{1}{2} Q^2 - M \psi_1 = \frac{P}{\rho_1} + \frac{1}{2} U^2, \quad \dots \quad (8)$$

$$\frac{p}{\rho_2} + \frac{1}{2} Q^2 = \frac{\Pi}{\rho_2} + \frac{1}{2} V^2, \quad \dots \quad (9)$$

when  $P$  and  $\Pi$  are the pressures at the centre and at infinity,  $U$  the velocity at the centre.

By using the values given for  $\psi_1$  and  $\psi_2$  we have at once for the pressure at  $r = a$

$$\frac{p}{\rho_1} + \frac{1}{50} M^2 a^4 \sin^2 \theta = \frac{P}{\rho_1} + \frac{1}{50} M^2 a^4,$$

$$\frac{p}{\rho_2} + \frac{9}{8} V^2 \sin^2 \theta = \frac{\Pi}{\rho_2} + \frac{1}{2} V^2,$$

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\* Basset, Hydrodynamics, Vol. II., p. 81, Equations (55) and (57) in spherical coordinates. The axis from which  $\theta$  is increased is parallel to the direction of  $V$

Therefore we must have  $V = -\frac{2}{15}\sqrt{\frac{\rho_1}{\rho_2}} \cdot Ma^2, \quad . \quad . \quad . \quad (10)$

$$P = \Pi - \frac{5}{8}\rho_2 V^2. \quad . \quad . \quad . \quad (11)$$

Further we find for the pressure at the point  $(r, \theta)$  inside the vortex

$$p + \frac{9}{8a^4} \cdot \rho_2 V^2 [(a^2 - r^2)^2 + r^2 \sin^2 \theta (3r^2 - 2a^2) + 5r^2 (a^2 - r^2) \sin^2 \theta] = P + \frac{9}{8}\rho_2 V^2.$$

This may be written

$$p = P - \frac{9}{32}\rho_2 V^2 + \frac{9}{8a^4}\rho_2 V^2 \left[ \left( r^2 - \frac{a^2}{2} \right) + r^2 (3a^2 - 2r^2) \cos^2 \theta \right] \quad . \quad (12)$$

Therefore  $p$  is least when  $r = \frac{a}{\sqrt{2}}$  and  $\theta = \pm \frac{\pi}{2}$ ,

$$\text{and this minimum pressure} = \Pi - \frac{29}{32}\rho_2 V^2 = P - \frac{9}{32}\rho_2 V^2. \quad . \quad (13)$$

We have thus found that, whether the density inside is the same or different from that outside, the pressure is least at the

points  $\left( \frac{a}{\sqrt{2}}, \pm \frac{\pi}{2} \right)$ : also a hollow will begin to form there if

$\Pi < \frac{29}{32}\rho_2 V^2$ : while the velocity of translation of the vortex is given

$$\text{by } V = \frac{2}{15}\sqrt{\frac{\rho_1}{\rho_2}} \cdot Ma^2.$$

In the circular cylindrical vortex and the vortex ring treated of by Hicks we find cases of hollow vortices. It might have been expected that a possible state of steady motion would be a spherical vortex with a concentric hollow. The fact that the points of minimum pressure are as found above dissipates this idea: the same conclusion might be reached by work resembling that in the next portion of this paper.

## § 2.

By means of a similar analysis it can be shown that with certain conditions between the vorticities, densities, and radii, it is possible

to have a vortex in which the liquid is arranged in spherical strata. The work may be generalised, but I take as sufficiently illustrative the case of two strata.

Here we have

$$\psi_3 = -\frac{1}{2}V\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta, \quad \dots \quad (14)$$

$$\psi_2 = \frac{1}{8}M_2r^4(-I_2 + I_4) + \left(\Lambda_2r^2 + \frac{B_2}{r}\right)I_2 + \left(A_4r^4 + \frac{B_4}{r^3}\right)I_4, \quad \dots \quad (15)$$

$$\psi_1 = \frac{1}{8}M_1r^4(-I_2 + I_4) + C_2r^2I_2 + C_4r^4I_4, \quad \dots \quad (16)$$

where  $\frac{dI_n}{d\mu} = P_{n-1}(\mu)$ .

These functions are treated of by Sampson in *Philosophical Transactions*, 1890, and, from the tables he gives, we see at once that  $I_4 - I_2 = \frac{5}{8}\sin^4\theta$ . For that reason we do not carry our expressions for  $\psi$  past  $I_4$ .

We shall have demonstrated the possibility of this case of motion if we can determine the constants to satisfy

$$\psi = \text{constant at } r = b \text{ and } r = a, \quad (b < a); \quad \dots \quad (17)$$

$$\text{Pressure continuous at } r = b \text{ and } r = a. \quad \dots \quad (18)$$

The results we now give follow at once from the expressions for  $\psi$  and those we obtain for the pressure

(a).  $\psi$  constant at  $r = b$  and  $r = a$ .

$$\left. \begin{aligned} \text{For this we have} \quad & \frac{1}{8}M_1b^4 + C_4b^4 = 0, \\ & -\frac{1}{8}M_1b^4 + C_2b^2 = 0, \\ & \frac{1}{8}M_2a^4 + A_4a^4 + \frac{B_4}{a^3} = 0, \\ & -\frac{1}{8}M_2a^4 + A_2a^2 + \frac{B_2}{a} = 0, \\ & \frac{1}{8}M_2b^4 + A_4b^4 + \frac{B_4}{b^3} = 0, \\ & -\frac{1}{8}M_2b^4 + A_2b^2 + \frac{B_2}{b} = 0. \end{aligned} \right\} \quad (19)$$

(β). The pressure continuous at  $r=b$  and  $r=a$ .

Hence we must have

$$\left. \begin{aligned} \frac{4}{5}M_2b^3 + 4A_4b^3 - 3\frac{B_4}{b^4} &= 0, \\ -\frac{4}{5}M_2a^2 + 2A_2 - \frac{B_2}{a^3} &= 3V\sigma_2, \\ -\frac{4}{5}M_2b^2 + 2A_2 - \frac{B_2}{b^3} &= -\frac{2}{5}M_1b^2\sigma_1^{-1}, \end{aligned} \right\} \quad (20)$$

where  $\sigma_2 = \sqrt{\frac{\rho_3}{\rho_2}} \quad ; \quad \sigma_1 = \sqrt{\frac{\rho_2}{\rho_1}}$ .

From these equations we have

$$\left. \begin{aligned} A_2 &= \frac{1}{5}M_2 \frac{a^5 - b^5}{a^3 - b^3} \quad ; \quad A_4 = -\frac{1}{5}M_2, \\ B_2 &= -\frac{1}{5}M_2a^3b^3 \frac{a^2 - b^2}{a^3 - b^3} \quad ; \quad B_4 = 0, \\ C_2 &= \frac{1}{5}M_1b^2 \quad ; \quad C_4 = -\frac{1}{5}M_1, \end{aligned} \right\} \quad (21)$$

$$\frac{M_2}{M_1}\sigma_1 = 2b^2 \cdot \frac{a^3 - b^3}{5a^3b^2 - 3a^5 - 2b^5} \quad . \quad . \quad . \quad (22)$$

$$V\sigma_2 = \frac{M_2}{15(a^3 - b^3)}(5a^2b^3 - 2a^5 - 3b^5) \quad . \quad . \quad . \quad (23)$$

Equation (23) gives the velocity of translation of this vortex ; (22), the necessary relation between  $M_1, M_2, \rho_1, \rho_2, a$  and  $b$ , that the motion may be possible ; (21), the expressions for the current function. Thus we have determined all the circumstances of the motion.