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A RING OF QUOTIENTS FOR GROUP RINGS WHICH IS EASY TO DESCRIBE

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1. Recently Luedeman studied certain idempotent topologizing families of left ideals in semi-group rings AS which arise from such families of left ideals of A. Let Σ be an idempotent topologizing family of left ideals in A and G a group, let ΣG denote the family of left ideals of AG which contain left ideals of the form LG, $L \in \Sigma$. Luedeman has shown that if G is finite the ring of quotients of AG corresponding to ΣG is the group ring with coefficients in the ring of quotients of A corresponding to Σ. In this note the theorem is proved for arbitrary groups but with a finiteness condition on Σ.

2. Throughout, A will be a ring with 1, G a group and Σ an idempotent topologizing family ([1]), σ-set in [5]) of left ideals of A. The notation generally is that of [2]. If G is a group, AG denotes the discrete group ring with elements $\Sigma_{g\in G}\, a_g g$, $a_g \in A$ all but finitely many of which are zero; if $r \in AG$ the coefficient of g in r is denoted by $r\langle g\rangle$. For a left A-module M we can similarly define an AG-module MG whose elements are sums $\Sigma_{g\in G}\, m_g g$, $m_g \in M$ all but finitely many of which are zero, if $n \in MG$ the coefficient of g in n is denoted by $n\langle g\rangle$. The family Σ is said to be of *finite type* if each $L \in \Sigma$ contains a finitely generated element of Σ (see [1]). This means that the finitely generated elements of Σ are cofinal in the filter.

Luedeman [5] has shown that the family of left ideals of AG, $\Sigma G = \{L \mid L$ contains KG for some $K \in \Sigma\}$ is again an idempotent topologizing family. Clearly if Σ is of finite type, $\{KG \mid K \in \Sigma, K$ finitely generated$\}$ is cofinal in ΣG.

Let $Z_\Sigma(M) = \{m \in M \mid Ann(m) \in \Sigma\}$. This is a submodule and $Z_\Sigma(A)$ is an ideal. Gabriel in [1] defined the module of quotients $Q_\Sigma(M)$ as

$$\varinjlim_{L\in\Sigma} \operatorname{Hom}_A(L, M/Z_\Sigma(M)).$$

$Q_\Sigma(A)$ is a ring and $Q_\Sigma(M)$ is a $Q_\Sigma(A)$-module. Clearly in the limit we may take a cofinal family from Σ.

THEOREM. *Let A be a ring, G a group, Σ an idempotent topologizing family of finite type. For any A-module M, $Q_{\Sigma G}MG \simeq Q_\Sigma(M)G$ as $Q_{\Sigma G}(AG)$-modules and $Q_{\Sigma G}(AG) \simeq Q_\Sigma(A)G$ as rings (this last isomorphism leaves $(A/Z_\Sigma(A))G$ fixed).*

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Proof. We remark first that $Z_{\Sigma G}(MG)=Z_\Sigma(M)G$. If $m\in Z_{\Sigma G}(MG)$ then for some $L\in\Sigma$, $LGm=0$. But $L\subseteq LG$ in a natural way so $Lm=0$ and all the coefficients of m are in $Z_\Sigma(M)$. Conversely, if $m\in Z_\Sigma(M)$, $Lm=0$ for some $L\in\Sigma$ so $LGm=0$. Hence, $MG/Z_{\Sigma G}(MG)\simeq(M/Z_\Sigma(M))G$ in a natural way, denote this by $\bar{M}G$. Then

$$Q_{\Sigma G}(MG)=\varinjlim_{L\in\Sigma}\operatorname{Hom}_{AG}(LG,\bar{M}G)$$

and we may assume that each L is finitely generated as a left ideal.

Let $\phi\in\operatorname{Hom}_{AG}(LG,\bar{M}G)$, L finitely generated in Σ, ϕ gives rise to a family of A-homomorphisms $L\to\bar{M}$ as follows. For $g\in G$, let $i:L\to LG$ be defined by $i(a)=a1$ and $\pi_g:\bar{M}G\to\bar{M}$ by $\pi_g(m)=m\langle g\rangle$. The composition $\phi_g=\pi_g\phi i\in\operatorname{Hom}_A(L,\bar{M})$. Since ϕ is an AG-homomorphism the maps ϕ_g determine ϕ. Indeed $\phi(ag)\langle h\rangle=(\phi(a)g)\langle h\rangle=\phi(a)\langle g^{-1}h\rangle=\phi_{g^{-1}h}(a)$. Suppose now that $L=Aa_1+\cdots+Aa_n$ then every ϕ_g is determined by its action on $a_1,\ldots,a_n$. For each a_i, $\phi(a_i)\langle g\rangle=\phi_g(a_i)$ is zero for all but finitely many $g\in G$. So $\{g\mid\phi_g(a_i)\neq 0$ for some $i=1,\ldots,n\}$ is finite. Hence, ϕ gives rise to a finite set $\phi_{g_1},\ldots,\phi_{g_r}$ of A-homomorphisms $L\to\bar{M}$ and ϕ is determined by this set.

Now let $m_1,\ldots,m_r$ be the elements of $Q_\Sigma(M)$ determined by $\phi_{g_1},\ldots,\phi_{g_r}$ respectively. In this fashion, ϕ determines an element $\Sigma_1^r m_ig_i$ of $Q_\Sigma(M)G$.

In the other direction, $m=m_1g_1+\cdots+m_rg_r\in Q_\Sigma(M)G$ determines an element $Q_{\Sigma G}(MG)$. In the direct limit which defines $Q_\Sigma(M)$ choose a finitely generated $L\in\Sigma$ so that each m_i, $i=1,\ldots,n$, is represented by some $\phi_i\in\operatorname{Hom}_A(L,\bar{M})$. For $a\in L$, $\phi_i(a)=am_i\in\bar{M}$ for $i=1,\ldots,n$. Thus if $r\in LG$, $r(m_1g_1+\cdots+m_rg_r)\in\bar{M}G$, so "multiplication" by m gives an element $\phi\in\operatorname{Hom}_{AG}(LG,\bar{M}G)$ and hence an element of $Q_{\Sigma G}(MG)$.

One can readily verify that these correspondences give an $Q_{\Sigma G}(AG)$-module isomorphism from $Q_{\Sigma G}(MG)$ to $Q_\Sigma(M)G$ and that, for $M=A$, we have a ring isomorphism.

Note that the same proof applies for any Σ if G is finite and this gives a proof of [5, Theorem p. 485] without the restriction that $Z_{\Sigma G}(AG)=0$. Further, for any G and any Σ, we have $Q_\Sigma(M)G\subseteq Q_{\Sigma G}(MG)$.

One can see also that the same proof applies to polynomial rings.

THEOREM. *If Σ is an idempotent topologizing family of finite type in A then $Q_{\Sigma[x]}(M[x])\simeq Q_\Sigma(M)[x]$ as $Q_{\Sigma[x]}(A[x])$-modules and $Q_{\Sigma[x]}(A[x])\simeq Q_\Sigma(A)[x]$ as rings.*

This theorem can be extended to other semigroup rings, at least to the case of a monoid which can be embedded in a group.

The following example shows that some restriction on Σ or G is essential for the theorem to be true. In what follows, the symbol y may be thought of as either the indeterminate of a polynomial ring or a generator of an infinite cyclic group. Thus we will have, at the same time, an example for the group ring and polynomial ring cases.

Let $F[x_1, x_2, \ldots]$ be the polynomial ring over a field in a countably infinite set of indeterminates and let R be the ring produced by dividing out the ideal generated by all expressions x_ix_j, $i\neq j$, and $x_i^2-x_i$. Denote the image in R of x_i by $\bar{x}_i$. Then, if Δ is the ideal of R generated by $\bar{x}_1, \bar{x}_2, \ldots$, we have that Δ is an idempotent maximal ideal. Hence, $\Sigma=\{R, \Delta\}$ is an idempotent topologizing family in R. $Q_\Sigma(R)$ is the ring of all expressions $\Sigma_1^\infty a_i\bar{x}_i$, $a_i \in F$ and the elements of R are identified with expressions $\Sigma a_i\bar{x}_i$ where for some n, $a_n=a_{n+1}=\cdots$.

Now $Q_{\Sigma[y]}(R[y])$ can be identified with a subring of $Q_\Sigma(R)[[y]]$. If we let $f_0+f_1y+\cdots$ denote a power series with $f_i \in Q_\Sigma(R)$ then $f_i=\Sigma a_{ij}\bar{x}_j$ for $a_{ij} \in F$. Then, $Q_{\Sigma[y]}(R[y])=\{f_0+f_1y+\cdots \mid \text{for each } j \text{ only finitely many } a_{ij} \text{ are nonzero}\}$. An example of an element of this ring which is not in $Q_\Sigma(R)[y]$ is $1+\bar{x}_1y+\bar{x}_2y^2+\cdots$.

3. At the end of [5] Luedeman asks if A is Σ-injective iff A_n is Σ_n-injective where Σ_n is the family of left ideals, of the matrix ring A_n, $\{I \mid I\supseteq J_n \text{ some } J \in \Sigma\}$ (here J_n means the set of matrices with entries from J). Luedeman remarks that the method of Utumi for ordinary injectivity does not readily generalize; however that of Kaye [3] does. More generally Turnidge [6] has studied the connections between idempotent topologizing families in Morita equivalent rings. If $G: {}_R\mathcal{M}\to {}_S\mathcal{M}$ and $H: {}_S\mathcal{M}\to {}_R\mathcal{M}$ are functors giving a category equivalence, there is a pairing between the hereditary torsion theories [6] in ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ and, hence, between the idempotent topologizing families of left ideals. Let $\mathcal{T}(R)$ be a torsion theory in ${}_R\mathcal{M}$ then the pairing is given by corresponding to $\mathcal{T}(R)$ the torsion class $\mathcal{T}(S)=\{M \in {}_S\mathcal{M} \mid H(M) \in \mathcal{T}(R)\}$. Turnidge shows that R is $\mathcal{T}(R)$-torsion free (corresponding singular ideal is zero) iff S is $\mathcal{T}(S)$-torsion free. Further he shows that M in ${}_R\mathcal{M}$ is $\mathcal{T}(R)$-injective iff $G(M)$ is $\mathcal{T}(S)$-injective.

The categories ${}_R\mathcal{M}$ and ${}_{R_n}\mathcal{M}$ are equivalent in the above sense with the equivalence (see [3]) given by $G(M)=M^n$ and $H(N)=e_{11}N$ (e_{11} the matrix unit). One can readily verify that the pairing of torsion theories pairs that generated by Σ with that generated by Σ_n. Hence R_n is Σ_n-injective iff R^n is Σ-injective iff R is Σ-injective.

Just as predicted by Luedeman, this last fact yields the following theorem, the proof of which is a modification of that of the last theorem of [5].

THEOREM. *If S is a finite inverse semigroup, A a ring, Σ an idempotent topologizing family of left ideals of A; then AS is ΣS-injective iff A is Σ-injective.*

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