Glasgow Math. J. **47** (2005) 155–165. © 2005 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504002216. Printed in the United Kingdom

ALGEBRAS DENSE IN L^2 SPACES: AN OPERATOR APPROACH

MICHAŁ WOJTYLAK
Instytut Matematyki, Uniwersytet Jagielloński, Kraków
e-mail: Michal.Wojtylak@im.uj.edu.pl

(Received 19 April, 2004; accepted 15 September, 2004)

Abstract. Let μ be a finite positive Borel measure defined on a σ-algebra of subsets of a set $\mathcal{X}$. Using operator techniques we provide several criteria for finitely generated algebras to be dense in the space $L^2(\mu)$.

2000 *Mathematics Subject Classification.* Primary 47A58. Secondary 41A65, 47B25.

Introduction. It turns out that methods from operator theory can be useful in approximation theory. Fuglede in **[2]** showed that the polynomials are dense in $L^2((\mathrm{x}_1^2 + \cdots + \mathrm{x}_n^2)\mu)$ provided that the multiplication operators $\mathbb{C}[\mathrm{x}_1, \dots, \mathrm{x}_n] \ni p \mapsto \mathrm{x}_j p \in \mathbb{C}[\mathrm{x}_1, \dots, \mathrm{x}_n]$ are jointly essentially selfadjoint in the closure of polynomials in $L^2(\mu)$. Putinar and Vasilescu proved in **[6]** that the algebra generated by the polynomials and the function $1/(1 + \mathrm{x}_1^2 + \cdots + \mathrm{x}_n^2)$ is dense in $L^2(\mu)$.

An inspiration for this paper was an example in **[8]**. Stochel and Sebestyén showed that if an algebra of functions on $\mathbb{R}$ contains polynomials and a function $1/p$ where p is a nonzero polynomial and for some polynomial q, the rational function q/p is nonconstant and bounded then this algebra is dense in $L^2(\mu)$. In Theorem 7 we generalize this fact in many ways:

- $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu)$ is replaced by an abstract finite measure space
- a result from domination theory enables us to deal with algebras generated by a finite set of functions (instead of algebras with only two generators)
- the algebra of polynomials is replaced by the algebra generated by the coordinates of an abstract measurable mapping $\boldsymbol{\phi} : \mathcal{X} \to \mathbb{R}^n$,
- the condition of boundedness is abandoned and a weaker μ-quasianalyticity condition is introduced.

As a consequence we get also the aforementioned result from **[6]**. Section 3 with Theorem 15 is a complement of Theorem 7. The situation where all the coordinates of the mapping $\boldsymbol{\phi}$ are of μ-quasianalytic type is considered therein. Theorem 18 is the most general result of this paper; Theorems 7 and 15 are in fact its consequences. However, the main ideas, examples and proofs are in Sections 3 and 4.

1. The multiplication operator and quasianalyticity. If Z is a topological space then the symbol $\mathfrak{B}(Z)$ will denote the σ-algebra of all Borel subsets of Z. In the whole paper we will assume that $(\mathcal{X}, \mathfrak{M}, \mu)$ is a *finite measure space*, i.e. $\mathcal{X}$ is a nonempty set, $\mathfrak{M}$ is a σ-algebra of subsets of $\mathcal{X}$ and μ is a positive finite measure defined on $\mathfrak{M}$. We will deal with the complex Hilbert space $L^2(\mu) := L^2(\mathcal{X}, \mathfrak{M}, \mu)$.

Let $\mathcal{H}$ be a (complex) Hilbert space. By an operator in $\mathcal{H}$ we mean a linear mapping $T:\mathcal{D}(T)\to\mathcal{H}$, where the domain $\mathcal{D}(T)$ of T is a linear subspace of $\mathcal{H}$. By $\mathcal{N}(T)$ we denote the kernel of T. If T is closable then $\overline{T}$ stands for the closure of T.

Let $\mathcal{D}$ be a dense linear subspace $\mathcal{H}$. We will denote by $\boldsymbol{L}(\mathcal{D})$ the algebra of all operators T in $\mathcal{H}$ such that $\mathcal{D}(T)=\mathcal{D}$ and $T(\mathcal{D})\subseteq\mathcal{D}$. By $I_{\mathcal{D}}$ we understand the identity operator on $\mathcal{D}$. We say that $T_1,\dots,T_n\in\boldsymbol{L}(\mathcal{D})$ *commute pointwise* if $T_iT_jf=T_jT_if$ for all $f\in\mathcal{D}$, $i,j=1,\dots,n$.

Let $S_1,\dots,S_n$ be selfadjoint operators in $\mathcal{H}$ and let $E_1,\dots,E_n$ be their spectral measures, respectively. We say that $S_1,\dots,S_n$ *spectrally commute* if for all $i,j=1,\dots,n$ and for all $\sigma,\tau\in\mathfrak{B}(\mathbb{R})$, we have $E_i(\sigma)E_j(\tau)=E_j(\tau)E_i(\sigma)$. In such case there exists a *joint spectral measure of the system* $(S_1,\dots,S_n)$, i.e. a spectral measure E on $\mathbb{R}^n$ satisfying

$$\int \mathrm{x}_j\,dE=S_j,\quad j=1,\dots,n, \tag{1.1}$$

where $\mathrm{x}_j(x):=x_j$. Conversely, if the condition (1.1) holds, then $S_1,\dots,S_n$ are spectrally commuting selfadjoint operators.

Let $(\mathcal{X},\mathfrak{M},\mu)$ be a finite measure space and let $\phi:\mathcal{X}\to\mathbb{R}$ be a measurable function. We define the operator M_ϕ in $L^2(\mu)$:

$$\begin{aligned}\mathcal{D}(M_\phi)&:=\{f\in L^2(\mu):\phi\cdot f\in L^2(\mu)\}\\ M_\phi f&:=\phi\cdot f,\quad f\in\mathcal{D}(M_\phi).\end{aligned}$$

M_ϕ is a well-defined selfadjoint operator.

Let us define the following set of mappings:

$$\mathcal{M}(\mathcal{X},\mathbb{R}^n,\mu)=\{\boldsymbol{\phi}:\mathcal{X}\to\mathbb{R}^n\mid\boldsymbol{\phi}\text{ is measurable},\ \forall_{\sigma\in\mathfrak{M}}\exists_{\tau\in\mathfrak{B}(\mathbb{R}^n)}\ \mu(\sigma\triangle\boldsymbol{\phi}^{-1}(\tau))=0\},$$

where $A\triangle B:=(A\backslash B)\cup(B\backslash A)$. (According to [**13**] we would say that $\boldsymbol{\phi}^{-1}(\mathfrak{B}(\mathbb{R}^n))$ is essentially all of $\mathfrak{M}$.) A result similar to the following Proposition, but in the context when $\boldsymbol{\phi}:\mathcal{X}\to\mathcal{X}$, appears in [**13**, Lemma 1].

PROPOSITION 1. *Let $\boldsymbol{\phi}:\mathcal{X}\to\mathbb{R}^n$ be measurable. Consider the following conditions:*

(i) *$\boldsymbol{\phi}$ is μ-a.e. injective and bimeasurable, i.e. there exists $Y\in\mathfrak{M}$ such that $\mu(\mathcal{X}\backslash Y)=0$, $\boldsymbol{\phi}|_Y$ is injective and $\boldsymbol{\phi}(\sigma\cap Y)\in\mathfrak{B}(\mathbb{R}^n)$ for $\sigma\in\mathfrak{M}$;*

(ii) *$\boldsymbol{\phi}\in\mathcal{M}(\mathcal{X},\mathbb{R}^n,\mu)$;*

(iii) *$\{f\circ\boldsymbol{\phi}:f\in L^2(\mu\circ\boldsymbol{\phi}^{-1})\}$ is dense in $L^2(\mu)$,* where $L^2(\mu\circ\boldsymbol{\phi}^{-1}):=L^2(\mathbb{R}^n,\mathfrak{B}(\mathbb{R}^n),\mu\circ\boldsymbol{\phi}^{-1})$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. To prove (i) $\Rightarrow$ (ii) it is enough to take $\tau=\boldsymbol{\phi}(\sigma\cap Y)$. Assume now (ii). The condition (iii) is fulfilled, since $\chi_\sigma=\chi_{\boldsymbol{\phi}^{-1}(\tau)}=\chi_\tau\circ\boldsymbol{\phi}$ (μ-a.e.) and the characteristic functions are linearly dense in $L^2(\mu)$.

Now let us assume (iii). Observe that the operator $U:L^2(\mu\circ\boldsymbol{\phi}^{-1})\ni f\mapsto f\circ\boldsymbol{\phi}\in L^2(\mu)$ is a unitary isomorphism, so for every $g\in L^2(\mu)$ there exists $f\in L^2(\mu\circ\boldsymbol{\phi}^{-1})$ such that $g=f\circ\boldsymbol{\phi}$ (μ-a.e.). Take $\sigma\in\mathfrak{M}$. Then there exists $f\in L^2(\mu\circ\boldsymbol{\phi}^{-1})$ such that $\chi_\sigma=f\circ\boldsymbol{\phi}$ (μ-a.e.). It is now easy to notice that $\chi_\sigma=\chi_\tau\circ\boldsymbol{\phi}$ (μ-a.e.) where $\tau=f^{-1}(1)\in\mathfrak{B}(\mathbb{R}^n)$. Since $\chi_\tau\circ\boldsymbol{\phi}=\chi_{\boldsymbol{\phi}^{-1}(\tau)}$, the proof of (iii) $\Rightarrow$ (ii) is completed. □

Notice that the unitary isomorphism appearing above might be used to reduce the proofs of Theorems 7 and 15 to the case when $\mathcal{X} = \mathbb{R}^{n+1}$, $\boldsymbol{\phi} = \mathrm{id}_{\mathbb{R}^{n+1}}$ ($\mathcal{X} = \mathbb{R}^n$, $\boldsymbol{\phi} = \mathrm{id}_{\mathbb{R}^n}$ respectively). However, this reduction is implicitly made in Proposition 2 and we will not repeat it later.

The implication (i)$\Rightarrow$(ii) of Proposition 1 will be used in applications. However, injectivity is not a necessary condition for a function to be in $\mathcal{M}(\mathcal{X}, \mathbb{R}^n, \mu)$. For example if $\mathcal{X} = \mathbb{R}$, $\mathfrak{M} = \{\cup_{k\in K}[k, k+1) \mid K \subseteq \mathbb{N}\}$ then ϕ defined by $\phi(x) := [x]$ belongs to $\mathcal{M}(\mathcal{X}, \mathbb{R}, \mu)$.

The following Proposition gives us a way of proving that a linear subspace is dense in $L^2(\mu)$. We slightly extend the method presented by Fuglede in **[2]** by introducing an abstract space $(\mathcal{X}, \mathfrak{M}, \nu)$ and a mapping $\boldsymbol{\phi}$.

Proposition 2. *Let $(\mathcal{X}, \mathfrak{M}, \nu)$ be a finite measure space, and let $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n) \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n, \nu)$ be such that $\phi_j \in L^2(\nu)$ for all $j = 1, \ldots, n$. If a closed linear subspace $\mathcal{K}$ of $L^2(\nu)$ reduces each operator M_{ϕ_j} $(j = 1, \ldots, n)$ and if $1 \in \mathcal{K}$, then $\mathcal{K} = L^2(\nu)$.*

Proof. Let E be the spectral measure given by the formula:

$$E(\sigma)f = \chi_\sigma f, \quad \sigma \in \mathfrak{M}, f \in L^2(\nu).$$

Let $j \in \{1, \ldots, n\}$. It is well known that $M_{\phi_j} = \int \phi_j \, dE$. On the other hand the measure transport theorem gives us

$$\int_{\mathcal{X}} \phi_j \, dE = \int_{\mathbb{R}^n} \mathrm{x}_j \, d(E \circ \boldsymbol{\phi}^{-1}).$$

Hence $M_{\phi_1}, \ldots, M_{\phi_n}$ are spectrally commuting selfadjoint operators and $E \circ \boldsymbol{\phi}^{-1}$ is their joint spectral measure. In consequence, if $\mathcal{K}$ reduces every M_{ϕ_j} then

$$P(E(\boldsymbol{\phi}^{-1}(\tau)) = (E(\boldsymbol{\phi}^{-1}(\tau))P, \quad \tau \in \mathfrak{B}(\mathbb{R}^n),$$

where P stands for the orthogonal projection from $L^2(\nu)$ onto $\mathcal{K}$. Take $\sigma \in \mathfrak{M}$. Since $\boldsymbol{\phi} \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n, \nu)$, there exists $\tau \in \mathfrak{B}(\mathbb{R}^n)$, such that $\chi_\sigma = \chi_{\boldsymbol{\phi}^{-1}(\tau)}$ (ν-a.e.). We have

$$\mathcal{K} \ni P(E \circ \boldsymbol{\phi}^{-1})(\tau)1 = (E \circ \boldsymbol{\phi}^{-1})(\tau)P1 = E(\boldsymbol{\phi}^{-1}(\tau))1 = \chi_\sigma.$$

Since the characteristic functions are linearly dense in $L^2(\nu)$, the proof is complete. $\square$

We introduce the set $\mathcal{Q}(A)$ of *quasianalytic vectors* of an operator A (in the whole paper $1/0 := +\infty$):

$$\mathcal{Q}(A) := \left\{ f \in \bigcap_{k=1}^{\infty} \mathcal{D}(A^k) \,\middle|\, \sum_{k=1}^{\infty} \|A^k f\|^{-1/k} = +\infty \right\}.$$

A real function $\phi \in \bigcap_{k=1}^{\infty} L^{2k}(\mu)$ is said to be *of μ-quasianalytic type* if

$$\sum_{k=1}^{\infty} \left(\int \phi^{2k} \, d\mu \right)^{-1/(2k)} = +\infty.$$

Observe, that ϕ is of μ-quasianalytic type if and only if $1 \in \mathcal{Q}(M_\phi)$. It is clear that real μ-a.e. bounded functions are of μ-quasianalytic type. If $|\psi| \leq |\phi|$ and ϕ is of μ-quasianalytic type then ψ is of μ-quasianalytic type as well.

https://doi.org/10.1017/S0017089504002216 Published online by Cambridge University Press

Let $\psi_n, \ldots, \psi_n$ be real measurable functions on $\mathcal{X}$ and let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The set

$$\mathbb{C}[\psi_1, \ldots, \psi_n] := \operatorname{lin}\{\, \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N}\}$$

is the complex algebra with unit generated by the functions $\psi_1, \ldots, \psi_n$. In this notation $\mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ stands for the algebra of all complex polynomials in n real variables; by $\mathbb{R}[\mathrm{x}]$ we will denote the set of all real polynomials in a single real variable. If $\mathcal{D}$ is an algebra of functions on $\mathcal{X}$, which are square-integrable with respect to the measure μ, then we will not distinguish between functions from $\mathcal{D}$ and their equivalence classes in $L^2(\mu)$ and we will write $\mathcal{D} \subseteq L^2(\mu)$. By $\mathcal{D}(\mu)$ we understand the closure of $\mathcal{D}$ in $L^2(\mu)$. For $\phi \in \mathcal{D}$ we define a densely defined operator in the Hilbert space $\mathcal{D}(\mu)$ via $X_\phi := M_\phi|_{\mathcal{D}}$. Observe that $X_\phi \in \boldsymbol{L}(\mathcal{D})$.

Lemma 3. *Assume that* $(\mathcal{X}, \mathfrak{M}, \nu)$ *is a finite measure space. Let* $\psi_1, \ldots, \psi_n$ *be real measurable functions on* $\mathcal{X}$ *such that* $\mathcal{D} := \mathbb{C}[\psi_1, \ldots, \psi_n] \subseteq L^2(\nu)$ *and let* $k \in \{1, \ldots, n\}$. *If* ψ_k *is of* ν*-quasianalytic type then* X_{ψ_k} *is essentially selfadjoint in* $\mathcal{D}(\nu)$.

Proof. The operator X_{ψ_k} is symmetric and commutes pointwise with X_{ψ_j} $(j = 1, \ldots, n)$. From [**11**, Proposition 2] obtain $X_{\psi_j}(\mathcal{Q}(X_{\psi_k})) \subseteq \mathcal{Q}(X_{\psi_k})$, $j = 1, \ldots, n$. Since $1 \in \mathcal{Q}(X_{\psi_k})$ we have that

$$\operatorname{lin}\mathcal{Q}(X_{\psi_k}) \supseteq \operatorname{lin}\left\{X_{\psi_1}^{\alpha_1} \cdots X_{\psi_n}^{\alpha_n} 1 \,\middle|\, \alpha_0, \ldots, \alpha_n \in \mathbb{N}\right\} = \mathcal{D}. \tag{1.2}$$

Now we can use Nussbaum's criterion for essential selfadjointness (cf. [**3**, Theorem 2]), which completes the proof. □

Since $\mathcal{D}(\mathcal{X}_{\psi_k}) = \mathcal{D}$ the inclusion in (1.2) is in fact an equality. We can obtain a stronger result here, namely $\mathcal{Q}(X_{\psi_k}) = \mathcal{D}$ because every linear combination of $X_{\psi_1}^{\alpha_1} \cdots X_{\psi_n}^{\alpha_n}$ $(\alpha_0, \ldots, \alpha_n \in \mathbb{N})$ commutes pointwise with X_{ψ_k}. Note that $\mathcal{Q}(A_0)$ need not be a linear space ([**7**]).

2. Log-convex sequences. We will call a sequence $(a_k)_{k=1}^\infty \subseteq [0, +\infty)$ *log-convex* if $a_k^2 \leq a_{k-1}a_{k+1}$ for $k > 1$.

Let us state now some simple facts about divergent sequences and functions of μ-quasianalytic type:

Proposition 4. (a) *If* $(b_k)_{k=1}^\infty \subseteq [0, +\infty)$ *and* $c > 0$ *then* $\sum_{k=1}^\infty b_k = +\infty$ *if and only if* $\sum_{k=1}^\infty c^{1/k} b_k = +\infty$.

(b) *Assume that* $(a_k)_{k=1}^\infty$ *is log-convex and* $n \in \mathbb{N}\backslash\{0\}$. *Then* $\sum_{k=1}^\infty a_k^{-1/k} = +\infty$ *if and only if* $\sum_{k=1}^\infty a_{kn}^{-1/kn} = +\infty$.

(c) *If* $a_k = \|A^k f\|$, *where* $A \in \boldsymbol{L}(\mathcal{D})$ *is symmetric and* $f \in \mathcal{D}$, *then the sequence* $(a_k)_{k=1}^\infty$ *is log-convex.*

(d) *If* $\phi \in \bigcap_{k=1}^\infty L^{2k}(\mu)$ *is real, then for all* $n \in \mathbb{N}\backslash\{0\}$

$$\sum_{k=1}^\infty \left(\int \phi^{2k}\, d\mu\right)^{-1/(2k)} = +\infty \quad\Longleftrightarrow\quad \sum_{k=1}^\infty \left(\int \phi^{2nk}\, d\mu\right)^{-1/(2nk)} = +\infty.$$

Proof. (cf. [**11**, Section 1]) Point (a) is in fact obvious, since $c^{1/k} \to 1$ $(k \to \infty)$. To prove point (b) observe that by induction we can obtain $a_k \leq a_0^{1/(k+1)} a_{k+1}^{k/(k+1)}$ for $k \in \mathbb{N}$.

https://doi.org/10.1017/S0017089504002216 Published online by Cambridge University Press

The case when $a_l = 0$ for some $l \in \mathbb{N}$ is trivial, assume now the contrary. Due to point (a) we can also assume, without loss of generality, that $a_0 = 1$ and so the sequence $(a_k^{-1/k})_{k=0}$ is decreasing. This completes the proof of (b). Point (c) is simple to prove and true even for paranormal operators; see [**9**] for consequences. Point (d) results straightforwardly from (b) and (c). $\square$

The following Lemma has been inspired by [**10**, Proposition 47].

LEMMA 5. *Let $(\mathcal{X}, \mathfrak{M}, \mu)$ be a finite measure space and let $\rho \in L^2(\mu)$, $\rho \geq 0$. If $\mathcal{D}$ is an algebra contained in $L^2(\mu)$ then $\mathcal{D}$ is contained in $L^2(\rho\mu)$. Moreover, if $\psi \in \mathcal{D}$ is of μ-quasianalytic type then it is also of $\rho\mu$-quasianalytic type.*

Proof. From Hölder's inequality we get

$$\int |\phi|^2 \rho \, d\mu \leq \left(\int |\phi^2|^2 d\mu \int \rho^2 d\mu \right)^{1/2}, \quad \phi \in \mathcal{D}.$$

So $\phi \in L^2(\rho\mu)$.

Let $\psi \in \mathcal{D}$ be of μ-quasianalytic type. Then, by the Schwarz inequality, we have

$$\left(\int \psi^{2k} \rho \, d\mu \right)^{1/(2k)} \leq \left(\int \psi^{4k} d\mu \right)^{1/(4k)} \left(\int \rho^2 d\mu \right)^{1/(4k)}.$$

which together with Proposition 4(a) and (d) gives us that ψ is of $\rho\mu$-quasianalytic type. $\square$

3. Finitely generated algebras: domination methods. The following Theorem (formulated in a slightly different form) was proved in [**4**]. See also [**10**, Theorem 10] for a result which is explicitly stronger than the Theorem below; also many consequences can be found in [**10**]. Even for $n = 1$ Theorem 6 is not trivial.

THEOREM 6. *Assume that $\mathcal{D}$ is a dense linear subspace of $\mathcal{H}$. Let $A_0, \dots, A_n \in \boldsymbol{L}(\mathcal{D})$ $(n \geq 1)$ be symmetric operators in $\mathcal{H}$ such that:*

(i) *the operators A_i, A_j commute pointwise for $i, j = 0, \dots, n$;*
(ii) *A_0 is essentially selfadjoint;*
(iii) *there exists $c > 0$, such that $\|A_j f\|^2 \leq c(\|f\|^2 + \|A_0 f\|^2)$ for $f \in \mathcal{D}, j = 1, \dots, n$.*

Then $\overline{A}_0, \dots, \overline{A}_n$ are spectrally commuting selfadjoint operators.

Now we will formulate one of the main results of this paper.

THEOREM 7. *Let $(\mathcal{X}, \mathfrak{M}, \mu)$ be a finite measure space. Assume that:*

(a1) *$\boldsymbol{\phi} = (\phi_0, \dots, \phi_n) \in \mathcal{M}(\mathcal{X}, \mathbb{R}^{n+1}, \mu)$;*
(a2) *there exists $c > 0$ such that $\phi_j^2 \leq c(1 + \phi_0^2)$ μ-a.e. for $j = 1, \dots, n$;*
(a3) *$p \in \mathbb{R}[x] \backslash \{0\}$ and the algebra $\mathcal{D} := \mathbb{C}[\phi_0, \phi_1, \dots, \phi_n, 1/(p \circ \phi_0)]$ is contained in $L^2(\mu)$;*
(a4) *the function $\dfrac{q \circ \phi_0}{p \circ \phi_0}$ is of μ-quasianalytic type for some $q \in \mathbb{R}[x] \backslash \{\lambda p : \lambda \in \mathbb{R}\}$.*

Then $\mathcal{D}$ is dense in $L^2(\rho\mu)$ for every $\rho \in L^2(\mu)$, $\rho \geq 0$. In particular $\mathcal{D}(\mu) = L^2(\mu)$.

Observe that the condition (a4) is implied by the following one:

(a4$'$) *p is a nonconstant polynomial, $1/(p \circ \phi_0)$ is μ-a.e. bounded.*

In particular, if $\xi \in \mathbb{R}\setminus\overline{\phi_0(\mathbb{R})}$, then the condition (a4′) holds with $p(x) := x - \xi$.

Note also that (a2) is equivalent to: There exists $d > 0$ such that $|P_j| \leq d(1 + |\phi_0|)$ μ-a.e. for $j = 1, \ldots, n$.

Theorem 7 does not exclude the case when $n = 0$. In such situation we have only one function $\phi_0 \in \mathcal{M}(\mathcal{X}, \mathbb{R}^1, \mu)$ and the condition (a2) disappears.

Proof of Theorem 7. Lemma 5 gives us that $\mathcal{D} \subseteq L^2(\rho\mu)$ and that the function $\frac{q\circ\phi_0}{p\circ\phi_0} \in \mathcal{D}$ is of $\rho\mu$-quasianalytic type. Hence, the operator $X_{\frac{q\circ\phi_0}{p\circ\phi_0}}$ is essentially selfadjoint in $\mathcal{D}(\rho\mu)$ (see Lemma 3). We will show now that the operator X_{ϕ_0} is essentially selfadjoint in $\mathcal{D}(\rho\mu)$.

Notice that, with $X_{\phi_0}^0 := I_{\mathcal{D}}$, $p(X_{\phi_0}) = X_{p\circ\phi_0}$ and $X_{\frac{1}{p\circ\phi_0}} p(X_{\phi_0}) = I_{\mathcal{D}}$. Thus the operator $p(X_{\phi_0})$ has the trivial kernel and a dense (in $\mathcal{D}(\rho\mu)$) range. Consequently, $\mathcal{N}(p(X_{\phi_0})^*) = \{0\}$ and

$$\left(\left(p\left(X_{\phi_0}\right)\right)^{-1} \right)^* = \left(\left(p\left(X_{\phi_0}\right)\right)^*\right)^{-1}. \tag{3.1}$$

Since q/p is a nonconstant meromorphic function with a finite number of poles (due to (a3) and (a4)) there exists $z \in \mathbb{C}\setminus\mathbb{R}$ such that $q(z)/p(z) \in \mathbb{C}\setminus\mathbb{R}$. Let us take a vector $f \in \mathcal{N}(z - X_{\phi_0}^*)$. Then

$$q\left(X_{\phi_0}^*\right)f = q(z)f, \quad p\left(X_{\phi_0}^*\right)f = p(z)f.$$

Because $p(X_{\phi_0}^*) \subseteq (p(X_{\phi_0}))^*$ the operator $p(X_{\phi_0}^*)$ has the trivial kernel as well. Moreover, we have

$$\left(p\left(X_{\phi_0}^*\right)\right)^{-1} f = \frac{1}{p(z)} f.$$

Since $X_{\frac{q\circ\phi_0}{p\circ\phi_0}} = (p(X_{\phi_0}))^{-1} q(X_{\phi_0})$ and

$$\left(p\left(X_{\phi_0}^*\right)\right)^{-1} q\left(X_{\phi_0}^*\right) \subseteq \left(\left(p\left(X_{\phi_0}\right)\right)^{-1} q\left(X_{\phi_0}\right)\right)^*,$$

we obtain:

$$X_{\frac{q\circ\phi_0}{p\circ\phi_0}}^* f = \left(\left(p\left(X_{\phi_0}\right)\right)^{-1} q\left(X_{\phi_0}\right)\right) f = \frac{q(z)}{p(z)} f.$$

The operator $X_{\frac{q\circ\phi_0}{p\circ\phi_0}}$ is essentially selfadjoint, hence $f = 0$ and consequently $\mathcal{N}(z - X_{\phi_0}^*) = \{0\}$. Because $\overline{q(z)/p(z)} = q(\bar{z})/p(\bar{z}) \in \mathbb{C}\setminus\mathbb{R}$ we can apply the same arguments and get $\mathcal{N}(\bar{z} - X_{\phi_0}^*) = \{0\}$. This completes the proof of essential selfadjointness of X_{ϕ_0} in $\mathcal{D}(\rho\mu)$.

Observe now that the assumptions (i) and (ii) of Theorem 6 hold with $\mathcal{H} := \mathcal{D}(\rho\mu)$, $A_j := X_{\phi_j}$ for $j = 0, \ldots, n$. The assumption (iii) of Theorem 6 is also fulfilled since for $j = 1, \ldots, n, f \in \mathcal{D}$ we have

$$\|A_j\|^2 = \int \phi_j^2 |f|^2 \rho\, d\mu \leq \int c(1 + \phi_0)^2 |f|^2 \rho\, d\mu = c(\|f^2\| + \|A_0 f\|^2).$$

As a consequence the operators $\overline{X}_{\phi_0}, \overline{X}_{\phi_1}, \ldots, \overline{X}_{\phi_n}$ are selfadjoint in $\mathcal{D}(\rho\mu)$. According to [**12**, Corollary 1] $\mathcal{D}(\rho\mu)$ reduces M_{ϕ_j} to $\overline{X}_{\phi_j}$ $(j = 0, \ldots, n)$. Observe that $\boldsymbol{\phi} \in$

https://doi.org/10.1017/S0017089504002216 Published online by Cambridge University Press

$\mathcal{M}(\mathcal{X}, \mathbb{R}^{n+1}, \rho\mu)$. This enables us to apply Proposition 2. As a consequence we get $\mathcal{D}(\rho\mu) = L^2(\rho\mu)$. □

The idea of the first part of the proof (essential selfadjointness of X_{ϕ_0}) is taken from Example 4.1 of **[8]**, which is stated below. We will show now that that Example is a special case of Theorem 7. Our r, s, ψ, $\mathcal{P}$ are denoted in **[8]** by p, q, ϕ, $\mathcal{D}$, respectively.

COROLLARY 8. [**8**, Example 4.1] *Let μ be a probability measure on $[0, +\infty)$ and let the algebra $\mathcal{P}$ of square integrable functions contain $\mathbb{C}[\mathrm{x}]$ and a nonconstant rational function $\psi = r/s$, where $r, s \in \mathbb{R}[\mathrm{x}]$. Suppose, moreover, that $1/(\lambda - \psi)$ is bounded and $1/(\lambda s - r) \in \mathcal{P}$ for some $\lambda \in \mathbb{R}$. Then $\mathcal{P}(\mu) = L^2(\mu)$.*

Proof. Consider the algebra $\mathcal{D} := \mathbb{C}[\mathrm{x}, 1/(\lambda r - s)] \subseteq \mathcal{P}$. $\mathcal{D}$ fulfills all the assumptions of Theorem 7 with $n = 0$, $\phi_0 = \mathrm{x}$, $p = \lambda s - r$, $q = s$, because $q/p = 1/(\lambda - \psi)$ is bounded (and hence of μ-quasianalytic type). Thus $\mathcal{D}$ and consequently $\mathcal{P}$ are dense in $L^2(\mu)$. □

COROLLARY 9. *Let $(\mathcal{X}, \mathfrak{M}, \mu)$ be a finite measure space and let $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n) \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n, \mu)$ be such that $\mathbb{C}[\psi_1, \ldots, \psi_n] \subseteq L^2(\mu)$. Then the algebra*

$$\mathcal{P} := \mathbb{C}\left[\psi_1, \ldots, \psi_n, \frac{1}{1 + \psi_1^{2k} + \cdots + \psi_n^{2k}}\right]$$

is dense in $L^2(\rho\mu)$ for every $\rho \in L^2(\mu)$, $\rho \geq 0$, $k \in \mathbb{N}\backslash\{0\}$.

Proof. We will apply Theorem 7 with $\phi_0 := \psi_1^{2k} + \cdots \psi_n^{2k}$, $\phi_j = \psi_j$ $(j = 1, \ldots, n)$, $p(x) := x + 1$. Notice that $\mathcal{P} = \mathbb{C}[\phi_0, \ldots, \phi_n]$. Consider the mapping

$$\kappa : \mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto \left(t_1^{2k} + \cdots + t_n^{2k}, t_1, \ldots, t_n\right) \in \mathbb{R}^{n+1}.$$

Observe that κ is a bimeasurable injection, $\boldsymbol{\phi} = \kappa \circ \boldsymbol{\psi}$ and $\boldsymbol{\psi} \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n, \mu)$. Hence $\boldsymbol{\phi} \in \mathcal{M}(\mathcal{X}, \mathbb{R}^{n+1}, \mu)$ and so the assumption (a1) (of Theorem 7) is satisfied. Point (a2) is obvious in this situation. Since the function $(1 + \psi_1^{2k} \cdots \psi_n^{2k})^{-1}$ is bounded and $\mathbb{C}[\psi_1, \ldots, \psi_n] \subseteq L^2(\mu)$, we have (a3). Observe that $-1 \notin \overline{\phi_0(\mathbb{R})}$, so the condition (a4′) is also fulfilled. Theorem 7 says now that $\mathcal{P}$ is dense in $L^2(\rho\mu)$. □

EXAMPLE 10. Recall that $\mathrm{x}_j(x) := x_j$. Putting $\mathcal{X} = \mathbb{R}^n$, $\mathfrak{M} = \mathfrak{B}(\mathbb{R}^n)$, $\psi_j := \mathrm{x}_j$ in the above Corollary we obtain the following fact. Let μ be a finite Borel measure on $\mathbb{R}^n$ such that the polynomials are absolutely integrable and let $k \in \mathbb{N}\backslash\{0\}$. Then the algebra $\mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n, \frac{1}{1 + \mathrm{x}_1^{2k} + \cdots + \mathrm{x}_n^{2k}}]$ is dense in $L^2(\rho\mu)$ for $\rho \in L^2(\mu)$, $\rho \geq 0$. (This result with $\rho = 1$ and $k = 1$ was proved in [**6**, Theorem 2.5]; see also [**10**, Proof of Theorem 45]).

EXAMPLE 11. Let μ be a finite Borel measure on $[\gamma, +\infty)^m$ $(\gamma \in \mathbb{R}, m \geq 2)$ such that the algebra $\mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_{m-1}, \mathrm{e}^{\mathrm{x}_1 + \cdots + \mathrm{x}_m}]$ is contained in $L^2(\mu)$. Consider the mapping $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_{m-1}) : [\gamma, +\infty)^m \to \mathbb{R}^m$, where $\phi_0 = \mathrm{e}^{\mathrm{x}_1 + \cdots + \mathrm{x}_m}$ and $\phi_j = \mathrm{x}_j$ $(j = 1, \ldots, m-1)$. Since $\boldsymbol{\phi}$ is a bimeasurable injection, the assumption (a1) of Theorem 7 holds with $n = m - 1$ (see Proposition 1). We can easily check the assumptions (a2), (a3) and (a4′) (with $p(x) = x + 1$). As a consequence we get that the algebra $\mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_{m-1}, \mathrm{e}^{\mathrm{x}_1 + \cdots + \mathrm{x}_m}, \frac{1}{1 + \mathrm{e}^{\mathrm{x}_1 + \cdots + \mathrm{x}_m}}]$ is dense in $L^2(\rho\mu)$ for every $\rho \in L^2(\mu)$, $\rho \geq 0$.

EXAMPLE 12. Let μ be a Borel measure whose support is contained in the set $\{x \in \mathbb{R}^{n+1} : |x_j|^2 \leq c(1 + |x_0|^{2k})\,(j = 1, \ldots, n),\ |x_0| > \varepsilon\}$ where $c, \varepsilon > 0$, $k \in \mathbb{N}\backslash\{0\}$.

Assume also that the algebra $\mathbb{C}[\mathrm{x}_0, \mathrm{x}_1, \ldots, \mathrm{x}_n]$ is contained in $L^2(\mu)$. Then for every $l \in \mathbb{N}$ such that $2l+1 \geq k$ the algebra $\mathbb{C}[\mathrm{x}_0^{2l+1}, \mathrm{x}_1, \ldots, \mathrm{x}_n, 1/\mathrm{x}_0^{2l+1}]$ is dense in $L^2(\rho\mu)$ ($\rho \in L^2(\mu)$, $\rho \geq 0$). Indeed, we put $\phi_0 := \mathrm{x}_0^{2l+1}$, $\phi_j := \mathrm{x}_j$ for $j = 1, \ldots, n$, $p(x) := x$ and check the assumptions of Theorem 7. The conditions (a1), (a3) and (a4′) are obvious. To see that (a2) holds observe that there exists $d > 0$ such that $c(1 + |x_0|^{2k}) \leq d(1 + |x_0|^{2(2l+1)})$ for all $x \in \mathbb{R}^n$.

In the applications of Theorem 7 presented above we used the condition (a4'). The following Proposition (which can be treated as a method of constructing examples) requires Theorem 7 with a stronger condition (a4).

PROPOSITION 13. *Let $(\mathcal{X}, \mathfrak{M}, \nu)$ be a finite measure space. Assume that:*

(i) $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_n) \in \mathcal{M}(\mathcal{X}, \mathbb{R}^{n+1}, \nu)$ *;*

(ii) *there exists $c > 0$ such that $\phi_j^2 \leq c(1 + \phi_0^2)$ ν-a.e. for $j = 1, \ldots, n$;*

(iii) *$p \in \mathbb{R}[\mathrm{x}]\backslash\{0\}$ and the algebra $\mathcal{D} := \mathbb{C}[\phi_0, \phi_1, \ldots, \phi_n, 1/(p \circ \phi_0)]$ is contained in $L^2(\nu)$.*

Let also $r \in \mathbb{R}[\mathrm{x}]\backslash\{\lambda p : \lambda \in \mathbb{R}\}$. Then the algebra $\mathcal{D}$ is contained and dense in $L^2(\rho\mu)$ for every $\rho \geq 0$, $\rho \in L^2(\mu)$, where $\mu := \exp(-|\frac{r \circ \phi_0}{p \circ \phi_0}|)\nu$.

Proof. It suffices to show that the system $(\mathcal{X}, \mathfrak{M}, \mu, \boldsymbol{\phi}, p)$ satisfies the assumptions (a1)–(a4) of Theorem 7.

The condition (a1) holds, because the measure μ is absolutely continuous with respect to ν. From the same reason the condition (a2) is fulfilled. Since the function $\exp(-|\frac{r \circ \phi_0}{p \circ \phi_0}|)$ is bounded, we have $\mathcal{D} \subseteq L^2(\mu)$, so the condition (a3) is also satisfied.

The only problem is now to show that the real function $\psi := \frac{q \circ \phi_0}{p \circ \phi_0}$ is of μ-quasianalytic type for some $q \in \mathbb{R}[\mathrm{x}]\backslash\{\lambda p : \lambda \in \mathbb{R}\}$. Put $q(x) := (e/4)r(x)$. Observe that

$$\forall_{x \in \mathcal{X}} \quad \left(\frac{\psi(x)}{k}\right)^{2k} \to 0 \text{ with } k \to \infty.$$

Moreover

$$\forall_{k \in \mathbb{N}\backslash\{0\}} \forall_{x \in \mathcal{X}} \quad \left(\frac{\psi(x)}{k}\right)^{2k} \leq e^{2|\psi(x)|/e}.$$

This equality is trivial if $\psi(x) = 0$. If $y := |\psi(x)| > 0$ it is enough to investigate the function $f_y : \xi \mapsto (y/\xi)^{2\xi}$ whose global maximum on $(0, +\infty)$ equals $\mathrm{e}^{2y/e}$.

Observe that $\mathrm{e}^{2|\psi|/e} = \exp(|\frac{r \circ \phi_0}{2p \circ \phi_0}|) \in L^2(\mu)$. So by the Lebesgue dominated convergence theorem

$$\int \left(\frac{\psi}{k}\right)^{2k} d\mu \to 0 \text{ with } k \to \infty.$$

So for k large enough

$$\left(\int \psi^{2k}\, d\mu\right)^{-1/(2k)} \geq 1/k,$$

and consequently ψ is of μ-quasianalytic type. $\square$

https://doi.org/10.1017/S0017089504002216 Published online by Cambridge University Press

4. Quasianalyticity without domination. The following Proposition is a special case of [**10**, Lemma 38]; we present here a simple proof.

Proposition 14. *If a function ψ is of μ-quasianalytic type and if for some $c > 0$ we have $\phi \le c(1 + |\psi|)$ (μ-a.e.) then ϕ is of μ-quasianalytic type.*

Proof. It is enough to prove that $(1 + |\psi|)$ is of μ-quasianalytic type. Applying the triangle inequality in the space $L^{2k}(\mu)$ we get

$$\left(\int (1+|\psi|)^{2k}\, d\mu\right)^{1/(2k)} \le \left[\left(\int 1\, d\mu\right)^{1/(2k)} + \left(\int \psi^{2k}\, d\mu\right)^{1/(2k)}\right].$$

Observe that the first summand on the right hand side tends to 1 with $k \to \infty$. We can apply now the following simple fact.

Let $(a_k)_{k=1}^{\infty} \subseteq [0, +\infty)$. If $\sum_{k=1}^{\infty} 1/a_k = +\infty$ and the sequence $(b_k)_{k=1}^{\infty} \subseteq [0, +\infty)$ is bounded then $\sum_{k=1}^{\infty} 1/(a_k + b_k) = +\infty$. □

Notice that if the function ϕ_0 from Theorem 7 is of μ-quasianalytic type, then the functions ϕ_j ($j = 1, \dots, n$) are also of μ-quasianalytic type. In this situation we can prove a similar result.

Theorem 15. *Let $(\mathcal{X}, \mathfrak{M}, \mu)$ be a finite measure space, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n) \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n, \mu)$ and let $\mathcal{D} := \mathbb{C}[\phi_1, \dots, \phi_n] \subseteq L^2(\mu)$. Suppose that ϕ_j is of μ-quasianalytic type for $j = 1, \dots, n$. Then $\mathcal{D}(\rho\mu) = L^2(\rho\mu)$ for every $\rho \in L^2(\mu)$, $\rho \ge 0$.*

Proof. Let $j \in \{0, \dots, n\}$. Lemma 5 gives us that $\mathcal{D} \subseteq L^2(\rho\mu)$ and that the function ϕ_j is of $\rho\mu$-quasianalytic type. So the operator X_{ϕ_j} is essentially selfadjoint in $\mathcal{D}(\rho\mu)$ (see Lemma 3). According to [**12**, Corollary 1] the space $\mathcal{D}(\rho\mu)$ reduces M_{ϕ_j}. Due to the Proposition 2 we get $\mathcal{D}(\rho\mu) = L^2(\rho\mu)$. □

Quasianalytic vectors have been investigated in many papers; for example in [**3**] or [**11**] one may find theorems similar to the one above (especially if $\rho \equiv 1$).

Corollary 16. *Let $\mathcal{X}$ be a Borel subset of a complete separable metric space. Let $\mathfrak{M} = \mathfrak{B}(\mathcal{X})$ and let μ be any finite Borel measure on $\mathcal{X}$. There exists a bounded function $\phi \in L^2(\mu)$ such that $\mathbb{C}[\phi]$ is dense in $L^2(\rho\mu)$ for every $\rho \in L^2(\mu)$, $\rho \ge 0$.*

Proof. Due to [**5**, p. 12, Theorem 2.8] $\mathcal{X}$ is countable or has the power of continuum. Due to Kuratowski's theorem (cf. [**5**, p. 14, Theorem 2.12]) if $\mathcal{X}$ has the power of continuum there exists a bimeasurable bijection $\phi : \mathcal{X} \to [0, 1]$. If $\mathcal{X}$ is enumerable ($\mathcal{X}$ is finite), then there exists a bimeasurable bijection $\phi : \mathcal{X} \to \{1/n : n \in \mathbb{N}\}$ ($\phi : \mathcal{X} \to \{1/n : n \in \mathbb{N}, n \le N\}$ where N is the number of elements of $\mathcal{X}$, respectively). It follows from Theorem 7 that the algebra with unit generated by this single function is dense in $L^2(\mu)$. □

This fact could be obtained by more elementary methods – the measure transport theorem and the Weierstrass theorem.

We will now show an application of Theorem 15 to moment problems. Let μ be a measure on $\mathbb{R}^n$ such that the polynomials are absolutely integrable. We will call μ *ultradeterminate* if the polynomials are dense in $L^2((1 + |\mathrm{x}|^2)\mu))$ where $|\mathrm{x}|^2 := \mathrm{x}_1^2 + \cdots \mathrm{x}_n^2$. If a measure μ is ultradeterminate then its moment sequence

$$c_\alpha = \int \mathrm{x}_1^{\alpha_1} \cdots \mathrm{x}_n^{\alpha_n}\, d\mu, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$$

https://doi.org/10.1017/S0017089504002216 Published online by Cambridge University Press

is determinate, i.e. if ν is any measure such that $c_\alpha = \int \mathrm{x}_1^{\alpha_1} \cdots \mathrm{x}_n^{\alpha_n}\, d\nu$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ then $\nu = \mu$ (cf. [**2**]). The following Corollary extends Theorem 9 from [**3**].

Corollary 17. *If* x_j *is of* μ*-quasianalytic type for* $j = 1, \ldots, n$ *then* $\eta\mu$ *is ultradeterminate for every* $\eta \in L^2((1 + |\mathrm{x}|^2)^2\mu)$*,* $\eta \geq 0$.

Proof. We apply Theorem 15 to $\phi_j := \mathrm{x}_j$, $j = 1, \ldots, n$, $\rho := \eta(1 + |\mathrm{x}|^2)$. □

We can write the fact that x_j is of μ-quasianalytic type in the language of sequences: $\sum_{k=1}^{\infty} c_{2ke_j}^{-1/(2k)} = +\infty$ where c_α is as above and e_j stands for the multiindex $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length n with 1 on the jth position. It appears that we do not need here that $(c_\alpha)_{\alpha \in \mathbb{N}^n}$ is a moment sequence – it is enough to assume that it is positive definite, cf [**2**].

5. The most general case. The following Theorem is a combination of Theorems 7 and 15.

Theorem 18. *Let* $(\mathcal{X}, \mathfrak{M}, \mu)$ *be a finite measure space and* $A = A_0 \cup \ldots \cup A_N$ *be a finite union of pairwise disjoint sets such that* $A_j \neq \emptyset$ *for* $j = 1, \ldots, n$*. Denote by* n *the number of elements of* A*. Assume that:*

(i) $\boldsymbol{\phi} = (\phi_\alpha)_{\alpha \in A} \in \mathcal{M}(\mathcal{X}, \mathbb{R}^n, \mu)$*;*

(ii) *for each* $\alpha \in A_0$ *the function* ϕ_α *is of* μ*-quasianalytic type;*

(iii) *for each* $j \in \{1, \ldots, N\}$ *there exists* $\beta_j \in A_j$ *and* $c_j > 0$ *such that for all* $\alpha \in A_j \backslash \{\beta_j\}$ *we have* $\phi_\alpha^2 \leq c_j(1 + \phi_{\beta_j}^2)$ μ*-a.e.;*

(iv) *for each* $j \in \{1, \ldots, N\}$ *there exists* $p_j \in \mathbb{R}[\mathrm{x}]\backslash\{0\}$ *such that the algebra*

$$\mathcal{D} := \operatorname{lin}\{\prod_{\alpha \in A} \phi_\alpha^{k_\alpha} \cdot \prod_{j=1,\ldots,N} \left(\frac{1}{p_j \circ \phi_{\beta_j}}\right)^{l_j} : k_\alpha, l_j \in \mathbb{N}\}$$

is contained in $L^2(\mu)$*;*

(v) *for each* $j \in \{1, \ldots, N\}$ *there exists* $q_j \in \mathbb{R}[x]\backslash\{\lambda p_j : \lambda \in \mathbb{R}\}$ *such that the function* $\frac{q_j \circ \phi_{\beta_j}}{p_j \circ \phi_{\beta_j}}$ *is of* μ*-quasianalytic type.*

Then $\mathcal{D}$ *is dense in* $L^2(\rho\mu)$ *for every* $\rho \in L^2(\mu)$*,* $\rho \geq 0$*. In particular* $\mathcal{D}(\mu) = L^2(\mu)$*.*

As in Theorem 7 the case when $A_j = \{\beta_j\}$ for some $j \in \{1, \ldots, N\}$ is not excluded. Also the case when $N = 0$ is possible; in this situation the condition (iii) disappears and the condition (iv) has the following form: *The algebra*

$$\mathcal{D} := \operatorname{lin}\{\prod_{\alpha \in A} \phi_\alpha^{k_\alpha} : k_\alpha \in \mathbb{N}\}$$

is contained in $L^2(\mu)$.

Observe that if we put $A_0 = \emptyset$, $N = 1$ then we get Theorem 7 and if we put $N = 0$ then we get Theorem 15.

Proof of Theorem 18. As in the proof of Theorem 7 we see that for all $j = 1, \ldots, N$, $\alpha \in A_j$ the operators X_{ϕ_α} are essentially selfadjoint in $\mathcal{D}(\rho\mu)$. Repeating the proof of Theorem 15 we get that X_{ϕ_α} is essentially selfadjoint in $\mathcal{D}(\rho\mu)$ for $\alpha \in A_0$. Now we use standard arguments to prove that $\mathcal{D}(\rho\mu) = L^2(\rho\mu)$. □

https://doi.org/10.1017/S0017089504002216 Published online by Cambridge University Press

EXAMPLE 19. Let μ be a Borel measure whose support is contained in the set $\{x \in \mathbb{R}^n : |x_j| > \varepsilon \ (j = 1, \ldots, n)\}$ where $\varepsilon > 0$. Assume also that the algebra $\mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ is contained in $L^2(\mu)$. Then for every $k \in \mathbb{N}$, k odd, the algebra $\mathcal{P}_k := \mathbb{C}[\mathrm{x}_1^k, 1/x_1^k, \ldots, \mathrm{x}_n^k, 1/\mathrm{x}_n^k]$ is dense in $L^2(\rho\mu)$ ($\rho \in L^2(\mu)$, $\rho \geq 0$). Indeed, we put $A_0 = \emptyset$, $A_j = \{j\}$, $\phi_j := \mathrm{x}_j^k$, $p_j(x) := x$, $q_j \equiv 1$ for $j = 1, \ldots, n$ and check the assumptions of Theorem 18. Observe that for $l \in \mathbb{N}\backslash\{0\}$ $\mathcal{P}_{k^l}$ is dense in $L^2(\rho\mu)$ and $\mathcal{P}_{k^{(l+1)}} \subseteq \mathcal{P}_{k^l}$. But $\bigcap_{l=1}^{\infty} \mathcal{P}_{k^l} = \mathbb{C} \cdot 1$.

ACKNOWLEDGEMENT. I would like to thank Professor Jan Stochel for the inspiration for this paper and for many important suggestions.

REFERENCES

1. N. I. Akhiezer and M. I. Glazman, *Theory of linear operators in Hilbert space* vol. II (The Pitman Press, Bath 1980).

2. B. Fuglede, The multidimensional moment problem, *Expo. Math.* **1** (1983), 47–65.

3. A. E. Nussbaum, Quasi-analytic vectors, *Ark. Mat.* **6** (1965), 179–191.

4. N. S. Poulsen, On the cannonical commutation relations, *Math. Scand.* **32** (1973), 112–122.

5. K. R. Parthasarathy, *Probability measures on metric spaces* (Academic Press, San Diego, 1967).

6. M. Putinar and F.-H. Vasilescu, Solving moment problems by dimensional extension, *Ann. of Math.* **149** (1999), 1087–1107.

7. J. Rusinek, Non linearity of the set of p-quasi analytic vectors for some essentially self-adjoint operators, *Bull. Polish Acad. Sci. Math.* **48** (2000), 287–292.

8. Z. Sebestyén and J. Stochel, On products of unbounded operators, *Acta Math. Hungar.* **100** (2003), 105–129.

9. J. Stochel and F. H. Szafraniec, $\mathcal{C}^\infty$-vectors and boundness, *Ann. Polon. Math.* **66** (1997), 223–238.

10. J. Stochel and F. H. Szafraniec, Domination of unbounded operators and commutativity, *J. Math. Soc. Japan* **55** (2003), 405–437.

11. J. Stochel and F. H. Szafraniec, On normal extensions of unbounded operators, I, *J. Operator Theory* **14** (1985), 31–55.

12. J. Stochel and F. H. Szafraniec, The normal part of an unbounded operator, *Nederl. Akad. Wetensch. Proc. Ser. A* **92** (1989), 495–503.

13. R. Whitley and Normal and quasinormal composition operators, *Proc. Amer. Math. Soc.* **70** (1978), 114–118.