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ALMOST NILPOTENT GAMMA RINGS

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In this paper we introduce the concept of almost nilpotence for Γ-rings, similar to the corresponding concept for rings, as defined by Van Leeuwen and Heyman. An almost nilpotent radical property $\mathcal{A}_0$ is introduced for Γ-rings, and shown to be supernilpotent. If M is a Γ-ring with left and right operator rings L and R respectively, then $\mathcal{A}(L)^+ = \mathcal{A}_0(M) = \mathcal{A}(R)^*$, where $\mathcal{A}()$ denotes the almost nilpotent radical of a ring. If M is a Γ-ring and m, n are positive integers, then $\mathcal{A}_0(M_{m,n})$ is the almost nilpotent radical of the $\Gamma_{n,m}$-ring $M_{m,n}$.

1. BASIC CONCEPTS

Let M and Γ be additive abelian groups. If we have a map from $M \times \Gamma \times M$ to M such that for all $x, y, z \in M$, $\gamma, \mu \in \Gamma$

(i) $x\gamma(y\mu z) = (x\gamma y)\mu z$;
(ii) $x\gamma(y + z) = x\gamma y + x\gamma z$; $x(\gamma + \mu)y = x\gamma y + x\mu y$;
$(x + y)\gamma z = x\gamma z + y\gamma z$

then M is called a Γ-*ring*. If $U \subseteq M$, $V \subseteq M$ and $\varphi \subseteq \Gamma$ then we define:

$$U\varphi V = \{u\gamma v : u \in U, \gamma \in \varphi, v \in V\}.$$

If A is a subgroup of M, and $A\Gamma M \subseteq A$, $M\Gamma A \subseteq A$, then A is an *ideal* of M, denoted by $A \trianglelefteq M$. Similar notation will be used for ideals of rings. If $Q \trianglelefteq M$, then Q is called a *semiprime ideal* of M if $A \trianglelefteq M$, $A\Gamma A \subseteq Q \Longrightarrow A \subseteq Q$. The next result is proved along the same lines as the corresponding result for rings:

PROPOSITION 1.1. *Let M be a Γ-ring and let $Q \trianglelefteq M$. Then the following are equivalent:*

(a) *Q is a semiprime ideal of M;*
(b) $\forall x \in M$, $x\Gamma M\Gamma x \subseteq Q \Longrightarrow x \in Q$.

If $A \trianglelefteq M$, the factor Γ-ring M/A is defined in the natural way.

Let M, M' be Γ-rings and let $f : M \to M'$ be a mapping. If for all $x, y \in M$, $\gamma \in \Gamma$, $f(x + y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$, then f is called a Γ-ring

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homomorphism. If, in addition, f is bijective, then f is called a Γ-ring isomorphism. We will denote this by $M \cong M'$. Similar notation will be used for rings.

If $A \trianglelefteq M$, and $0 \neq I \trianglelefteq M$ implies $I \cap A \neq 0$, then A is called an *essential ideal* of M, denoted A by $\triangleleft \cdot M$.

Let $x \in M$, $\gamma \in \Gamma$. Define $[x, \gamma]: M \to M$ by $[x, \gamma]y = x\gamma y$ for all $y \in M$. The subring L of $\mathrm{End}(M)$ consisting of all sums $\sum_i [x_i, \gamma_i]$, $x_i \in M$, $\gamma_i \in \Gamma$, is called the *left operator ring* of M. A *right operator ring* R of M is defined similarly, and consists of all sums of the form $\sum_i [\gamma_i, x_i]$, $\gamma_i \in \Gamma$, $x_i \in M$.

$$\begin{aligned}
\text{If } A \subseteq L, A^* &= \{x \in M \mid \forall\gamma \in \Gamma; [x,\gamma] \in A\}. \\
\text{If } B \subseteq R, B^* &= \{x \in M \mid \forall\gamma \in \Gamma; [\gamma,x] \in B\}. \\
\text{If } C \subseteq L, C^{+'} &= \{\ell \in L \mid \ell M \subseteq C\} \text{ and} \\
C^{*'} &= \{r \in R \mid Mr \subseteq C\}.
\end{aligned}$$

It is easily seen that all of these mappings take ideals to ideals, and preserve intersections.

If $A \subseteq M$, then $[\Gamma, A] = \{\sum [\gamma_i, x_i] \mid \gamma_i \in \Gamma \,\&\, x_i \in M\}$. It is easily shown that, if $A \trianglelefteq M$, then $[\Gamma, A] \trianglelefteq R$ and that $[\Gamma, A] \subseteq A^{*'}$. If $B \subseteq R$, then $MB = \{\sum x_i b_i \mid x_i \in M, b_i \in B\}$. If $B \trianglelefteq R$, then $MB \trianglelefteq M$, and $MB \subseteq B^*$.

All classes of rings considered will be for some fixed Γ, and abstract, that is closed under isomorphisms. A *radical property* $\mathcal{R}$ for Γ-rings is defined as for the corresponding concept for rings (see [**7**, p.3]). If $\mathcal{U}$ is a class of rings, the *lower radical* determined by $\mathcal{U}$ is constructed as for rings [**7**, p.13]). A class $\mathcal{C}$ of Γ-rings is called *weakly special*, if:

(a) $\mathcal{C}$ consists of semiprime Γ-rings;
(b) if $m \in \mathcal{C}$ and $A \trianglelefteq M$, then $A \in \mathcal{C}$;
(c) if M is a Γ-ring and $A \trianglelefteq \cdot M$, then $A \in \mathcal{C}$ implies $M \in \mathcal{C}$.

A weakly special class of rings is defined in the same way.

Proposition 1.2. *([**5**, Theorem 2.5]). Let $\mathcal{C}$ be a weakly special class of rings. Let $\overline{\mathcal{C}}$ be the class of all Γ-rings M such that the right operator ring $\mathcal{R}$ of M is in $\mathcal{C}$, and $M\Gamma x = 0$ implies $x = 0$, for all $x \in M$. Then $\overline{\mathcal{C}}$ is a weakly special class of gamma rings.*

If $\mathcal{C}$ is a weakly special class of Γ-rings, the *upper radical* $\mathcal{R}$ determined by $\mathcal{C}$ is

$$\mathcal{R} = \mathcal{U}\mathcal{C} = \{M \mid M \text{ has no nonzero homomorphic image in } \mathcal{C}\}.$$

A radical property $\mathcal{R}$ for Γ-rings is called *supernilpotent* if $\mathcal{R}$ includes all Γ-rings M such that $M\Gamma M = 0$, and if $M \in \mathcal{R}$, then $A \trianglelefteq M$ implies $A \in \mathcal{R}$. As for rings we have:

Proposition 1.3. *A radical property $\mathcal{R}$ for Γ-rings is supernilpotent if and only if $\mathcal{R} = \mathcal{UC}$ for some weakly special class $\mathcal{C}$ of Γ-rings M, $\mathcal{R}(M) = \cap\{Q \trianglelefteq M \mid M/Q \in \mathcal{C}\}$.*

2. Almost nilpotent rings

Almost nilpotent rings were introduced by Van Leeuwen and Heyman [**12**]. A ring R is called *almost nilpotent (anp)* if $(\forall 0 \neq A \trianglelefteq R)(\exists n \in \mathbb{N})(R^n \subset A)$. This definition, which first appeared in [**8**], is slightly stronger than the one given in [**12**]. Nevertheless, all the results concerning anp rings which are discussed in this paper remain valid for this definition. The lower radical property $\mathcal{A}$ determined by the class of all anp rings is supernilpotent, and coincides with the upper radical property determined by the class $\mathcal{C}$ of rings which contain no nonzero anp ideals (see [**12**, Theorems 2 and 4]). $\mathcal{A}$ is called the *almost nilpotent radical property* in the variety of rings.

A Γ-ring M is called *anp* if $(\forall 0 \neq A \trianglelefteq M)(\exists n \in \mathbb{N})(M\Gamma)^n M = M\Gamma M\Gamma \ldots \Gamma M \subset A$. Let $\mathcal{A}_0$ be the lower radical property determined by the class of anp Γ-rings. Since the class of anp Γ-rings contains all Γ-rings M such that $M\Gamma M = 0$, the construction of $\mathcal{A}_0$ terminates at the second step ([**10**, Lemma 2.4]), that is $\mathcal{A}_0$ is the set of all M such that every nonzero homomorphic image of M has a non-zero ideal which is anp.

$\mathcal{A}_0$ is called *almost nilpotent radical property* for Γ-rings. Let $\mathcal{C}_0$ denote the class of all Γ-rings which contain no nonzero anp ideals. Throughout this section, let M denote a Γ-ring with left and right operator rings L and R, respectively.

Proposition 2.1. *Suppose that A is an anp ideal of M. Then $[\Gamma, A]$ is an anp ideal of R.*

Proof: Let $0 \neq B \trianglelefteq [\Gamma, A]$. We claim that $MB \trianglelefteq A$. For if $x \in M$, $b \in B$, then $b = \sum_i [\alpha_i, a_i]$ for some $\alpha_i \in \Gamma$, $a_i \in A$. Hence $xb = \sum_i x\alpha_i a_i \in A$ since $A \trianglelefteq M$. Thus $MB \subseteq A$. Since $B \neq 0$, $MB \neq 0$. Now suppose $x \in M$, $b \in B$, $a \in A$ and $\gamma \in \Gamma$. Then $a\gamma(xb) = (a\gamma x)b \in MB$, and $(xb)\gamma a = x(b[\gamma, a]) \in MB$ since $B \trianglelefteq [\Gamma, A]$. Thus $0 \neq MB \trianglelefteq A$. Since A is an anp ideal of M, $(A\Gamma)^n A \subset MB$ for some $n \in \mathbb{N}$. Hence, $A[\Gamma, A]^n \subset MB$, whence $[\Gamma, A]^{n+1} \subseteq [\Gamma, MB] = RB$. It follows that $[\Gamma, A]^{n+2} \subseteq [\Gamma, A]RB \subseteq [\Gamma, A]B \subseteq B$ since $B \trianglelefteq [\Gamma, A]$.

Since $(A\Gamma)^n A \subset MB$, there exist $x \in M$, $b \in B$ such that $xb \notin (A\Gamma)^n A$. Suppose that $b \in [\Gamma, A]^{n+2}$. Then there exist $a_{ij} \in A$, $\gamma_{ij} \in \Gamma$, $1 \leqslant i \leqslant n+2$, $1 \leqslant j \leqslant m$ such that

$$b = \sum_{j=1}^{m} [\gamma_{1j}, a_{1j}] \ldots [\gamma_{n+2j}, a_{n+2j}].$$

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Hence

$$xb = \sum_{j=1}^{m} (x\gamma_{1j}a_{1j})\gamma_{2j}a_{2j} \dots \gamma_{n+2j}a_{n+2j} \in (A\Gamma)^n A,$$

which contradicts our choice of x and b. Hence $b \notin [\Gamma, A]^{n+2}$, so $[\Gamma, A]^{n+2} \subset B$, as required. ∎

Proposition 2.2. *Suppose that M has the property that $(\forall x \in M)(M\Gamma x = 0 \Longrightarrow x = 0)$. Let A be an anp ideal of R. Then A^* is an anp ideal of M.*

Proof: It is easily verified that $[\Gamma, A^*] \subseteq A$. Since $[\Gamma, A^*] \triangleleft R$, $[\Gamma, A^*] \triangleleft A$. But the class anp rings is hereditary (see [**12**]), the Lemma before Theorem 2). Hence, $[\Gamma, A^*]$ is an anp ideal of R. Suppose $0 \neq B \triangleleft A^*$. If $[\Gamma, B] = 0$, then $M\Gamma B = 0$, whence $B = 0$, contradicting our choice of B. Thus $0 \neq [\Gamma, B] \triangleleft [\Gamma, A^*]$. Hence there exists $n \in \mathbb{N}$ such that $[\Gamma, A^*]^n \subset [\Gamma, B]$. It follows that $(A^*\Gamma)^n A^* \subseteq A^*\Gamma B \subseteq B$ since $B \triangleleft A^*$. Since $[\Gamma, A^*]^n \subset [\Gamma, B]$, there exist $\gamma \in \Gamma$, $b \in B$ such that $[\gamma, b] \notin [\Gamma, A^*]^n$. Then $b \notin (A^*\Gamma)^n A^*$, otherwise $[\gamma, b] \in [\gamma, A^*]^{n+1} \subseteq [\Gamma, A^*]^n$, contradicting our choice of γ and b. Hence $(A^*\Gamma)^n A^* \subset B$, as required. ∎

Theorem 2.3. *The following are equivalent:*

(a) *M contains no nonzero anp ideals;*
(b) *R contains no nonzero anp ideals, and $(\forall x \in M)(M\Gamma x = 0 \Longrightarrow x = 0)$;*
(c) *L contains no nonzero anp ideals, and $(\forall x \in M)(x\Gamma M = 0 \Longrightarrow x = 0)$.*

Proof:

(a) $\Longrightarrow$ (b): Suppose M contains no nonzero anp ideals. Clearly, M is a semiprime Γ-ring. Suppose $x \in M$ and $M\Gamma x = 0$. Then $x\Gamma M\Gamma x = 0$, whence $x = 0$. Suppose that A is an anp ideal of R. Then A^* is an anp ideal of M, by Proposition 2.2. Hence, $A^* = 0$. Now $A \subseteq (A^*)^{*'} = 0^{*'} = \{r \in R : Mr = 0\} = 0$, as required.

(b) $\Longrightarrow$ (a): Suppose that R contains no nonzero anp ideals, and that $(\forall x \in M)(M\Gamma x = 0) \Rightarrow x = 0$. Let A be an anp ideal of M. Then $[\Gamma, A]$ is an anp ideal of R, by Proposition 2.1. Hence $[\Gamma, A] = 0$, whence $M\Gamma A = 0$. It follows that $A = 0$.

(a) $\Longrightarrow$ (c) follows by symmetry. ∎

Let $\mathcal{C}_0$ denote the class of all Γ-rings without anp ideals. As an immediate consequence of Proposition 1.2 and Theorem 3.3, we have

Corollary 2.4. *$\mathcal{C}_0$ is a weakly special class of Γ-rings.*

Proposition 2.5. $\mathcal{A}_0 = \mathcal{U}\mathcal{C}_0$.

The proof is identical to that of the corresponding result for rings ([**12**, Theorem 4]). ∎

From Corollary 2.4 and Proposition 2.5, we have

THEOREM 2.6. *$\mathcal{A}_0$ is a supernilpotent radical property in the variety of Γ-ring.*

It follows that $\mathcal{A}_0(M) = \cap\{Q \trianglelefteq M \mid M/Q \in \mathcal{C}_0\}$.

LEMMA 2.7. *([3, Corollary 2.2]). Let $A \triangleleft M$, and let R' denote the right operator ring of the factor Γ-ring M/A. Then*

$$R' \cong R/A^{*'}.$$

LEMMA 2.8. *The mapping $Q \to Q^*$ defines a one-to-one correspondence between ideals Q of R such that $Q/R \in \mathcal{C}$ and ideals S of M such that $M/S \in \mathcal{C}_0$.*

PROOF: Let $Q \trianglelefteq R$ be such that $R/Q \in \mathcal{C}$. Then Q is a semiprime ideal of R. Hence $Q = (Q^*)^{*'}$ (see [9, Theorem 1]). The right operator ring R' of the factor Γ-ring M/Q^* is isomorphic to $R/(Q^*)^{*'} = R/Q$, by Lemma 2.7. Suppose $x \in M$ is such that $(M/Q^*)\Gamma(x + Q^*) = 0$. It follows that $M\Gamma x \subseteq Q^*$. Since Q is a semiprime ideal of R, Q^* is a semiprime ideal of M ([9, Lemma 1]). Now $x\Gamma M\Gamma x \subseteq Q^*$, whence $x \in Q^*$, that is $x + Q^* = 0$. By Theorem 2.3, M/Q^* contains no nonzero anp ideals, whence $M/Q \in \mathcal{C}_0$.

Conversely, suppose $S \triangleleft M$ is such that $M/S \in \mathcal{C}_0$. Since S is a semiprime ideal of $M, S = \left(S^{*'}\right)^*$. Since the right operator ring of the factor Γ-ring M/S is isomorphic to $R/S^{*'}$, by Lemma 2.7, it follows from Theorem 2.3 that $R/Q^{*'} \in \mathcal{C}$, as required. ∎

THEOREM 2.9. $\mathcal{A}(R)^* = \mathcal{A}_0(M) = \mathcal{A}_0(L)^+$.

PROOF: $\mathcal{A}(R) = \mathcal{A}_0(M)$ follows easily from Lemma 2.8. It follows by symmetry that $\mathcal{A}(L)^+ = \mathcal{A}_0(M)$. ∎

3. MATRIX Γ-RINGS

Let M be a Γ-ring, and let m, n be positive integers. Denote by $M_{m,n}$ and $\Gamma_{n,m}$, the sets of $m \times n$ matrices with entries from R, and $n \times m$ matrices with entries from Γ, respectively. Then $M_{m,n}$ is a $\Gamma_{n,m}$-ring with the matrix addition, and multiplication defined by

where $$(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij}), c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}.$$

LEMMA 3.1. *([9, Lemma 3]). Let m be a Γ-ring, and let M, n be_ integers. Denote by R' the right operator ring of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$. Then $R' \cong R_n$, the ring of $n \times n$ matrices with entries from R. Moreover, if $A \subseteq M$, then $(A_n)^* = (A^*)_{m,n}$.*

LEMMA 3.2. *([8, Theorem 3]). Let R be a ring, and let n be a positive integer. Then*

$$\mathcal{A}(R_n) = (\mathcal{A}(R))_n.$$

Theorem 3.3. *Let M be a Γ-ring, and let m,n be positive integers. Then*

$$\mathcal{A}_0(M_{m,n}) = (\mathcal{A}_0(M))_{m,n}.$$

Proof:

$$\begin{aligned}\mathcal{A}_0(M_{m,n}) &= (\mathcal{A}(R_n))^* && \text{(by Theorem 2.9)}\\ &= ((\mathcal{A}(R))_n)^* && \text{(by Lemma 3.2)}\\ &= (\mathcal{A}(R)^*)_{m,n} && \text{(by Lemma 3.1)}\\ &= (\mathcal{A}_0(M))_{m,n} && \text{(by Theorem 2.9).}\end{aligned}$$

Every ring R is a Γ-ring with $\Gamma = R$ and the usual addition and multiplication operations on R. Let R' denote the right operator ring of R considered as a Γ-ring with $\Gamma = R$. Define $f\colon R^2 \to R'$ by

$$f\left(\sum_i x_i y_i\right) = \sum_i [x_i, y_i].$$

It is easily shown that f is well-defined, and maps the ring R^2 homomorphically onto R'. Moreover, the kernel K of f is equal to $R^2 \cap r(R)$, where $r(R)$ denotes the right annihilator of R. Hence,

$$R' \cong \frac{R^2}{K} = \frac{R^2}{R^2 \cap r(R)}.$$

□

Lemma 3.4. *Let $\mathcal{R}$ be any supernilpotent radical property. Then*

$$\mathcal{R}(R') = \frac{R^2 \cap \mathcal{R}(R)}{R^2 \cap r(R)} = \frac{R^2 \cap \mathcal{R}(R)}{K}.$$

Proof: Note firstly that K is a zero ring. Since $\mathcal{R}$ is a supernilpotent radical property, $K \in \mathcal{R}$ and hence $K \subseteq \mathcal{R}(R^2) = R^2 \cap \mathcal{R}(R)$. Let $\mathcal{R}(R') = A/K$ where $K \subseteq A \trianglelefteq R^2$. Let f be the natural homomorphism of R^2 onto the factor ring R^2/K. Then $f(\mathcal{R}(R^2)) \subseteq \mathcal{R}(R')$, whence $R^2 \cap \mathcal{R}(R) \subseteq A$. But $K \in \mathcal{R}$ and $A/K = \mathcal{R}(R') \in \mathcal{R}$, whence by the extension property, $A \in \mathcal{R}$. It follows that $A \subseteq \mathcal{R}(R^2)$. Hence $A = R^2 \cap \mathcal{R}(R)$, and the result follows. □

Theorem 3.5. *Let R be a ring. Then $\mathcal{A}(R) = \mathcal{A}_0(R)$, where $\mathcal{A}$ and $\mathcal{A}_0$ denote, respectively, the anp radical of the ring R and the anp radical of R considered as a Γ-ring with $\Gamma = R$.*

Proof: Let $R' = R^2/K$ be the right operator ring of R considered as a Γ-ring with $\Gamma = R$, where $K = R^2 \cap r(R)$. Then by Theorem 2.9, $\mathcal{A}_0(R) = \mathcal{A}(R')^*$.

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We will show that $\mathcal{A}(R')^* = \mathcal{A}(R)$. Let $x \in \mathcal{A}(R)$. Then, for all $r \in R$, $rx \in R^2 \cap \mathcal{A}(R)$, whence $rx + K \in R^2 \cap \mathcal{A}(R)/K = \mathcal{A}(R')$, by Lemma 3.4. Hence $x \in \mathcal{A}(R')^*$. Conversely, suppose that $y \in \mathcal{A}(R')^*$. Then $ry + K \in \mathcal{A}(R') = R^2 \cap \mathcal{A}(R)/K$, for all $r \in R$. It follows that $Ry \subseteq \mathcal{A}(R)$, whence $yRy \subseteq \mathcal{A}(R)$. Since $\mathcal{A}(R)$ is a semiprime ideal of R, this implies that $y \in \mathcal{A}(R)$. Thus $\mathcal{A}(R')^* = \mathcal{A}(R)$. □

Let R be a ring, and let m, n be positive integers. Let $R_{m,n}$ denote the set of $m \times n$ matrices with entries from R. Then clearly $R_{m,n}$ is an $R_{n,m}$-ring with the usual operations of matrix addition and multiplication.

THEOREM 3.6. *Let R be a ring, and let m, n be positive integers. Then*

$$\mathcal{A}_0(R_{m,n}) = (\mathcal{A}(R))_{m,n}$$

where $\mathcal{A}_0(R_{m,n})$ denotes the anp radical of the $R_{n,m}$-ring $R_{m,n}$.

The proof is an easy consequence of Theorems 3.3 and 3.5, and will be omitted.

4. Position of the anp radical

The prime, Jacobson and antisimple radicals β, $\mathcal{J}$ and β_ϕ were defined for Γ-rings in [6] and [4], respectively. We will use the same notation for the corresponding radical properties of rings.

PROPOSITION 4.1. *Let $\mathcal{R}$ denote the prime, Jacobson or antisimple radical property for a ring, and let $\mathcal{R}_0$ denote the corresponding radical property for a Γ-ring. Let M be an arbitrary Γ-ring with right operator ring R. Then $\mathcal{R}(R)^* = \mathcal{R}_0(M)$.*

For the proof, we refer to [**6**, Theorems 4.1 and 8.2], and [**4**, Corollary 2.10].

It is known [**12**, Theorem 5] that for a ring R, $\beta(R) \subseteq \mathcal{A}(R) \subseteq \beta_\phi(R)$. As an easy consequence of this result, Theorem 2.9 and Proposition 4.1, we have

PROPOSITION 4.2. *Let M be a Γ-ring. Then $\beta(M) \subseteq \mathcal{A}_0(M) \subseteq \beta_\phi(M)$.*

Analogues of Theorem 3.5 are known for the prime and Jacobson radicals ([**1**, Theorem 4.7] and [**6**, Theorem 10.1] respectively). In [**4**, Theorem 2.4 and Corollary 2.10] respectively, it is shown that β_ϕ is a special radical property in the variety of Γ-rings, and that for a Γ-ring M with right operator ring R, $\beta_\phi(R)^* = \beta_\phi(M)$. Using the techniques of proof employed in Lemma 3.4 and Theorem 3.5, it is easy to prove the following result

LEMMA 4.3. *Let R be a ring and let $\beta_\phi(R)$ and $\beta'_\phi(R)$ denote, respectively, the antisimple radicals of the ring R and of R considered as a Γ-ring with $\Gamma = R$. Then $\beta_\phi(R) = \beta'_\phi(R)$.*

For rings it is known (see [**12**, Theorem 5] and [**11**, Example 2]) that the inclusion $\beta \subset \mathcal{A} \subset \beta_\phi$ is strict and that $\mathcal{A}$ and $\mathcal{J}$ are independent. In view of Theorem 3.5, its analogues for the prime and Jacobson radicals and Lemma 4.3, we have for Γ-rings that the inclusion $\beta \subset \mathcal{A}_0 \subset \beta_\phi$ is strict and that the radical properties $\mathcal{A}_0$ and $\mathcal{J}$ are independent.

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