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ARTICLE

Comparative Learning

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Abstract

This article concerns the diachronic rationality norms for *comparative confidence judgments*, that is, judgments of the form "I am at least as confident in *p* as I am in *q*." Specifically, it identifies, characterizes, and evaluates an intuitively compelling learning rule called *comparative conditionalization* that specifies how agents should revise their comparative confidence judgments in the face of novel evidence.

1 Introduction

We humans are prone to believing things, like when I believe that Olivia is on the sofa. We are also prone to lending credence to things, like when I lend a credence of around 0.1 to there being rain in Windhoek tomorrow. Finally, we are also prone to making comparative confidence judgments, like when I am more confident that Olivia is on the sofa than I am that it will rain in Windhoek tomorrow. Whereas doxastic attitudes of the first two kinds (qualitative belief and numerically graded credence) are widely taken to play a crucial role in framing the fundamental norms by which the rationality of an agent's doxastic states are to be assessed, comparative confidence judgments have attracted much less attention in the contemporary philosophical literature. This is somewhat surprising, given that several eminent figures in the history of inductive inference—for example, Keynes (1921), de Finetti (1937), Koopman (1940), and Fine (1973)—have contended that comparative confidence judgments are the most fundamental, intuitive, and psychologically basic of all our doxastic attitudes.[1]

Over the years, numerous authors have attempted to identify synchronic rationality requirements for comparative confidence orderings (see, e.g., Halpern [2003] for a thorough overview). However, the philosophical foundations of this

[1] Thus, we read, for example,

> The fundamental viewpoint of the present work is that the primal intuition of probability expresses itself in a (partial) ordering of eventualities: A certain individual at a certain moment considers the propositions $a, b, h, k, \ldots$. Then the phrase '*a* on the presumption that *h* is true is equally or less probable than *b* on the presumption that *k* is true' conveys a precise meaning to his intuition . . . . This is, as we see it, a first essential in the thesis of intuitive probability, and contains the ultimate answer to the question of the meaning of the notion of probability. (Koopman 1940, 270)

project have, until recently, been largely neglected,[2] and there is still little consensus regarding what kinds of comparative confidence structures are characteristic of rational agents. Happily, this situation is beginning to improve. Icard (2016) has shown that money-pump-style arguments can be used to provide a prospective pragmatic justification of the requirement that a rational agent's comparative confidence judgments should always be representable by a probability function. Meanwhile, Fitelson and McCarthy (2014) have shown that accuracy dominance arguments can be used to provide epistemic justifications for some significantly weaker synchronic coherence requirements (e.g., the principle that a rational agent's comparative confidence judgments should always be representable by a Dempster–Shafer belief function).

But despite recent progress in identifying the synchronic coherence norms that constrain the comparative confidence judgments of rational agents at a time, relatively little has been written on the question of how rational agents should change their comparative confidence judgments *over time* as they gather new evidence.[3] This is the problem with which I'll be concerned in this article.

Before moving on, it is worth briefly clarifying an important point. My aim in this article is *not* to justify the claim that the doxastic states of ideally or boundedly rational agents should be conceived of in terms of comparative confidence judgments rather than qualitative beliefs or (precise or imprecise) numerical credences. Rather, I assume in the background that there are at least some scenarios in which such a conception is principled and then consider the question of how doxastic states, thus conceived, should evolve over time. After all, it is surely at least possible to conceive of a creature whose doxastic state is characterized purely by comparative confidence judgments, and it is surely philosophically interesting to ask what kinds of doxastic norms would determine the rationality of such a creature's reasoning. As I mentioned earlier, the comparative conceptualization of doxastic states has numerous illustrious champions, and I mainly take it for granted that the reader will agree that a proper understanding of the dynamics of rational comparative confidence is a worthy philosophical goal.

The structure of the article is as follows. In section 2, I introduce the standard formalism for analyzing comparative confidence judgments and provide a concise summary of some of the most important synchronic coherence constraints from the literature. In section 3, I turn to the central question of the article: How should a rational agent revise their comparative confidence judgments over time as they acquire new evidence? I address this question by studying the way in which a Bayesian agent's comparative confidence judgments change when they conditionalize on new evidence. I then show that the resulting revision rule (which I call *comparative*

[2] See Fine (1973) for a critical assessment of the philosophical motivations behind several synchronic rationality requirements from the literature.

[3] There is some extant work that is closely related to the problem of revising confidence orderings, but it tends to focus on either entrenchment orderings in the context of belief revision (see, e.g., Booth and Meyer 2011) or revising general preference orderings (as opposed to comparative confidence orderings in particular) (see, e.g., Freund 2005). There is also some related work on formalizing the notion of conditional comparative confidence (e.g., Koopman 1940; Suppes 1994), but as I argue here, the relationship between diachronic updating and conditional comparative confidence is more complicated than it initially appears.

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conditionalization [CC]) is intuitively compelling, even outside of the context of probabilistic Bayesian epistemology. In section 4, I provide an evidentialist motivation for CC and argue that this motivation is on far sounder footing than an analogous argument that is commonly given for Bayesian conditionalization. In section 5, I illustrate two important senses in which the comparative rule requires less epistemic structure for its application than its numerical counterpart and prove that it preserves some salient synchronic coherence constraints. Section 6 concludes.

2 Coherence conditions for confidence orderings

2.1 Preliminaries

Beginning with some technical preliminaries, I assume that agents always make comparative confidence judgments about "propositions" drawn from the Boolean algebra $\mathfrak{B}$ of equivalence classes of logically equivalent sentences of some language $\mathcal{L}$.[4] Intuitively, an agent A can make two kinds of comparative confidence judgments about propositions in $\mathfrak{B}$. Firstly, they can be strictly more confident in the truth of p than they are in the truth of q. I denote this kind of judgment with the notation $p \succ q$. Alternatively, A can be equally confident in the truth of p and q. I denote this second kind of judgment with the notation $p \sim q$.[5] Together, the set of all A's comparative confidence judgments define a *confidence ordering*, $\succsim$, over some subset of the propositions in $\mathfrak{B}$. I write $p \succsim q$ to indicate the disjunction "$p \succ q$ or $p \sim q$." I turn now to briefly outlining some of the most important basic structural properties that authors typically assume are satisfied by $\succsim$. Firstly, I follow orthodoxy in assuming that $\succ$ always satisfies the following conditions:

Irreflexivity of $\succ$: For every $p \in \mathfrak{B}$, A does not make the judgment $p \succ p$ (i.e., $p \not\succ p$).

Transitivity of $\succ$: For every $p, q, r \in \mathfrak{B}$, if $p \succ q$ and $q \succ r$, then $p \succ r$.

Secondly, I assume that $\sim$ is an equivalence relation—that is:

Reflexivity of $\sim$: For every $p \in \mathfrak{B}$, $p \sim p$.

Transitivity of $\sim$: For every $p, q, r \in \mathfrak{B}$, if $p \sim q$ and $q \sim r$, then $p \sim r$.

Transitivity of $\sim$: For every $p, q \in \mathfrak{B}$, if $p \sim q$, then $q \sim p$.

When all of these assumptions are satisfied, I say that the ordering $\succsim$ is a "partial preorder" over $\mathfrak{B}$. For the remainder of the article, I will assume that the confidence orderings being considered are partial preorders over $\mathfrak{B}$, unless otherwise stated.

The following two coherence norms are also widely accepted:

($A1$) $\top \succ \bot$.

($A2$) For any $p, q \in \mathfrak{B}$, if $p \vdash q$, then $q \succsim p$.

[4] For simplicity, I assume that $\mathfrak{B}$ and L are always finite. The assumption that the relata of comparative confidence judgments are logical equivalence classes rather than simple sentences can be seen as a logical omniscience assumption—that is, that the agent is always aware of all logical equivalences.

[5] To be clear, $p \sim q$ denotes the judgment that p and q are equally plausible. Depending on one's view of doxastic indifference, this may or may not be distinct from simply being doxastically indifferent between p and q. See Eva (2019) and Eva and Stern (2022) for a discussion of comparative conceptions of doxastic indifference.

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$A1$ requires that rational agents always be strictly more confident in the tautology than they are in the contradiction, and $A2$ is a general monotonicity requirement stipulating that agents should never be strictly more confident in p than they are in the logical consequences of p. As well as being intuitively compelling, these rationality constraints have been given a range of pragmatic justifications (see, e.g., Fishburn 1986; Halpern 2003). Following orthodoxy, I will assume both A1 and A2 as constraints on the confidence orderings of rational agents. The final coherence norm that I'll assume here is the following:

($A3$) For every $p, q, r \in \mathfrak{B}$, if $p \wedge r \sim r$, then $(p \wedge q \wedge r) \sim (q \wedge r)$.

A3 simply requires that if you are exactly as confident in $p \wedge r$ as you are in r, then you should also be exactly as confident in $p \wedge q \wedge r$ as you are in $q \wedge r$. Intuitively, making the judgment $(p \wedge r) \sim r$ amounts to ruling out the possibility of r being true without p being true. And once you've ruled out that possibility, it seems wrong to be more confident in $q \wedge r$ than you are in $p \wedge q \wedge r$.[6] A3 is less familiar than A1/A2, but it seems equally compelling and is directly entailed by many of the stronger norms that have been proposed in the literature (including two of the representability norms discussed in sec. 2.2).

I turn now to reviewing two popular synchronic norms that I do not assume in this article.[7] Firstly, many authors assume the following constraint on rational confidence orderings:

Opinionation: For any $p, q \in \mathfrak{B}$, A makes exactly one of the judgments $p \succ q$, $q \succ p$, or $p \sim q$.

In what follows, I call an agent A's confidence ordering $\succsim$ a "total preorder over $\mathfrak{B}$" if and only if it is a partial preorder that satisfies the opinionation assumption.[8] Intuitively, this means that there are "no gaps" in A's confidence judgments; that is, A makes a comparative confidence judgment about every pair of propositions in $\mathfrak{B}$. The opinionation assumption, although controversial, is standard in the extant literature on comparative confidence orderings.[9] I will not generally assume opinionation for the rest of this article, and the learning rule I introduce in section 3 (as well as its evidentialist justification in section 4) is perfectly applicable in non-opinionated settings.

[6] Of course, $A2$ ensures that you can't be any *less* confident in $q \wedge r$ than you are in $p \wedge q \wedge r$.

[7] Note that the literature is replete with possible synchronic coherence constraints for confidence orderings, and it would be impossible to provide an exhaustive survey here (the interested reader should consult, e.g., Halpern [2003] and Wong et al. [1991]). I review only those synchronic constraints that play a crucial role in what follows.

[8] Fitelson and McCarthy (2014) work in a more general setting than that described here. Specifically, they consider an agent's comparative confidence over arbitrary (possibly proper) subsets of $\mathfrak{B}$. which they call "agendas." They then assume only that $\succsim$ is opinionated with respect to the given agenda.

[9] For philosophical critiques of the opinionation assumption, see, for example, Keynes (1921) and Forrest (1989). One might plausibly contend that one of the primary advantages of conceiving of an agent's doxastic states in terms of comparative confidence judgments rather than numerical credences or qualitative beliefs is that it allows us to study the epistemological consequences of failures of opinionation.

In cases where opinionation fails and there exist $p, q \in \mathfrak{B}$ such that $\neg(p \succsim q)$ and $\neg(q \succsim p)$, I will write "$p \odot q$" and say that p and q are "incomparable" in the agent's confidence ordering. I emphasize that this does not constitute an additional category of comparative confidence judgment but rather the absence of any comparative confidence judgment whatsoever. Finally, the following additional constraint is also sometimes assumed (see, e.g., Fitelson and McCarthy 2014):

Regularity of $\succsim$: For any contingent $p \in \mathfrak{B}$, $\top \succ p \succ \bot$.

Regularity requires that A is always strictly more confident in the tautology than they are in any contingent proposition and that they are always strictly less confident in the contradiction than they are in any contingent proposition. This is a generalization of the controversial regularity condition from Bayesian epistemology, which requires that an agent never assign credence 1 to any contingent proposition (see, e.g., Lewis [1980] and Skyrms [1980] for philosophical justifications of the Bayesian regularity condition). Both formulations intuitively capture the idea that no matter how good your evidence is for the truth of a contingent proposition p, it's always, in principle, possible that your evidence is misleading and that p is in fact false. Assigning credence 1 to p (or being equally confident in p and the tautology) seems to unduly neglect this possibility. Now, one could reject the Bayesian formulation of regularity while accepting the comparative formulation. This would amount to accepting that it can sometimes be rational to have credence 1 in a contingent proposition p while still insisting that it is always irrational to be as confident in p as one is in the tautology (see Easwaran [2014] for a discussion of some views in this neighborhood). This kind of view requires one to reject the standard presupposition that having equal credence in two propositions entails being equally confident in those propositions, which in turn suggests that credence cannot be thought of as a straightforward quantitative representation of confidence. Unfortunately, the nuances of this kind of view lie beyond the ambit of this article.

For current purposes, I am interested in studying the diachronic norms that govern the rational evolution of comparative confidence judgments when an agent learns the truth of a piece of contingent evidence *with certainty*. Because this kind of learning experience is explicitly ruled out by the comparative formulation of regularity (if we identify certainty in p with the judgment $p \sim \top$), this means that I am committed to rejecting regularity. Again, a full defense of this rejection lies well beyond the scope of the present inquiry, but it is worth noting that there is a significant philosophical precedent for rejecting regularity and that the theoretical motivations for doing so are numerous and diverse. For instance, many philosophers have been tempted to claim that rational agents can never doubt the contents of their own present phenomenal experience (e.g., Ayer 1946; Chalmers 2003; Descartes 1637), whereas others have argued that we should always think of confidence as a notion that is tied to a specific inquiry, and thus certainty only ever means something like "practical certainty for the purposes of the present inquiry" (see, e.g., Levi 1980).[10] I don't commit to any specific philosophical argument against regularity here but simply note that the implicit rejection of the comparative formulation of regularity fits cleanly into multiple influential epistemological traditions.

[10] Levi (1980) uses the example of an observer watching a coin flip who "rules out" the possibility of the coin suddenly breaking the laws of physics and floating off into space.

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2.2 Representability

Given a comparative confidence ordering $\succsim$ over $\mathfrak{B}$ and a set S of functions $\mu : \mathfrak{B} \to [0,1]$, say that $\succsim$ is "fully represented" by S if and only if for every $p, q \in \mathfrak{B}$:

(i) $p \succsim q \Leftrightarrow (\forall \mu \in S)\big(\mu(p) \gtrsim \mu(q)\big)$, and
(ii) $\mathfrak{D} \odot q \Leftrightarrow (\exists \mu_1, \mu_2 \in S)((\mu_1(p) > \mu_1(q)) \wedge (\mu_2(q) > \mu_2(p)))$.

If $\succsim$ is fully represented by the set $S = \{\mu\}$, say that S is fully represented by the function μ. It is easy to see that $\succsim$ is opinionated (satisfies opinionation) if and only if there exists a function μ such that $\succsim$ is fully represented by μ.

Call a function $\mu : \mathfrak{B} \to [0,1]$ a "plausibility function" if it satisfies the following two conditions:

($PL1$) $\mu(\mathrm{T}) = 1$, and $\mu(\bot) = 0$.
($PL2$) For any $p, q \in \mathfrak{B}$, if $p \vdash q$, then $\mu(p) \leq \mu(q)$.

It is easy to see that if $\succsim$ is a partial preorder over $\mathfrak{B}$, then $\succsim$ will satisfy $A1$ and $A2$ if and only if $\succsim$ is fully representable by a set S of plausibility functions on $\mathfrak{B}$. Thus, an important prospective synchronic coherence requirement for comparative confidence judgments is as follows:

($\mathfrak{C}_1$) $\succsim$ should be fully representable by a set of plausibility functions, or equivalently, $\succsim$ should satisfy $A1$ and $A2$.[11]

By accepting $A1, A2$, and $A3$, I implicitly commit to the normative force of $\mathfrak{C}1$. One might also consider representability by other kinds of numerical functions as prospective synchronic norms for comparative confidence. For instance, consider the following:

($\mathfrak{C}_{DS}$) $\succsim$ should be fully representable by a set of Dempster–Shafer belief functions (see, e.g., Wong et al. 1991).

($\mathfrak{C}_R$) $\succsim$ should be fully representable by a set of ranking functions (see, e.g., Spohn 2012).

($\mathfrak{C}_P$) $\succsim$ should be fully representable by a set of possibility functions (see, e.g., Zadeh 1978).

Finally, the strictly strongest prospective synchronic rationality constraint that I will consider here is as follows:

($\mathfrak{C}2$) $\succsim$ should be fully representable by a set of probability functions.[12]

It is easy to show that a confidence ordering that satisfies $\mathfrak{C}2$ automatically satisfies all the other synchronic constraints considered here. In what follows, I don't

[11] In the presence of the opinionation assumption, $\mathfrak{C}1$ is equivalent to $\succsim$ being representable by a single plausibility function, and similarly for the other representability requirements.
[12] The qualitative statements of $\mathfrak{C}3$. standardly referred to as *cancellation axioms*) are rather technical, so I omit them here (but see, e.g., Harrison-Trainor et al. 2016; Konek 2019; Scott 1964).

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assume any representability norms beyond $\mathfrak{C}1$, which is equivalent to the conjunction of the simple qualitative constraints $A1$ and $A2$.

3 Comparative conditionalization

We are now ready to address the central question of this article: How should a rational agent revise their comparative confidence judgments after learning the truth of some evidential proposition e? Before going further, it is worth spelling out a couple of important background assumptions.

Firstly, I assume here that the evidential proposition e is learned *with certainty*. Thus, the kind of learning I am interested in is the same as that described by standard Bayesian conditionalization, where the agent assigns a posterior probability of 1 to the learned proposition. In the context of comparative confidence judgments, the analogous requirement is that after the learning experience, the agent makes the judgment $e \sim \top$—that is, that they become equally confident in the truth of the learned proposition and the tautology.

Secondly, I assume that upon learning e, the agent needs to reorganize their comparative confidence judgments in a way that (i) ensures that they become certain in the truth of e and (ii) defines a confidence ordering that preserves all the relevant synchronic rationality requirements that were satisfied by their initial ordering. So, for example, if we assume $\mathfrak{C}1$ and $\mathfrak{C}2$ as synchronic rationality requirements and the agent's initial ordering satisfies all these requirements, then their ordering should still satisfy those requirements after they have revised their comparative confidence judgments to accommodate the new evidence. Whatever the synchronic rationality norms are, learning new evidence should not lead one to violate them.

Thirdly, I assume that the agent initially makes the judgment $e \succ \perp$—that is, that the agent is learning something that they didn't previously consider to be certainly false.

It is clear that there are generally many ways that an agent can revise their confidence orderings while satisfying these basic requirements (for any fixed specification of the synchronic norms). How to choose between them? It is instructive here to take inspiration from a key structural property of Bayesian conditionalization. Specifically, given a probability distribution P, let $\succsim_P$ be the confidence ordering defined by $q \succ_P p$ if and only if $P(q) > P(p)$ and $p \sim_P q$ if and only if $P(p) = P(q)$. By definition, P fully represents $\succsim_P$, and we can think of $\succsim_P$ as encoding the comparative confidence judgments of a Bayesian agent whose credal state is given by the probability function P. Now we can ask, What is the relationship between $\succsim_P$ and $\succsim_{P(-|e)}$, where $P(-|e)$ is the probability function obtained by conditionalizing P on e? Less formally: How does conditionalizing on e change the comparative confidence judgments implicit in P? Happily, this question has a simple answer:

$$\begin{aligned} q \succsim_{P(-|e)} p &\Leftrightarrow P(q|e) \geq P(p|e), \\ &\Leftrightarrow P(q|e)P(e) \geq P(p|e)P(e), \\ &\Leftrightarrow P(e \wedge q) \geq P(e \wedge p), \\ &\Leftrightarrow e \wedge q \succsim_P e \wedge p. \end{aligned}$$

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Thus, if we let $\succsim_e$ denote the ordering that results from revising the initial ordering $\succsim$ after learning e, a Bayesian agent will always revise their confidence orderings according to the following rule:

$$q \succsim_e p \Leftrightarrow (e \wedge q) \succsim (e \wedge p), \tag{CC}$$

where, as before, *CC* stands for *comparative conditionalization*.[13] The question now is whether there is anything special about CC as opposed to other revision rules for comparative confidence judgments. One might be tempted here to simply invoke the observation that there are numerous philosophical justifications for viewing Bayesian conditionalization as the uniquely rational rule for updating numerical credences and thus conclude that the revision rule defined by Bayesian conditionalization must therefore be the correct one. However, this kind of justification is clearly flawed because it assumes at the outset that an agent's comparative confidence judgments are defined by a specific credal state and that the way in which an agent revises those judgments will be entirely determined by the rule they use to update that credal state. But as I noted in the introduction, there is a significant minority of authors who contend that comparative confidence judgments are philosophically and psychologically more fundamental than assignments of numerical credence, and these authors will reject the implicit assumption that an agent's comparative confidence judgments are always determined by some specific credal state. It may be that the content of an agent's epistemic state is exhausted by their confidence ordering and that they simply have no well-defined credal state.[14] Again, it's at least coherent to conceive of such an agent. And in this context, rejecting CC in favor of another revision rule does not bring one into conflict with Bayesian conditionalization because the way in which an agent revises their comparative confidence judgments will have *no* implications regarding the way in which they update their credences if they have no well-defined credences in the first place.[15]

If one hopes to justify CC, then one must do so within the context of the epistemology of comparative confidence judgments. My aim in the rest of this article is to explore the possibility of systematically justifying CC within the context of a comparativist epistemology. But before doing so, it is worth emphasizing the intuitive rationality of CC as opposed to alternative revision rules. Toward this end, consider the following example:

[13] Note that CC is also implicitly assumed by alternative quantitative models of inductive learning, including, for example, rank conditionalization (Spohn 1988) and possibility conditionalization (Zadeh 1978). Interestingly, it turns out that CC does *not* cohere with Dempster's rule for updating Dempster–Shafer belief functions. However, it is easy to observe that it does cohere with Fagin and Halpern's (1991) rule for updating belief functions. Thus, the arguments presented here in favor of CC have significant implications for evaluating competing updating rules for Dempster–Shafer belief functions.

[14] Note that this is true even if their confidence ordering is fully representable by a probability function because there will generally be infinitely many different probability functions that can be used to represent the ordering.

[15] It is worth stressing again that my aim in this article is not to defend the position that we should conceive of an agent's epistemic state purely in terms of comparative confidence judgments. Rather, I start from the assumption that there are at least some situations in which such a conception is desirable and then address the question of how agents in situations of this sort should revise their epistemic states over time in light of this assumption.

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> Mufasa is sitting in a soundproof room with no windows, and he has no idea what the weather is like outside. The room is equipped with a speaker that will occasionally announce some partial information about the weather outside. Based on past experience, he judges that it is more likely to be raining and thundering outside than it is to be sunny and thundering outside; that is, he makes the judgments $(r \wedge t) \succ (s \wedge t)$. The speaker then announces that it's thundering outside. Mufasa subsequently revises his comparative confidence judgments in a way that leads him to judge that it is more likely to be sunny outside than it is to be rainy outside.

I take it that there is something intuitively bizarre about the dynamics of Mufasa's confidence judgments here. The question is whether this intuitive strangeness is indicative of diachronic irrationality. In the next section, I turn to providing a formal evidentialist justification of CC that vindicates its apparent normative force.

4 An argument from evidential relevance

The primary argument I will present in support of CC here is evidentialist, in the sense that it relies on doxastic norms that stipulate how an agent's doxastic attitudes should relate to the evidence they have at their disposal, and assumes that agents should always aim to base their judgments on relevant evidence. I begin with the following basic norm, which makes a compelling stipulation about the relationship between an agent's evidence, on the one hand, and their comparative confidence judgments, on the other.

Evidential Norm (EN): An agent A should make the initial judgment $q \succsim p$ if and only if they would retain that judgment upon learning (only) the truth of a proposition e that is entailed by both p and q. Formally, for any $p, q, e \in \mathfrak{B}$ with $p \vdash e$, $q \vdash e$, letting $\succsim^*$ denote the agent's confidence ordering after learning (only) e:

$$q \succsim p \Leftrightarrow q \succsim^* p.$$

To illustrate the intuition behind EN,[16] consider the following example. Let p and q be the propositions "Alice is in Falmouth" and "Alice is in Redruth," respectively, both of which entail the proposition $e =$ "Alice is in Cornwall."[17] Suppose that we are initially at least as confident that Alice is in Falmouth as we are that she is in Redruth. If we subsequently learn only that Alice is in Cornwall, then we have not learned anything about *where* in Cornwall she is. If we were now to change (or simply abandon) our comparative confidence judgment regarding whether she is more likely to be in Falmouth or Redruth, we would, by definition, be changing our judgments in a way that is not warranted by any relevant evidence. So if we assume that one should only change one's judgments when one has relevant evidence that explicitly warrants the change, then violating EN in this way will never be permissible. Generalizing, learning that the actual world $w_@$ is in some region e of the possible world space does not give one any evidence concerning *where* in e $w_@$ is. In particular, it does not say anything about whether $w_@$ is more likely to be in any subregion p of e than it is to be in any other subregion q of e. So any change in how one compares the plausibility of

[16] EN is reminiscent of a belief revision postulate presented by Darwiche and Pearl (1997).

[17] Redruth and Falmouth are both towns in Cornwall.

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different subregions of e upon learning only that e is true would be arbitrary and evidentially unjustified. And such changes are exactly what is prohibited by EN. Importantly, we have the following result:

Proposition 1. *Assuming the synchronic coherence constraints* $A1, A2$*, and* $A3$*, CC is the only updating rule for comparative confidence judgments that satisfies EN in full generality.*

Proposition 1 establishes a clear sense in which agents who want their judgments to be guided by evidence should always abide by CC. If one deviates from CC, then one is committed to sometimes changing one's judgments in the absence of any relevant evidence that actually warrants the change. And this result relies only on the synchronic constraints $A1, A2$, and $A3$, which do *not* entail either opinionation or any form of probabilistic representability. In fact, the justification for CC given by proposition 1 doesn't rely on an agent's confidence ordering being representable by any kind of numerical function other than plausibility functions.

At this stage, it is instructive to compare the evidentialist argument for CC given here with an analogous argument that is often given in favor of Bayesian conditionalization (see, e.g., Lin 2022).[18] Specifically, it is often observed that Bayesian conditionalization is the unique updating rule for probabilistic credences with the properties that (i) $P^*(e) = 1$, and (ii) $\frac{P^*(e \wedge p)}{P^*(e \wedge q)} = \frac{P(e \wedge p)}{P(e \wedge q)}$, $\forall p, q, e \in \mathfrak{B}$ (where P^* denotes the posterior credence function obtained after learning e). By virtue of property (ii), many authors refer to Bayesian conditionalization as the unique rule that "preserves probability ratios." And there is an intuitive sense in which preserving probability ratios captures the requirement that agents should not change their credences in ways that are not licensed by the evidence. To see this, note that failing to preserve probability ratios means (assuming that P^* is probabilistic) that there exist p, q, e such that $\frac{P^*(e \wedge p)}{P^*(e \wedge q)} \neq \frac{P(e \wedge p)}{P(e \wedge q)}$. Again, because learning that the actual world $w_@$ is in e doesn't tell us anything about where in e $w_@$ is likely to lie, changing the ratio $\frac{P(e \wedge p)}{P(e \wedge q)}$ after learning only e seems unwarranted. If the ratio gets bigger, we seem to be favoring $e \wedge p$ in a way that is not warranted by the evidence. If it gets smaller, we are likewise favoring $e \wedge q$ in a way that is not warranted by the evidence. So, like CC, Bayesian conditionalization is also supported by an intuitively compelling argument from evidential relevance.

However, upon closer inspection, it is easy to see that the evidentialist argument for CC is significantly stronger than the analogous argument for Bayesian conditionalization. Note first that the evidential arguments for CC and Bayesian conditionalization both start from the premise that upon learning e, we should not favor any subregions of e when we update our doxastic state because the evidence tells us nothing about where in e the actual world is. Specifically, for any p, q, we should not favor $e \wedge p$ against $e \wedge q$, or vice versa, when we learn e. In the comparative

[18] Indeed, Lin (2022) writes that "the essence of conditionalisation is the preservation of certain probability ratios."

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setting, this requirement has an obvious unique interpretation—namely, that learning e shouldn't change the comparative confidence judgment we initially made about the pair $(e \wedge p, e \wedge q)$. But in the numerical credence setting, it's not at all obvious that there is a single correct interpretation of the requirement that neither $e \wedge p$ nor $e \wedge q$ should be favored against its counterpart. When we interpret this "not favoring" requirement in terms of preserving the ratio $\frac{P(e \wedge p)}{P(e \wedge q)}$, we obtain an argument for conditionalization. But there are other equally natural interpretations of the requirement that do *not* lead to Bayesian conditionalization. To see this, consider the following example, where $\mathfrak{B}$ is the algebra generated by the three worlds w_1, w_2, w_3. Let $P(w_1) = \frac{1}{2}, P(w_2) = \frac{1}{3}$, and $P(w_3) = \frac{1}{6}$. Upon learning $\neg w_1$, we should not favor either w_2 or w_3 because both entail the evidence. Bayesian conditionalizing respects this constraint by preserving the fact that w_2 is viewed as twice as probable as w_3. Specifically, it yields the new posterior function $P^*(w_1|\neg w_1) = 0, P^*(w_2|\neg w_1) = \frac{2}{3}, P^*(w_3|\neg w_1) = \frac{1}{3}$. But there is also a clear sense in which this response to the evidence *does* favor w_2 over w_3 because the increase in w_2's probability $(\frac{1}{3})$ is greater than the increase in w_3's probability $(\frac{1}{6})$. And it seems perfectly reasonable to interpret the requirement that neither w_2 nor w_3 should be favored upon learning $\neg w_1$ as requiring that they should both increase in probability to the same degree. This interpretation identifies the following posterior function as the correct response to the evidence: $P^*(w_1) = 0, P^*(w_2) = \frac{7}{12}, P^*(w_3) = \frac{5}{12}$. Importantly, this response to the evidence yields the same posterior confidence ordering over $\mathfrak{B}$ as Bayesian conditionalization does, and therefore it coheres perfectly with CC and satisfies EN. So the simple evidentialist requirement that upon learning e, one should not favor any subregion of e over any other is enough to justify CC, but it is *not* enough to justify Bayesian conditionalization because in the comparative setting, this requirement has a single obvious interpretation (EN). In the context of numerical credences, it can be plausibly interpreted in multiple ways, some of which single out Bayesian conditionalization as a privileged updating rule and some of which are in direct conflict with Bayesian conditionalization (despite yielding the same posterior confidence orderings as Bayesian conditionalization).

Of course, the preceding analysis does not show that the standard evidentialist arguments for Bayesian conditionalization are fundamentally flawed in any sense. What it shows is that these arguments require significantly stronger premises than the evidentialist justification for CC presented earlier, which relies only on the premise that upon learning only e, one should not favor any subregion of e over any other. To the extent that an argument's strength is inversely proportional to the strength of its premises, this shows that the evidentialist justification of CC is meaningfully stronger than analogous justifications of Bayesian conditionalization. It should also be noted that extant evidentialist justifications for Bayesian conditionalization that are framed in terms of, for example, entropy maximization (Williams 1980; Skyrms 1985) rely crucially on the assumption that a rational agent's credences should always be probabilistic. So if one really wants to derive Bayesian conditionalization from evidentialist norms alone, then one probably needs to begin with an evidentialist justification of probabilism (the thesis that rational credence is always probabilistic). But there exists a significant minority of authors who argue that

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aligning one's credences with the available evidence sometimes *precludes* the possibility of probabilistic credences altogether (see, e.g., Shafer 1976; Spohn 2012). In this context, it is also salient to note that the only substantive synchronic norms presupposed by the evidentialist justification of CC given here are that a rational agent's comparative confidence judgments should always satisfy $A1$, $A2$, and $A3$. This requirement is compatible both with failures of probabilistic representability and failures of opinionation. Thus, the evidentialist justification for CC given here is also noteworthy insofar as it dispenses with many of the controversial synchronic norms that are presupposed by extant evidentialist arguments for Bayesian conditionalization.

5 Some extra details

Before concluding, it will be instructive to briefly highlight a couple of further important properties of CC. Firstly, as noted in section 3, it is important that an updating rule should never lead an agent from a coherent prior confidence ordering to an incoherent posterior ordering; that is, it should preserve the relevant coherence norms. The following results show that CC preserves all those coherence norms that play a salient role in this article:

Proposition 2. *Let* $\succsim$ *satisfy* $\mathfrak{C}1$. *Then* $\succsim_e$ *satisfies* $\mathfrak{C}1$ *if and only if* $e \succ \bot$.

As noted earlier, I always assume that the agent is initially strictly more confident in the learned evidential proposition e than they are in the contradiction (i.e., $e \succ \bot$).[19] Given this assumption, we can show that CC preserves all the relevant synchronic rationality constraints described in section 2. Most importantly, we have the following:

Proposition 3. *If* $\succsim$ *satisfies* $A3$, *then* $\succsim_e$ *satisfies* $A3$.

Proposition 4. *If* $\succsim$ *satisfies* $\mathfrak{C}2$, *then* $\succsim_e$ *satisfies* $\mathfrak{C}2$.

Thus, we know that for several influential conceptions of synchronic coherence, revising by CC will never lead an agent to replace a coherent confidence ordering with an incoherent one.[20]

[19] This assumption is, of course, reminiscent of the fact that a Bayesian agent can never condition on a probability 0 event. Critics of Bayesian epistemology typically take this feature to be problematic and unmotivated. I don't address this issue here, but it is certainly worth noting that this aspect of Bayesian inference generalizes so naturally to the comparative setting (and so can't be straightforwardly attributed to the ratio definition of conditional probabilities, as is often suggested).

[20] It is also worth noting that, perhaps unsurprisingly, CC shares many of the key structural properties of Bayesian conditionalization. For example, CC defines a commutative revision procedure; that is, the order in which the agent receives novel evidence makes no difference to the comparative confidence judgments that they end up with at the end of the learning process. To see this, let $\succsim_{e_1,e_2}$ the result of revising $\succsim$ sequentially by e_1 and then e_2. Then $p \succ_{e_1,e_2} q \Leftrightarrow e_2 \wedge p \succ_{e_1} e_2 \wedge q \Leftrightarrow e_1 \wedge e_2 \wedge p \succ e_1 \wedge e_2 \wedge q \Leftrightarrow p \succ_{e_1 \wedge e_2} q$. The commutativity of CC is, of course, of fundamental importance because it ensures that there is always a well-defined and intuitively rational way to iterate the revision procedure in sequential learning scenarios.

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5.1 Opinionation failures and conditional judgment

I turn now to briefly describing two important points regarding the scope of CC's applicability. Firstly, it is important to note that the definition of CC given earlier does *not* assume that the prior ordering satisfies opinionation, even though it was inspired by standard Bayesian conditioning, an updating rule that does implicitly assume opinionation.[21] To see this, note that CC can be equivalently formulated as follows, where $p \odot_e q$ denotes the case in which the agent makes no judgment regarding the pair (p, q) after learning e:

(CC*) $q \succsim_e p \Leftrightarrow (e \wedge q) \succsim (e \wedge p)$, and $q \odot_e p \Leftrightarrow (e \wedge q) \odot (e \wedge p)$.

The second biconditional (absent from the initial definition) is, of course, implied by the first, and it is irrelevant when opinionation is assumed. At this stage, it is instructive to consider the theory of imprecise credences, where it is often assumed that an agent's credences are represented by a *set* of precise probabilistic credence functions, often referred to as the agent's "representor" (see, e.g., Joyce 2010; Levi 1974, 1985; Weatherson 2007). In (some influential variants of) this view, an agent's comparative confidence ordering can be derived through the following supervaluationist semantics. Firstly, the agent makes the judgment $p \succsim q$ if and only if every function in their representor assigns p a credence that is at least as high as what it assigns to q. Secondly, if there are two functions P_1, P_2 in the agent's representor such that $P_1(p) > P_1(q)$ and $P_2(q) > P_2(p)$, then the agent makes no comparative confidence judgment regarding p and q; that is, their confidence ordering satisfies $p \odot q$. By definition, the ordering identified by this semantics always satisfies $\mathfrak{C}2$. Typically, these imprecise models assume that upon learning a proposition e, a rational agent will replace their prior representor $\mathcal{P}$ by the set $\mathcal{P}(-|e) = \{P(-|e)|P \in \mathcal{P}\}$—that is, that they will simply condition every function in their prior representor on e and take the set of updated functions as their new representor. Now it's easy to see that if $\succsim$ is fully represented by the agent's representor, then the posterior ordering obtained by applying CC* will always be fully represented by the agent's posterior representor. So just as CC coheres perfectly with standard conditionalization, CC* coheres perfectly with its imprecise counterpart. Thus, because CC and CC* are equivalent, CC can be straightforwardly and naturally applied to the nonopinionated setting, and it is in fact directly entailed by the most influential extant attempt to codify the norms of inductive inference in the absence of the opinionation assumption.

The second important point to note regarding the scope of CC's applicability concerns the rule's relation to supposition and conditional judgment. Here, it is significant that the definition of standard Bayesian conditionalization relies on the availability of *conditional degrees* of belief. In order to calculate my new credence in q after conditionalizing on p, I need to know my prior conditional degree of credence in q given p, $P(q|p)$, which is standardly interpreted as representing my credence in q

[21] Opinionation's status as a doxastic norm is contested by many authors (see, e.g., Kaplan 1983; Keynes 1921; Eva 2019).

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under the (indicative) supposition that p is true. In the comparative context, Koopman (1940) forwarded a set of axioms whose satisfaction allowed for the definition of an analogous notion of *comparative conditional confidence*.[22] It is significant that the definition (and justification) of CC does not involve reference to any such notion. The rule can be straightforwardly and intuitively applied without any appeal to representations of conditional or suppositional judgment. This suggests that the close relationship between learning, supposition, and conditional judgment that is familiar from Bayesian epistemology is likely to be fundamentally different in the comparative setting.

6 Conclusion and future work

Let's recap. In section 3, I introduced and characterized CC as a rule for updating one's comparative confidence judgments on the basis of novel evidence. In section 4, I showed that CC follows directly from a fundamental norm regarding the relation between an agent's judgments and their evidence and demonstrated that this evidentialist argument for CC is more general and in some ways stronger than analogous arguments for Bayesian conditionalization. In section 5, I showed that CC preserves some salient synchronic coherence norms and explored the connection between CC, the opinionation assumption, and the notion of comparative conditional confidence.

In closing, I draw the reader's attention to some open questions that I aim to address in sequels to this article. Firstly, one might hope to generalize the diachronic norm CC to deal with a broader range of possible evidence. In its current form, CC applies only to agents who learn the truth of a proposition $e \in \mathfrak{B}$ *with certainty*. It says nothing about how agents should revise their confidence orderings upon acquiring more equivocal evidence. For example, an agent might learn only that p is more likely to be true than q is or that e is evidentially independent of p. In the Bayesian setting, subtle evidential constraints like these can be integrated by means of Jeffrey conditionalization and distance-minimization methods, both of which reduce to Bayesian conditionalization in the special case where a proposition is learned with certainty. Of course, no analogous techniques exist for comparative confidence judgments, and the task of generalizing CC to obtain methods like these is a pressing one that I will return to in a sequel to this article. Secondly, I have explored the possibility of generalizing one of the most influential epistemic arguments for Bayesian conditionalization to the comparative setting. In two further sequels to this article, I explore the possibility of generalizing both pragmatic and epistemic utility theoretic justification for Bayesian conditionalization to obtain analogous purely comparative justifications of CC.

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[22] Importantly, Koopman's axioms encode a substantive array of synchronic norms that are not assumed here.

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Appendix

Proofs

Proof of Proposition 1: Fix $\mathfrak{B}$ and let U be an updating rule for $\mathfrak{B}$, that is, a function that takes a partial preorder $\gtrsim$ over $\mathfrak{B}$ and a proposition $e \in \mathfrak{B}$ and returns a partial preorder $\gtrsim_{U(\gtrsim,e)}$ on $\mathfrak{B}$ such that $e \sim_{U(\gtrsim,e)} \mathrm{T}$ (where it is assumed that $\gtrsim$ and $\gtrsim_{U(\gtrsim,e)}$ satisfy $A1, A2, A3$). I show that U satisfies EN if and only if $\gtrsim_{U(\gtrsim,e)} = \gtrsim_e$ for all $e \in \mathfrak{B}$ (where $\gtrsim_e$ denotes the posterior ordering produced by CC).

First of all, suppose that U does coincide with CC. Then for any $p, q, e \in \mathfrak{B}$ with $p \vdash e$, $q \vdash e$,

$$p \gtrsim_{U(\gtrsim,e)} q \Leftrightarrow (e \wedge p) \gtrsim (e \wedge q) \Leftrightarrow p \gtrsim q,$$

which shows that U satisfies EN. Conversely, let U satisfy EN. Because $\gtrsim_{U(\gtrsim,e)}$ satisfies $A3$, $e \sim_{U(\gtrsim,e)} \mathrm{T}$ implies $p \sim_{U(\gtrsim,e)} (e \wedge p)$ for all $p \in \mathfrak{B}$. And because $e^\wedge p \vdash e$, $e^\wedge q \vdash e$ for all $p, q \in \mathfrak{B}$ and U satisfies EN, we get $p \gtrsim_{U(\gtrsim,e)} q \Leftrightarrow (e \wedge p) \gtrsim_{U(\gtrsim,e)} (e \wedge q) \Leftrightarrow (e \wedge p) \gtrsim (e \wedge q) \Leftrightarrow p \gtrsim_e q$. ∎

Proof of Proposition 2: $\top \succ_e \bot \Leftrightarrow (e \wedge \top) \succ (e \wedge \bot) \Leftrightarrow e \succ \bot$. So $\succsim_e$ satisfies A1 if and only if $e \succ \bot$. To see that $\succsim_e$ satisfies A2 as long as $\succsim$ does, let $p \vdash q$. Then $(e \wedge p) \vdash (e \wedge q)$. So $p \succsim_e q \Leftrightarrow (e \wedge p) \succsim (e \wedge q)$, which is guaranteed by $\succsim$ satisfying A2. ∎

Proof of Proposition 3: Let $\succsim$ satisfy $A3$; let p, q, r be arbitrary; and let $p \wedge r \sim_e r$—that is, $(e \wedge p \wedge r) \sim (e \wedge r)$. Because $\succsim$ satisfies $A3$, it follows that $(e \wedge p \wedge q \wedge r) \sim (e \wedge q \wedge r)$, which entails that $(p \wedge q \wedge r) \sim_e (q \wedge r)$ and hence that $\succsim_e$ satisfies $A3$. ∎

Proof of Proposition 4: By definition, if $\succsim$ is fully representable by a set S of probability functions, then $\succsim_e$ is fully representable by the set $S_e = \{P(-|e)|P \in S\}$, which proves the proposition. ∎

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