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Condensed and Strongly Condensed Domains

Dedicated to Maryam Fassi Fehri on her twenty-ninth birthday

Abdeslam Mimouni

Abstract. This paper deals with the concepts of condensed and strongly condensed domains. By definition, an integral domain R is condensed (resp. strongly condensed) if each pair of ideals I and J of R, $IJ = \{ab/a \in I, b \in J\}$ (resp. $IJ = aJ$ for some $a \in I$ or $IJ = Ib$ for some $b \in J$). More precisely, we investigate the ideal theory of condensed and strongly condensed domains in Noetherian-like settings, especially Mori and strong Mori domains and the transfer of these concepts to pullbacks.

1 Introduction

The concept of a condensed domain was introduced by D. F. Anderson and D. E. Dobbs [4] and further developed in [5]. An integral domain R is condensed if for each pair of ideals I and J of R, $IJ = \{ab/a \in I, b \in J\}$. They showed that a condensed domain R has $\mathrm{Pic}(R) = (0)$ and that a Noetherian condensed domain R has $\dim R \leq 1$. Later, D. F. Anderson, J. T. Arnold and D. E. Dobbs [5] showed that an integrally closed domain is condensed if and only if it is Bézout. Next, C. Gottlieb introduced a class of condensed domains, the strongly condensed domains [20]. An integral domain R is strongly condensed (or SC domain for short) if for each pair of ideals I and J of R, either $IJ = aJ$ for some $a \in I$, or $IJ = Ib$ for some $b \in J$. In 2003, D. D. Anderson and T. Dumetriscu developed the concepts of condensed and strongly condensed domains for various classes of integral domains, namely, Noetherian, integrally closed and local cases [2, 3]. In this paper, we continue the investigation of the condensed and strongly condensed domains. The second section is devoted to the ideal-theoretic of condensed and strongly condensed domains in Noetherian-like settings, especially Mori and strong Mori domains. We first prove that a condensed Mori domain R has $\dim R \leq 1$ and we characterize strongly condensed Mori domains. The third section deals with the transfer of the above concepts to pullbacks in order to provide original examples.

Throughout R is an integral domain, L its quotient field, R' its integral closure and $\bar{R}$ its complete integral closure. For nonzero (fractional) ideals I and J of a domain R, we denote by $(I{:}J) = \{x \in K/xJ \subseteq I\}$ and $I^{-1} = (R{:}I)$. The v-closure of I is defined by $I_v = (I^{-1})^{-1}$, and I is said to be a v-ideal (or divisorial) if $I = I_v$.

A Mori domain is a domain R satisfying the ascending chain condition on v-ideals. Noetherian and Krull domains are Mori. A nonzero ideal I is said to be stable (or

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Sally–Vasconcelos stable) (respectively strongly stable) if I is invertible (respectively principal) in its endomorphisms ring $E(I) = (I:I)$, and a domain R is said to be stable (respectively strongly stable) if each nonzero ideal is stable (respectively strongly stable). Finally, we recall the following useful result [2, Proposition 3.3]: a domain R is an SC-domain if and only if R is strongly stable and $[R, \bar{R}]$, the set of rings between R and $\bar{R}$, is linearly ordered by inclusion.

2 Condensed and Strongly Condensed Mori Domains

Proposition 2.1 *Let R be a domain with the ascending chain condition on principal ideals. If R is condensed, then every maximal ideal is a t-ideal.*

Proof Let M be a maximal ideal of R and suppose that $M_t = R$. Then there exists an fg ideal I of R, such that $I \subseteq M$ and $I^{-1} = R$. Since $\bar{R}$ is t-linked over R, $(\bar{R}:I\bar{R}) = \bar{R}$ (see [16, Corollary 2.3], we recall that an overring T of a domain R is said to be t-linked over R if for each fg ideal I of R such that $I^{-1} = R$, one has $(T:IT) = T$). Hence $I\bar{R}(\bar{R}:I\bar{R}) = I\bar{R}$. On the other hand, since $\bar{R}$ is an integrally closed condensed domain (as an overring of R), then $\bar{R}$ is a Bézout domain. So $I\bar{R}$ is a principal ideal of $\bar{R}$ and therefore $I\bar{R}(\bar{R}:I\bar{R}) = \bar{R}$. Hence $I\bar{R} = \bar{R}$. So $1 = \sum_{i=1}^{i=n} b_i x_i$, for some $b_i \in I$ and $x_i \in \bar{R}$. Now, for each i, there is an ideal A_i of R such that $x_i \in (A_i:A_i)$ (since $\bar{R} = \bigcup(F:F)$, where F ranges over all nonzero (fractional) ideals of R). Set $A = \prod_{i=1}^{i=n} A_i$. Then $x_i \in (A:A)$ for each $i = 1, \dots, n$. So $1 = \sum_{i=1}^{i=n} b_i x_i \in I(A:A)$. Since R is condensed (and I and $(A:A)$ are fractional ideals of R), $1 = ax$ for some $a \in I$ and $x \in (A:A)$. So for each $y \in A$, $y = a(yx) \in aA$. Hence $A \subseteq aA \subseteq A$ and therefore $A = aA$. By induction on n, $A = a^n A$. Hence $A = \bigcap_{n\geq 0} a^n A \subseteq \bigcap_{n\geq 0} a^n R$. But, since $a \in I$, a is non unit of R, so $\bigcap_{n\geq 0} a^n R = (0)$. (Otherwise, if $0 \neq b \in \bigcap_{n\geq 0} a^n R$, then for each n, $b = \alpha_n a^n = \alpha_{n+1} a^{n+1}$ for some α_n and α_{n+1} in R. Then $\alpha_n = \alpha_{n+1} a$. So the sequence $\{\alpha_n R\}_{n\geq 0}$ is an increasing sequence of principal ideals of R. Then it stabilizes since R satisfies the ascending chain condition on principal ideals. So there exists $s \geq 0$ such that $\alpha_s R = \alpha_n R$ for each $n \geq s$. In particular, $\alpha_s R = \alpha_{s+1} R$. Hence $\alpha_{s+1} = c\alpha_s$ for some nonzero $c \in R$. Then $\alpha_s = a\alpha_{s+1} = ca\alpha_s$. So $1 = ca \in I$, a contradiction.) Hence $A = (0)$, which is absurd. It follows that $M = M_t$. ■

We recall that the w-closure of an ideal is defined by $I_w := \bigcup(I : J)$ where the union is taken over all the finitely generated ideals J such that $J^{-1} = R$. An ideal I is said to be a w-ideal if $I = I_w$ and a domain R is said to be a strong Mori domain if R satisfies the ascending chain condition on w-ideals. Noetherian and Krull domains are strong Mori, and strong Mori domains are Mori domains.

Corollary 2.2 *Let R be a strong Mori domain. If R is condensed, then* $\dim R = 1$, *and so R is Noetherian.*

Proof Let M be a maximal ideal of R. By Proposition 2.1, M is t-maximal. So R_M is a Notherian domain [1, Corollary 4.3], [17, Theorem 1.9]. Since R_M is a condensed domain, then $\operatorname{ht} M = \dim R_M = 1$ [4]. Hence $\dim R = 1$ and therefore R is Noetherian [17, Corollary 1.10]. ■

We recall that a domain R is semi-Krull if $R = \bigcap R_P$, where P ranges over the set of height one primes of R, the intersection has a finite character, and every nonzero ideal of R_P contains a power of PR_P, for every height one prime ideal P of R [24, Proposition 4.5].

Corollary 2.3 *Let R be a semi-Krull domain. If R is condensed, then* $\dim R = 1$.

Proof By [10, Theorem 1.10], R satisfies the ascending chain condition on principal ideals. By Proposition 2.1, every maximal ideal is t-maximal, that is, $\operatorname{Max}(R) = \operatorname{Max}_t(R)$. Now, by [10, Proposition 1.2], $\operatorname{Max}_t(R) = X^1(R)$, where $X^1(R)$ is the set of height-one prime ideals of R. Hence $\operatorname{ht} M = 1$ for every maximal ideal M of R and therefore $\dim R = 1$. ■

It is well known that for a Mori domain R and a prime ideal P of R, if $\operatorname{ht} P = 1$, then P is divisorial and if $\operatorname{ht} P \geq 2$, then either P is a strongly divisorial ideal or $P^{-1} = R$, *i.e.*, $P_v = R$ [8, Theorem 3.1]. The following corollary asserts that for a condensed Mori domain, each prime ideal is divisorial.

Corollary 2.4 *Let R be a Mori domain. If R is condensed, then each prime ideal of R is divisorial.*

Proof Let P be a prime ideal of R. Since R_P is a condensed Mori domain, by Proposition 2.1, PR_P is a t-maximal ideal of R_P. Since R_P is a TV-domain (*i.e.*, the t- and v-operations are the same [22]), then PR_P is divisorial. Now, let $x \in P_v = P_t$. Then there is an fg ideal I of R such that $I \subseteq P$ and $x \in I_v$, that is, $xI^{-1} \subseteq R$. Since I is fg, then $(IR_P)^{-1} = I^{-1}R_P$. So $x(IR_P)^{-1} = xI^{-1}R_P \subseteq R_P$. So $x \in (IR_P)_{v_1} = (IR_P)_{t_1} \subseteq (PR_P)_{v_1} = PR_P$, (where t_1 and v_1 are the t- and v-operations with respect to R_P). Hence $x \in R \cap PR_P = P$. It follows that $P_v = P$. ■

Proposition 2.5 *Any strongly stable prime ideal is divisorial. In particular, any prime ideal of an SC domain is divisorial.*

Proof Let P be a prime ideal of R and suppose that $P \subset P_v$. Let $x \in P_v \setminus P$. Since $xP^{-1} \subseteq R$, then $xPP^{-1} \subseteq P$. So $PP^{-1} \subseteq P$ and therefore $PP^{-1} = P$. Hence $P^{-1} = (P:P)$. Since P is strongly stable, then $P = a(P:P)$ for some nonzero $a \in P$. So $P = a(P:P) = aP^{-1}$, and then $P^{-1} = a^{-1}P$. Hence $P_v = (R:P^{-1}) = (R:a^{-1}P) = a(R:P) = aP^{-1} = P$, which is absurd. Hence $P = P_v$. ■

We recall that a domain is divisorial if each ideal is divisorial. W. Heinzer [21] characterized such domains in the context of the integrally closed case; as h-local Prüfer domains, their maximal ideals are finitely generated. Also it is well known that an integrally closed SC domain is a generalized Dedekind domain and such a domain is divisorial. For the convenience of the reader, we include it here as a corollary of Proposition 2.5.

Corollary 2.6 *Any integrally closed SC domain is divisorial.*

Proof By [2, Theorem 3.7], any nonzero ideal I of R is of the form $I = aP$ for some prime ideal P of R. Since P is divisorial (Proposition 2.5), then so is I. Hence R is divisorial. ■

Theorem 2.7 *Let R be a Mori domain satisfying one of the following conditions:*

(i) *the conductor $(R{:}R')$ is nonzero;*

(ii) *R is seminormal.*

If R is condensed, then $\dim R = 1$.

Proof (i) Assume that $A = (R{:}R') \neq 0$. Then $R' \subseteq (A{:}A) \subseteq (A_v{:}A_v) = (AA^{-1})^{-1} = T$. Since R is condensed, then R' is Bézout. So T is Bézout. Since T is a Mori domain, then T is a Dedekind domain. Now, since $(R{:}T) = (AA^{-1})_v$ is nonzero, then T and R have the same complete integral closure, that is, $\bar{R} = \bar{T} = T$ (since T is Dedekind, so completely integrally closed). Hence $\bar{R}$ is a Dedekind domain and $(R{:}\bar{R}) = (AA^{-1})_v$. Hence $\dim \bar{R} = 1$. By [11, Corollary 3.4.1], $\dim R = 1$. (Note that $\bar{R}$ is a condensed domain (as an overring of R). So $\mathrm{Pic}(\bar{R}) = (0)$ and therefore $\bar{R}$ is a *PID*).

(ii) Assume that R is seminormal and suppose that $\dim R \geq 2$. Let $0 \subset P \subset Q$ be a chain of prime ideals of R with $\mathrm{ht}\, Q \geq 2$. Since R_Q is a condensed Mori domain which is also seminormal, without loss of generality, we may assume that R is local with maximal ideal M, $\mathrm{ht}\, M \geq 2$. By Corollary 2.4, M is divisorial. Since $\mathrm{ht}\, M \geq 2$, then $M^{-1} = (M{:}M)$ [8, Theorem 3.1]. Set $T = (R{:}M) = (M{:}M)$ and let $Q = (P{:}M)$. Then Q is a prime ideal of T and $P \subseteq Q \subseteq Q + M$. Since $Q \cap R \subseteq M$ (R is local), then $Q + M \subset T$. Otherwise, if $Q + M = T$, then $1 = a + m$, where $a \in Q$ and $m \in M$. So $a = 1 - m \in Q \cap R \subseteq M$, which is absurd. Hence there is a maximal ideal N of T such that $Q + M \subseteq N$. So $0 \subset Q \subset N$ is a chain of prime ideals of T. Then $\mathrm{ht}_T\, N \geq 2$. By [7, Lemma 2.3], N is not a divisorial ideal of T, which is absurd by Corollary 2.4, since T is a condensed Mori domain. ■

Proposition 2.8 *Let R be a Mori domain with $(R{:}\bar{R}) \neq 0$. Then R is condensed if and only if* $\mathrm{Pic}(R) = 0$*, R_M is condensed for each maximal ideal M of R and $\bar{R}$ is a PID.*

Proof ($\Rightarrow$) By [4], $\mathrm{Pic}(R) = (0)$ and R_M is condensed for every maximal ideal M of R. By the proof of Theorem 2.7, $\bar{R}$ is a PID.

($\Leftarrow$) By [2, Lemma 2.2], it suffices to show that R is h-local. Since $\bar{R}$ is a PID and $(R{:}\bar{R}) \neq (0)$, by [11, Corollary 3.4 (1)], $\dim R = \dim \bar{R} = 1$. So it suffices to show that R has finite character. Let x be a nonzero non-unit of R and $\{M_\alpha\}_{\alpha\in\Omega}$ the set of all maximal ideals that contain x. Since $\mathrm{ht}\, M_\alpha = 1$, there exists a prime ideal N_α of $\bar{R}$ such that $N_\alpha \cap R = M_\alpha$ [11, Proposition 1.1]. Since $\bar{R}$ is a PID, $\{N_\alpha\}_{\alpha\in\Omega}$ is finite and so is $\{M_\alpha\}_{\alpha\in\Omega}$, as desired. ■

The following Theorem is an analogue of [2, Theorem 3.8]. However, we show in Example 2.10 that the last statement of [2, Theorem 3.8] cannot be extended to a Mori domain.

Theorem 2.9 *Let R be a Mori domain. Then R is an SC domain if and only if* (i) *R is is a PID or* (ii) $\dim R = 1$ *and R has a unique non principal maximal ideal M, and R_M is an SC domain.*

Proof ($\Rightarrow$) If R is a PID, there is nothing to prove. Assume that R is not a PID. Let M be a maximal ideal of R. Since R_M is an SC Mori domain, without loss of generality, we may assume that R is local with maximal ideal M. Two cases are then possible.

(i) $(R:R') \neq (0)$. By Theorem 2.7, $\operatorname{ht} M = \dim R = 1$.
(ii) $(R:R') = (0)$. By [25, Corollary 4.17], $\operatorname{ht} M = \dim R = 1$. It follows that $\operatorname{ht} M = 1$ and so $\dim R = 1$.

Now, since R is not a PID, there exists a nonzero ideal I of R which is not principal. Since R is condensed, then $\operatorname{Pic}(R) = (0)$. So I cannot be invertible, that is, $II^{-1} \subset R$. Then there exists a maximal ideal M such that $II^{-1} \subseteq M$. Since II^{-1} is a trace ideal, then so is M, that is, $M = MM^{-1}$. So M is divisorial and $M^{-1} = (M:M)$. Now, if N is a non principal maximal ideal of R, then N cannot be invertible (since $\operatorname{Pic}(R) = (0)$). Then $N = NN^{-1}$. So N is divisorial and $N^{-1} = (N:N)$. Since M^{-1} and N^{-1} are overrings of R between R and $\bar{R}$, by [2, Proposition 3.3], M^{-1} and N^{-1} are comparable. If $M^{-1} \subseteq N^{-1}$, then $N = N_v \subseteq M_v = M$ and by maximality $M = N$. The same holds if $N^{-1} \subseteq M^{-1}$, and therefore R has a unique non principal maximal ideal M. Clearly R_M is an SC domain as a quotient ring of R.

($\Leftarrow$) If R is a PID, then clearly R is an SC domain. Assume that the assertion (ii) holds. By [2, Theorem 3.4], it suffices to show that $\operatorname{Spec}(R)$ is Noetherian, *i.e.*, R satisfies the ascending chain condition on radical ideals. But, let I be a radical ideal of R and let P be a minimal prime ideal of I. Since $\dim R = 1$, then P is divisorial. So $I_v \subseteq P$. Hence $I_v \subseteq \bigcap\{P/P \text{ minimal over } I\} = I$. Hence I is a v-ideal. So every radical ideal of R is divisorial and since R is Mori, then R satisfies the ascending chain condition on divisorial ideals and therefore on radical ideals, as desired. ∎

The condition (c) in [2, Theorem 3.8] is not sufficient to make R an SC domain in the case of Mori domain as is shown by the following example.

Example 2.10 Let k be a field and X and Y indeterminates over k. Set $V = k(X)[[Y]] = k(X) + M$, where $M = YV$ and $R = k + M$. By [18, Theorem 4.18], R is an integrally closed Mori domain which is local and $\dim R = 1$. Since R is local, then $\operatorname{Pic}(R) = (0)$ and $R/R' = (0)$ is serial. However, R is not even condensed (since R is not Bézout, or even Prüfer).

3 Classical "$D + M$" Constructions

We start this section with the following result which is a generalization of [2, Proposition 2.6] and which leads us to construct a family of condensed domains. For any D-submodules U and W of K, we denote by $\mathcal{P}(U, W) = \{ab/a \in U \text{ and } b \in W\}$ and UW the D-submodule of K generated by $\mathcal{P}(U, W)$.

Theorem 3.1 *For the classical "$D + M$" construction, the following conditions are equivalent:*

(i) *R is condensed;*
(ii) *$\mathcal{P}(U, W) = UW$ for each D-submodules U and W of K containing D.*

Proof (i) $\Rightarrow$ (ii) Let U and W be D-submodules of K. Let $0 \neq m \in M$ and set $I_1 = m(U + M)$ and $I_2 = m(W + M)$. Let $z \in UW$ and write $z = \sum_{i=1}^{i=n} x_i y_i$, where $x_i \in U$ and $y_i \in W$ for each $i = 1, \dots, n$. So $m^2 z = \sum_{i=1}^{i=n} (mx_i)(my_i) \in I_1 I_2$. Then there is $x = m(a_1 + m_1) \in I_1$ and $y = m(a_2 + m_2) \in I_2$, where $a_1 \in U$, $a_2 \in W$

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and $m_1, m_2 \in M$ such that $m^2 z = xy = m^2(a_1a_2 + b)$, for some $b \in M$. Hence $z = a_1a_2 \in \mathcal{P}(U, W)$. It follows that $\mathcal{P}(U, W) = UW$.

(ii) $\Rightarrow$ (i) Let I_1 and I_2 be ideals of R and $x \in I_1I_2$.

Case 1: $M \subset I_1$ and $M \subset I_2$. Set $I_1 = J_1 + M$, and $I_2 = J_2 + M$, for some nonzero ideals J_1 and J_2 of D. Then $M \subset I_1I_2$ (since each ideal of R is comparable to M). If $x \notin M$, then $x = a + m$ for some $0 \neq a \in J_1J_2$ and $m \in M$. Since D is condensed, then $a = a_1a_2$ for some $0 \neq a_1 \in J_1$ and $0 \neq a_2 \in J_2$. So $x = a + m = a_1a_2 + m = a_1(a_2 + a_1^{-1}m)$, with $a_1 \in I_1$ and $(a_2 + a_1^{-1}m) \in I_2$, as desired. Assume that $x \in M$ and let $0 \neq a \in J_1$. Then $x = a(a^{-1}x)$ with $a \in I_1$ and $a^{-1}x \in M \subseteq I_2$, as desired.

Case 2: $M \subset I_1$ and $I_2 \subseteq M$. Then set $I_1 = J + M$ for some nonzero ideal J of D. If I_2 is an ideal of V, then let $0 \neq a \in J$. Since $a^{-1} \in K$ and $x \in I_1I_2 \subseteq I_2$, then $xa^{-1} \in I_2$. So $x = a(xa^{-1})$ with $a \in I_1$ and $xa^{-1} \in I_2$, as desired. Assume that I_2 is not an ideal of V. Then $I_2 = c(W + M)$ for some D-submodule W of K with $D \subseteq W \subset K$. Write $x = \sum_{i=1}^{i=n} x_iy_i$, where $x_i = a_i + m_i \in I_1$ and $y_i = c(b_i + m'_i) \in I_2$, with $a_i \in J$, $b_i \in W$, and $m_i, m'_i \in M$ for each $i = 1, \dots, n$. Then $x = c(\sum_{i=1}^{i=n} a_ib_i + m)$ for some $m \in M$. If $\sum_{i=1}^{i=n} a_ib_i = 0$, then $x = cm$, with $m \in M \subseteq I_1$ and $c \in I_2$, as desired. Assume that $\sum_{i=1}^{i=n} a_ib_i \neq 0$. Since $\mathcal{P}(J, W) = JW$, then $\sum_{i=1}^{i=n} a_ib_i = ab$ for some nonzero $a \in J$ and $b \in W$. Hence $x = c(ab + m) = ac(b + a^{-1}m)$, with $a \in I_1$ and $c(b + a^{-1}m) \in I_2$, as desired.

Case 3: $I_1 \subseteq M$ and $I_2 \subseteq M$. Then three subcases are possible.

(i) I_1 and I_2 are ideals of V. Then the result follows from the fact that V is condensed.

(ii) Neither I_1 nor I_2 is an ideal of V. Then $I_1 = c(U + M)$ and $I_2 = d(W + M)$, where U, W are D-submodules of K with $D \subseteq U$ (resp. W)$\subset K$ and $c \in I_1$, $d \in I_2$. Write $x = \sum_{i=1}^{i=n} x_iy_i$, where $x_i = c(a_i + m_i) \in I_1$ and $y_i = d(b_i + m'_i) \in I_2$, with $a_i \in U$, $b_i \in W$, and $m_i, m'_i \in M$ for each $i = 1, \dots, n$. Then $x = cd(\sum_{i=1}^{i=n} a_ib_i + m)$ for some $m \in M$. If $\sum_{i=1}^{i=n} a_ib_i = 0$, then $x = cdm$, with $c \in I_1$ and $dm \in I_2$, as desired. Assume that $\sum_{i=1}^{i=n} a_ib_i \neq 0$. Since $\mathcal{P}(U, W) = UW$, then $\sum_{i=1}^{i=n} a_ib_i = ab$ for some nonzero $a \in U$ and $b \in W$. So $x = cd(ab + m) = (ca)d(b + a^{-1}m)$ with $ca \in I_1$ and $d(b + a^{-1}m) \in I_2$, as desired.

(iii) One of them is an ideal of V while the other is not. Assume that I_1 is an ideal of V and I_2 is not an ideal of V. Then $I_2 = c(W + M)$ for some nonzero $c \in I_2$ and W a D-submodule of K with $D \subseteq W \subset K$. Since $x \in I_1I_2 \subseteq I_2$, then $xc^{-1} \in W + M \subseteq V$. If $xc^{-1} \notin I_1$, then $I_1 \subset xc^{-1}V$. So $cx^{-1}I_1 \subseteq M$. Hence $cx^{-1}I_1I_2 \subseteq I_2M = I_2VM = cM$. Since $x \in I_1I_2$, then $c = cx^{-1}x \in cM$. So $1 \in M$, which is absurd. Hence $xc^{-1} \in I_1$ and therefore $x = (xc^{-1})c$, as desired. It follows that R is condensed. ∎

We recall that a domain R is conducive if for each overring T of R other than L (quotient field of R), the conductor $(R{:}T) = \{x \in L / xT \subseteq R\}$ is nonzero.

Corollary 3.2 *Let D be a conducive domain which is condensed, K its quotient field and V a valuation domain of the form $V = K + M$ (for instance $V = K[[X]]$, or $K[X]_{(X)}$) and $R = D + M$. Then R is condensed.*

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Proof Since D is conducive, each D-submodule W of K (with $W \subset K$) is a fractional ideal of D. Since D is condensed, for all factional ideals I and J of D, $\mathcal{P}(I, J) = IJ$. So for all D-submodules U and W of K (that are fractional ideals of D), $\mathcal{P}(U, W) = UW$. ■

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References

[1] D. D. Anderson, *Star-operations induced by overrings.* Comm. Algebra **16**(1988), no. 12, 2535–2553.
[2] D. D. Anderson and T. Dumetriscu, *Condensed domains.* Canad. Math. Bull. **46**(2003), no. 1, 3–13.
[3] ———, *Condensed local domains which are not strongly condensed.* Math. Rep. **5(55)**(2003), no. 3, 205–209.
[4] D. F. Anderson and D. E. Dobbs, *On the product of ideals.* Canad. Math. Bull, **26**(1983), no. 1, 106–114.
[5] D. F. Anderson, J. T. Arnold and D. E. Dobbs, *Integrally closed condensed domains are Bézout.* Canad. Math. Bull, **28**(1985), no. 1, 98–102.
[6] V. Barucci, *Strongly divisorial ideals and complete integral closure of an integral domain.* J. Algebra **99**(1986), no. 1, 132–142.
[7] ———, *Seminormal Mori domains.* In: Commutative Ring Theory. Lecture Notes in Pure and Appl. Math. 153, Dekker, New York, 1994, pp. 163–170.
[8] ———, *Mori domains.* In: Non-Noetherian Commutative Ring Theory. Math. Appl. 520 Kluwer, Dordrecht, 2000, pp. 57–73.
[9] V. Barucci, D. E. Dobbs, and M. Fontana, *Conducive integral domains as pullbacks.* Manuscripta Math. **54**(1986), no. 3, 261–277.
[10] V. Barucci, S. Gabelli, and M. Roitman, *On semi-Krull domains.* J. Algebra **145**(1992), 306–328.
[11] V. Barucci and E. Houston, *On the prime spectrum of a Mori domain.* Comm. Algebra, **24**(1996), no. 11, 3599–3622.
[12] E. Bastida and R. Gilmer, *Overrings and divisorial ideals of rings of the form $D + M$.* Michigan Math. J. **20**(1973), 79–95.
[13] J. W. Brewer and E. A. Rutter, *$D + M$ constructions with general overrings.* Michigan Math. J. **23**(1976), 33–42.
[14] D. Costa, J. L. Mott, and M. Zafrullah, *The construction $D + XD_s[X]$.* J. Algebra **53**(1978), no. 2, 423–439.
[15] D. E. Dobbs and R. Fedder, *Conducive integral domains.* J. Algebra **86**(1984), no. 2, 494–510.
[16] D. E. Dobbs, E. G. Houston, T. G. Lucas, and M. Zafrullah, *t-linked overrings and Prüfer v-multiplication domains.* Comm. Algebra **17**(1989), no. 11, 2835–2852.
[17] W. Fanggui and R. L. McCasland, *On strong Mori domains.* J. Pure Appl. Algebra **135**(1999), no. 2, 155–165.
[18] S. Gabelli and E. Houston, *Coherentlike conditions in pullbacks.* Michigan Math. J. **44**(1997), no. 1, 99–123.
[19] R. Gilmer, *Multiplicative Ideal Theory.* Pure and Applied Mathematics 12, Marcel Dekker, New York, 1972.
[20] C. Gottlieb, *On condensed Noetherian integral domains whose integral closures are discrete valuation rings.* Canad. Math. Bull. **32**(1989), no. 2, 166–168.
[21] W. Heinzer, *Integral domains in which each non-zero ideal is divisorial.* Mathematika **15**(1968), 164–170.
[22] E. G. Houston and M. Zafrullah, *Integral domains in which each t-ideal is divisorial.* Michigan. Math. J. **35**(1988), no. 2, 291–300.
[23] I. Kaplansky, *Commutative Rings.* The University of Chicago Press, Chicago, 1974.
[24] E. Matlis, *Some properties of commutative ring extensions.* Illinois J. Math. **31**(1987), no. 3, 374–418.
[25] B. Olberding, *On the structure of stable domains.* Comm. Algebra **30**(2002), no. 2, 877–895.

Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
e-mail: amimouni@kfupm.edu.sa