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REGULARITY OF MEAN-VALUES

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Dedicated to Robert Edwards in recognition of
25 years' distinguished contribution to mathematics in Australia,
on the occasion of his retirement

Abstract

Let X be either the d-dimensional sphere or a compact, simply connected, simple, connected Lie group. We define a mean-value operator analogous to the spherical mean-value operator acting on integrable functions on Euclidean space. The value of this operator will be written as $\mathscr{M}f(x,a)$, where $x \in X$ and a varies over a torus A in the group of isometries of X. For each of these cases there is an interval $p_0 < p \leq 2$, where the p_0 depends on the geometry of X, such that if f is in $L^p(X)$ then there is a set of full measure in X and if x lies in this set, the function $a \mapsto \mathscr{M}f(x,a)$ has some Hölder continuity on compact subsets of the regular elements of A.

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1. Introduction

Suppose that X is a compact symmetric space with G its group of isometries, acting transitively on the left. Fix an origin, say x_0, in X and let K be its isotropy subgroup in G. In this way we can identify X with G/K. Let A be a torus in G so that the corresponding Cartan decomposition is $G = KAK$. Equip X, G, and K with their normalized invariant measures. In [3] Leonardo Colzani defined the mean-value operator acting on integrable functions on X. When f

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is such a function and if $x = g \cdot x_0 \in X$ and $a \in A$ then the value of this operator is formally given by

$$\mathscr{M} f(x,a) = \int_K f(gka \cdot x_0)\, dk. \tag{1.1}$$

For almost every $x \in X$, $a \mapsto \mathscr{M} f(x,a)$ is defined almost everywhere on A.

Let $\hat{G}_K$, φ_Λ, and d_Λ have the same meaning as in [3], so that φ_Λ is a zonal spherical function and d_Λ is the dimension of the corresponding irreducible representation of G. An integrable function f on X has a spherical harmonic expansion

$$\sum_{\Lambda \in \hat{G}_K} Y_\Lambda(f : x).$$

For each $\Lambda \in \hat{G}_K$ the Λth component of this expansion is

$$Y_\Lambda(f : x) = d_\Lambda \int_G f(g \cdot x_0) \varphi_\Lambda(g^{-1} g')\, dg \tag{1.2}$$

for all $g' \in G$ such that $x = g' \cdot x_0$. Then

$$\mathscr{M} f(x,a) = \sum_{\Lambda \in \hat{G}_K} Y_\Lambda(f : x) \varphi_\Lambda(a). \tag{1.3}$$

The first part of [3] was concerned with demonstrating that this mean-value operator has regularity properties analogous to the spherical mean-values operator in Euclidean space, as studied by Peyrière and Sjölin in [5]. However, the regularity was given in terms of weighted l^2-spaces of expansions in the zonal spherical functions restricted to A, rather than in the usual Sobolev spaces on the torus A. In the cases which we treat in this paper we will consider regularity in terms of these latter spaces.

1.4 DEFINITION. *For a positive integer l and real number $s \geq 0$, let $W_S(\mathbf{T}^l)$ denote the space of those elements $f \in L^2(\mathbf{T}^l)$ with*

$$\|f\|_{W_S} = \left(\sum_{m \in \mathbf{Z}^l} |\hat{f}(m)|^2 (|m| + 1)^{2s} \right)^{1/2} < \infty.$$

The following imbedding theorem relates these Sobolev spaces with Hölder spaces. See page 19 in [7].

1.5 PROPOSITION. *Fix a positive integer l. If $s > (l/2) + k$ then $W_S(\mathbf{T}^l) \subset C^k(\mathbf{T}^l)$. If $s = (l/2) + \alpha$ and $0 < \alpha < 1$ then $W_S(\mathbf{T}^l) \subset C^\alpha(\mathbf{T}^l)$.*

This work began in conversations with Leonardo Colzani and I am very grateful for his valuable comments during the preparation of this paper. Some of these

results were described in a talk at the November 1985 meeting of the American Mathematical Society in Columbia, Missouri.

2. The unit sphere

Fix $d > 1$ and let $X = S^d$ be the unit sphere in $\mathbf{R}^{d+1}$, equipped with its usual Riemannian structure. Write the elements of X as columns. Then the group $G = SO(d+1)$ acts transitively on the left and its elements are isometries of X. Set $x_0 = (1, 0, \dots, 0)^t$, so that

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k_{22} \end{pmatrix} : k_{22} \in SO(d) \right\}.$$

We can take

$$A = \left\{ a(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & \cdots & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{d-1} & \\ 0 & 0 & & & \end{pmatrix} : 0 \le \theta \le 2\pi \right\}.$$

With this notation, $\mathscr{M} f(x, a(\theta))$ is the mean-value of the function f taken over the subset

$$\{y \in S^d : y^t x = \cos(\theta)\}.$$

When $\theta \notin \pi\mathbf{Z}$, this is a $(d-1)$-dimensional smooth submanifold of S^d.

We can identify $\hat{G}_K$ with $\{0, 1, 2, \dots\}$ and for each nonnegative integer n, the corresponding zonal spherical function is

$$\varphi_n(a(\theta)) = R_n^{(\alpha,\alpha)}(\cos(\theta)). \tag{2.1}$$

Here we are following the notation of [2], so that $R_n^{(\alpha,\alpha)}$ is the normalized Jacobi polynomial of degree n and index (α, α). Furthermore,

$$\alpha = (d-2)/2. \tag{2.2}$$

It is also known that the degrees d_n of the corresponding representations of G have the asymptotic behaviour

$$d_n \sim c_d \cdot (n+1)^{d-1}.$$

The functions $\{R_n^{(\alpha,\alpha)}(\cos(\theta)) : n = 0, 1, 2, \dots\}$ form a complete orthogonal family on $[0, \pi]$ w.r.t. the measure $|\sin(\theta)|^{2\alpha+1}\, d\theta$ and each $R_n^{(\alpha,\alpha)}(\cos(\theta))$ is orthogonal to $\cos(k\theta)$ for $0 \le k < n$.

For the moment, assume that d is an odd integer, so that $2\alpha + 1$ is an even integer. Then $|\sin(\theta)|^{2\alpha+1}$ is an even trigonometric polynomial of degree $d - 1$.

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Combining these facts enables us to write

$$(2.3)\qquad |\sin(\theta)|^{d-1}\varphi_n(a(\theta)) = \sum_{k=n}^{n+d-1} c(n,k)\cos(k\theta)$$

and

$$\sum_{k=n}^{n+d-1} |c(n,k)|^2 \le \int_0^{\pi} |R_n^{(\alpha,\alpha)}(\cos(\theta))|^2(\sin(\theta))^{d-1}\,d\theta.$$

This shows that

$$(2.4)\qquad |c(n,k)| = O((1+n)^{(1-d)/2})$$

for $n \le k \le n+d-1$.

Now consider $f \in L^1(S^d)$ and its mean-value

$$\mathscr{M} f(x, a(\theta)) = \sum_{n=0}^{\infty} Y_n(f:x)\varphi_n(a(\theta)).$$

For almost every x this is defined almost everywhere on $\mathbf{T}$ and is integrable there. The Fourier series of $(\sin(\theta))^{d-1}\mathscr{M} f(x,a(\theta))$ is equal to

$$(2.5)\qquad \sum_{k=0}^{\infty}\left(\sum_{n=\max(k-d+1,0)}^{k} c(n,k)Y_n(f:x)\right)\cos(k\theta).$$

If we are to measure the norm of this in a Sobolev space on $\mathbf{T}$ then we must estimate sums of the form

$$\sum_{k=0}^{\infty}(k+1)^{2s}\left|\sum_{n=\max(k-d+1,0)}^{k} c(n,k)Y_n(f:x)\right|^2.$$

The estimate (2.4) shows that this is less than or equal to

$$(2.6)\qquad c_d\sum_{n=0}^{\infty}(n+1)^{2s+1-d}|Y_n(f:x)|^2.$$

The following proposition was proved in [3] and is based on [1].

2.7 PROPOSITION. *If $1 < p \le 2$ and $f \in L^p(S^d)$ then*

$$\left\{\sum_{n=0}^{\infty}(1+n)^{-2d((1/p)-(1/2))}||Y_n(f)||_2^2\right\}^{1/2} \le c_{d,p}\cdot||f||_p.$$

In particular, for such an f and almost every x in S^d,

$$\sum_{n=0}^{\infty}(1+n)^{-2d(1/p-1/2)}|Y_n(f:x)|^2 < \infty.$$

Going back to the expression (2.6), we see that we can take $s = d(1-1/p)-\frac{1}{2}$, which will be positive provided $p > 2d/(2d-1)$.

2.8 THEOREM. *Suppose d is an odd integer greater than one. If*

$$(2d/(2d-1)) < p \le 2 \quad \textit{and} \quad s = d(1-(1/p)) - (1/2)$$

then for every $f \in L^p(S^d)$,

$$\left\{ \int_{S^d} \| |\sin(\cdot)|^{d-1} \mathscr{M} f(x, a(\cdot)) \|^2_{W_S(\mathbf{T})} \, dx \right\}^{1/2} \le c_{d,p} \cdot \|f\|_p.$$

Proposition 1.5 then tells us that this theorem has the following corollary, which matches Corollary 2 in [5]. First note that $d(1-1/p) - \frac{1}{2} > \frac{1}{2}$ when $p > d/(d-1)$.

2.9 COROLLARY. *Fix an odd integer $d > 1$ and $2 \ge p > d/(d-1)$. To each $f \in L^p(S^d)$ there is a set of full measure in S^d so that for all x in this set and for all $0 < \gamma < d(1-1/p) - 1$ the function $\theta \mapsto \mathscr{M} f(x, a(\theta))$ agrees almost everywhere with a function of class C^γ on each compact subinterval of $(0, \pi)$.*

Next we must consider the case of even dimensional spheres. In fact, we consider general ultraspherical expansions and the Sobolev norms of functions of the form

$$|\sin(\theta)|^{2\alpha+1} \sum_{k=0}^{\infty} a_k R_k^{(\alpha,\alpha)}(\cos(\theta)) \tag{2.10}$$

when there is a weighted-l^2 condition on the sequence $\{a_k\}_{k=0}^{\infty}$ and when $2\alpha+1$ is not an even integer. In (2.30) of [2] we saw that

$$|\sin(\theta)|^{2\alpha+1} R_k^{(\alpha,\alpha)}(\cos(\theta)) = \sum_{n=k}^{\infty} b(\alpha, n, k) \cos(n\theta),$$

where $b(\alpha, n, k) = 0$ for $n-k$ odd and otherwise

$$b(\alpha, n, k) \sim c_\alpha \cdot (n+1) \left(\frac{n+k}{2} + 1 \right)^{-\alpha-3/2} \cdot \left(\frac{n-k}{2} + 1 \right)^{-\alpha-3/2}.$$

Hence, to estimate the Sobolev norm of (2.10), we must examine

$$\sum_{n=0}^{\infty} (n+1)^{2s} \left| \sum_{k=0}^{n} b(\alpha, n, k) \cdot a_k \right|^2$$

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which is less than or equal to

(2.11)

$$c\sum_{n=0}^{\infty}(n+1)^{2s}\left\{\sum_{k=0}^{n}(n+1)^2\left(\frac{n+k}{2}+1\right)^{-2\alpha-3}\cdot|a_k|^2\left(\frac{n-k}{2}+1\right)^{-2\alpha-3/2}\right\}$$
$$\times\left\{\sum_{l=0}^{n}\left(\frac{n-l}{2}+1\right)^{-3/2}\right\}$$
$$\le c\sum_{k=0}^{\infty}|a_k|^2\sum_{n=k}^{\infty}(n+1)^{2s+2}\left(\frac{n+k}{2}+1\right)^{-2\alpha-3}\cdot\left(\frac{n-k}{2}+1\right)^{-2\alpha-3/2}.$$

Now $(n+1)^{2s+2}(n+k+2)^{-2\alpha-3}$ is equal to

$$\left(\frac{n+1}{n+k+2}\right)^{2s+2}\cdot(n+k+2)^{2s-2\alpha-1}$$

which will be $O((k+1)^{2s-2\alpha-1})$ provided $2s\le 2\alpha+1$. This shows that (2.11) is dominated by

$$c\cdot\sum_{k=0}^{\infty}|a_k|^2(k+1)^{2s-2\alpha-1} \tag{2.12}$$

when $2\alpha>-\frac{1}{2}$ and $2s<2\alpha+1$.

2.13 THEOREM. *Suppose that $2\alpha>-\frac{1}{2}$ and that $s=\alpha+\frac{1}{2}$. If a sequence $\{a_k\}$ satisfies $\sum_{k=0}^{\infty}|a_k|^2<\infty$ then the function*

$$|\sin(\theta)|^{2\alpha+1}\sum_{k=0}^{\infty}a_k\cdot R_k^{(\alpha,\alpha)}(\cos(\theta))$$

is in $W_s(\mathbf{T})$. In addition, if $r>0$ and $\sum_{k=0}^{\infty}|a_k|^2\cdot(k+1)^{-2r}<\infty$ then

$$|\sin(\theta)|^{2\alpha+1}\sum_{k=0}^{\infty}a_k\cdot R_k^{(\alpha,\alpha)}(\cos(\theta))$$

is in $W_{s-r}(\mathbf{T})$.

Combining this with Proposition 2.7 proves the following theorem.

2.14 THEOREM. *If d is even, $2d/(2d-1)<p\le 2$, and $f\in L^p(S^d)$ then*

$$\left\{\int_{S^d}\||\sin(\cdot)|^{d-1}\mathscr{M}f(x,a(\cdot))\|^2_{W_s(\mathbf{T})}\,dx\right\}^{1/2}\le c_{d,p}\cdot\|f\|_p$$

with $s=d(1-(1/p))-(1/2)$.

This means that we can remove the hypothesis that d is odd from the statement of Corollary 2.9.

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3. Compact simple Lie groups

Now let U denote a d-dimensional compact, connected, simply connected, simple Lie group, with Lie algebra $\mathfrak{u}$. The rank of U is denoted by r. Equip U with its bi-invariant Riemannian structure, coming from the Killing form on U. Set $G = U \times U$, so that G acts on U by $(g_1, g_2) \cdot u = g_1 u g_2^{-1}$. Take 1, the identity element of U, as the origin. Its isotropy subgroup in G is

$$K = \{(g, g) \in G : g \in U\},$$

and we see that $U = G/K$. Next, let T be a fixed maximal torus in U and put $A = \{(t, t^{-1}) : t \in T\}$. The corresponding Cartan decomposition of G is $G = KAK$. Let Δ denote the roots of $(\mathfrak{u}_C, \mathfrak{t}_C)$ and fix an order on them once and for all, with Δ^+ denoting the positive roots. Furthermore, let $\mathscr{L}$ denote the lattice of dominant integral weights, with $\omega_1, \omega_2, \ldots, \omega_r$ the basis of fundamental weights. Every $\Lambda \in \mathscr{L}$ is of the form $\Lambda = \sum_{j=1}^r \Lambda_j \omega_j$ with each Λ_j a nonnegative integer. In particular, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \sum_{j=1}^r \omega_j$. Set $\mathfrak{W}$ to be the Weyl group for $(\mathfrak{u}_C, \mathfrak{t}_C)$. It is knowns that the zonal spherical functions for the pair (G, K) are parameterized by $\mathscr{L}$ with

$$\varphi_\Lambda(g_1, g_2) = d_\Lambda^{-1} \cdot \chi_\Lambda(g_1 g_2^{-1}). \tag{3.1}$$

Here χ_Λ is the character of the representation of U with highest weight Λ and dimension d_Λ. The mean-value operator in this case takes the following special form. If f is an integrable function on U and if $a(t) = (t, t^{-1}) \in A$ then

$$\mathscr{M} f(x, a(t)) = \sum_{\Lambda \in \mathscr{L}} f * \chi_\Lambda(x) \chi_\Lambda(t^2) \tag{3.2}$$

where $*$ means convolution on U. This is (1.3) in this case and equation (1.2) takes the form

$$\mathscr{M} f(x, a(t)) = \int_U f(x g t^2 g^{-1})\, dg, \tag{3.3}$$

so that it is the average of the values of f taken over the translate by x of the conjugacy class of t^2. When t^2 is regular this submanifold has codimension equal to the rank of U.

For $H \in \mathfrak{t}$ and $\Lambda \in \mathscr{L}$ define the alternating sum to be the following trigonometric polynomial on T,

$$A_\Lambda(\exp(H)) = \sum_{\sigma \in \mathfrak{W}} \det(\sigma) e^{(\sigma\Lambda)(H)}.$$

The Weyl character formula states that for all $t \in T$ and $\Lambda \in \mathscr{L}$,

$$A_\rho(t) \cdot \chi_\Lambda(t) = A_{\Lambda+\rho}(t). \tag{3.4}$$

The Weyl dimension formula states that

$$d_\Lambda = \prod_{\alpha\in\Delta^+} \frac{(\Lambda+\rho|\alpha)}{(\rho|\alpha)}. \tag{3.5}$$

In [4] Ermanno Giacalone proved the following inequality for elements of the Hardy space $H^1(U)$, generalizing Hardy's inequality. For $f \in H^1(U)$

$$\sum_{\Lambda\in\mathscr{L}} d_\Lambda \|f * \chi_\Lambda\|_2 (1+|\Lambda|)^{-(d+r)/2} \le c\|f\|_{H^1(U)}. \tag{3.6}$$

Since $\|\chi_\Lambda\|_2 = 1$ and $\|f*\chi_\Lambda\|_2 \le \|f\|_1$ we can rewrite (3.6) as

$$\left\{\sum_{\Lambda\in\mathscr{L}} d_\Lambda \|f * \chi_\Lambda\|_2^2 (1+|\Lambda|)^{-(d+r)/2}\right\}^{1/2} \le c\|f\|_{H^1(U)}.$$

The Plancherel formula states that for $f \in L^2(U)$,

$$\|f\|_2^2 = \sum_\Lambda d_\Lambda^2 \|f*\chi_\Lambda\|_2^2.$$

Interpolating between $H^1(U)$ and $L^2(U)$ yields the analogue of Proposition (2.7).

3.7 PROPOSITION. *If $1 < p \le 2$ and $f \in L^p(U)$ then*

$$\left\{\sum_{\Lambda\in\mathscr{L}} (1+|\Lambda|)^{-(d+r)((1/p)-(1/2))} d_\Lambda^{3-(2/p)} \|f*\chi_\Lambda\|_2^2\right\}^{1/2} \le c_p\|f\|_p.$$

In particular, for almost every $x \in U$

$$\sum_{\Lambda\in\mathscr{L}} (1+|\Lambda|)^{-(d+r)((1/p)-(1/2))} d_\Lambda^{3-(2/p)} |f*\chi_\Lambda(x)|^2 < \infty.$$

Following the style of argument used in Section 2, we wish to estimate Sobolev norms on T of expressions

$$A_\rho(t^2)\mathscr{M}f(x,a(t)) = \sum_\Lambda f*\chi_\Lambda(x) A_{\Lambda+\rho}(t^2)$$

which means looking at

$$\left\{|\mathfrak{W}| \sum_\Lambda |f*\chi_\Lambda(x)|^2 (1+4|\Lambda+\rho|)^{2s}\right\}^{1/2}. \tag{3.8}$$

The next step is to compare d_Λ with $|\Lambda+\rho|$. Equation (3.5) tells us that d_Λ is a polynomial of degree $(d-r)/2$ in the variables $\Lambda_1+1, \Lambda_2+1, \ldots, \Lambda_r+1$, it is homogeneous, and each monomial has nonnegative coefficients. In addition, the appendix in the paper of Stanton and Tomas [6] states that there is an r-tuple $(k_1, k_2, \ldots, k_r)$ such that for every permutation $s \in \mathfrak{S}_r$,

$$(\Lambda_{s(1)}+1)^{k_1}(\Lambda_{s(2)}+1)^{k_2}\cdots(\Lambda_{s(r)}+1)^{k_r} \tag{3.9}$$

occurs in d_Λ. These r-tuples are listed here, tabulated in terms of the type of Δ.

Type of roots	Rank	Dimension of U	r-tuple
A_r	r	$r(r+2)$	$(1,2,3,\ldots,r)$
B_r	r	$r(2r+1)$	$(1,3,\ldots,2r-1)$
C_r	r	$r(2r+1)$	$(1,3,\ldots,2r-1)$
$D_r (r \geq 6)$	r	$r(2r-1)$	$(1,3,4,5,7,10,\ldots,2r-2)$
D_4	4	28	$(1,2,4,5)$
D_5	5	45	$(1,2,4,6,7)$
G_2	2	14	$(1,5)$
F_4	4	52	$(1,5,7,11)$
E_6	6	78	$(1,4,5,7,8,11)$
E_7	7	133	$(1,5,7,9,11,13,17)$
E_8	8	248	$(1,7,11,13,17,19,23,29)$

Let $k_U = \max\{k_j : 1 \leq j \leq r\}$, which can be read off from the table. Then $d_\Lambda \geq c(1+|\Lambda|)^{k_U}$ for all $\Lambda \in \mathscr{L}$. Feeding this inequality into Proposition 3.7 we see that if $f \in L^p(U)$ then for almost every $x \in U$,

$$\sum_{\Lambda \in \mathscr{L}} (1+|\Lambda|)^{(3-(2/p))k_U-(d+r)((1/p)-(1/2))}|f * \chi_\Lambda(x)|^2 < \infty.$$

The exponent of $(1+|\Lambda|)$ can be rewritten as

$$\left(3k_U + \frac{d+r}{2}\right) - \frac{(2k_U+d+r)}{p}.$$

This will be positive provided

$$p > \frac{(4k_U+2d+2r)}{(6k_U+d+r)}.$$

3.10 THEOREM. *Maintain U and T as above. In addition, suppose*

$$\frac{(4k_U+2d+2r)}{(6k_U+d+r)} < p \leq 2 \quad \textit{and} \quad 0 < s \leq \left(\frac{3}{2}-\frac{1}{p}\right)k_U - \frac{(d+r)}{2}\left(\frac{1}{p}-\frac{1}{2}\right).$$

For every $f \in L^p(U)$,

$$\left\{\int_U ||A_\rho(\bullet^2)\cdot \mathscr{M}f(x,a(\bullet^2))||^2_{W_s(T)}\,dx\right\}^{1/2} \leq c_p||f||_p.$$

Now suppose that $2 \geq p > (4k_U + 2d + 2r)/(6k_U + d - r)$ *and that*

$$0 < \gamma < \left(\frac{3}{2} - \frac{1}{p}\right) k_U - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{r}{2}\left(\frac{1}{p} + \frac{1}{2}\right).$$

For each $f \in L^p(U)$ *there is a set of full measure on* U *so that for every* x *in this set, the function* $t \mapsto \mathscr{M} f(x, a(t^2))$ *is of Hölder class* C^γ *on all compact subsets of* $\{t\colon t^2$ *is regular in* $T\}$.

This is significant in the case of higher rank groups, since it then gives examples of regularity of mean-values formed over submanifolds of high codimension. For example, if $U = SU(r+1)$ (the case of A_l in the table), the interval of p values which lead to some regularity, as described above, is

$$2 - \frac{4}{r+7} < p \leq 2.$$

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