Anallagmatic Curves. I.

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1. The theory of Inversion presents one of the simplest examples of those Birational Transformations of plane figures, whose general theory is due to Cremona.* It has a distinguishing feature to which it owes its name. If the point P "inverts" into Q, then Q inverts into P. It is therefore a simple case of these involutive point transformations much of the general theory of which was developed by the late Admiral de Jonquières in a paper † printed as late as 1864 in the *Nouvelles Annales*, but which had originally been addressed to the Institute of France in 1859. This memoir is not only highly interesting, but is eminently readable and very ingenious.

In such an involutive point transformation if the point P is transformed into Q, then the repetition of the *same* transformation transforms Q into P. Well-known examples are the Hirst transformation in which to a point P corresponds the point in which the line PO joining P to a fixed point O cuts the polar of P with respect to a fixed conic S (Hirst, *Proc. R. S. L.*, 1865), and the Beltrami transformation in which to a point P corresponds the intersection of the polars of P with respect to two fixed conics (Beltrami, *Mem. della Acad. di Bologna*, Tomo. II., 1863).

The number of simple involutive transformations is by no means thereby exhausted, but these are the only cases of involutive point transformations in which the points of a straight line are transformed into a curve of the second degree (Bertini, *Annali di Mat.*, II., 8; Tweedie, *Trans.*, *R.S.E.*, Vol. XL.).

2. The analytical characteristic of such a transformation is that

* Cremona, Mem. sulle transf. geom. delle figure plane, Mem. di Bologna, 1863 and 1865.

† Jonquières; De la transf. géom. des figures planes, Nouv. Annales, 1864, also, Giorn. di Matem., 23, 1885.

if P, Q are the points (x, y), (ξ, η), and if the corresponding operational equations give

$$\phi(x, y) = \xi\,; \quad \psi(x, y) = \eta,$$

where ϕ and ψ are rational functions of their arguments, then must

$$\phi(\xi, \eta) = x, \quad \psi(\xi, \eta) = y.$$

The self-corresponding points, if any, are given by the equations

$$\phi(x, y) = x, \quad \psi(x, y) = y.$$

Thus in the case of inversion we have, for a circle of inversion of radius unity, the equations

$$\frac{x}{x^2+y^2} = \xi, \quad \frac{y}{x^2+y^2} = \eta\,;$$

$$\frac{\xi}{\xi^2+\eta^2} = x, \quad \frac{\eta}{\xi^2+\eta^2} = y\,;$$

while the self-corresponding points are the common solutions to

$$\frac{x}{x^2+y^2} = x, \quad \frac{y}{x^2+y^2} = y,$$

i.e., the points on $x^2 + y^2 - 1 = 0$.

3. In any such transformation the question may be asked: Are there any curves which, though not transforming into themselves point for point, may still be transformed into themselves on the whole? In the case of inversion any circle cutting the circle of inversion orthogonally has this property. Another familiar example of the same is the Right Strophoid (the pedal of a parabola with respect to the pole which is the point of intersection of directrix and axis). In the latter case the centre of inversion is the point of contact of the tangent which is parallel to the real asymptote.

To this class of curves belong all the Bicircular Quartics.

It was Moutard who first suggested that curves (and surfaces) possessing this property of invariability by a particular inversion, should be called Anallagmatic: "Je propose de leur donner le nom *d'anallagmatique*; 'α privatif; 'αλασσω, je change" (*Nouv. Ann.*, 1864). It is surprising how local, even in these days, the development of any special branch of mathematics may be. The researches of men

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like Moutard, Laguerre,* Darboux,† upon these curves have enriched geometrical science with many interesting theorems which form part of the stock-in-trade of any French writer on analytical geometry (Niewenglowski, Picquet, Koenigs), and even enter into their Traités d'Analyse (*e.g.*, Laurent, who dedicates his Traité to "M. Moutard, mon Beaupère").

On the other hand, so far as I know, they have been comparatively neglected in English text-books. On this account I venture to give an account of the most fundamental properties of anallagmatic curves, referring for more detailed information to Koenigs, Leçons de l'Agrégation Classique de Mathématiques, where a very exhaustive treatment is given.

4. A good method of obtaining an anallagmatic is given by the following theorem (Maleyx, *Propriétés de la Strophoide*, *Nouv. Ann.*, 1875).

THEOREM I. *The inverse of any curve possessing a line of symmetry is, in general, anallagmatic. The centre of inversion for the latter is the inverse by the first transformation of the image of the first centre of inversion in the line of symmetry.*

Let L be the line of symmetry, P′ the image of P in L; O the given centre of inversion, O′ its image in L. (Fig. 14.)

The quadrilateral OPP′O′ is clearly cyclic, and therefore the inverse of the circle OPP′O′ for O as pole, is a straight line. Hence the inverses Ω, Π, Π' of O′, P, P′ are in a straight line. The inverse of the straight line PP′ is a circle tangent at O to the parallel line OO′. Hence $\Omega\Pi \, . \, \Omega\Pi' = \Omega O^2$, which is constant. Therefore, etc. Q.E.D.

If, however, O is on L, the inverse remains symmetrical only.

Cor. 1. Conversely any anallagmatic may be inverted into a figure which has a given straight line for line of symmetry.

Cor. 2. The inverse of a central conic is in general anallagmatic in at least *two* different ways, for a central conic possesses two lines

* Laguerre, Mém. sur l'emploi des imaginaires dans la Géométrie de l'espace, *Nou. Ann.*, 1872.

† Darboux, Mém. sur une classe remarquable de Courbes et Surfaces Algébriques et sur la théorie des imaginaires, Mém. de Bordeaux, 1873.

of symmetry; and so is the pedal of a conic, for the pedal is in general the inverse of another conic. (A curve is often anallagmatic in several different ways.)

5. THEOREM II. More generally, *the inverse of any anallagmatic curve is in general anallagmatic.*

Let the curve be anallagmatic with respect to the centre O, and let P and Q be corresponding points. (Fig. 15.)

Invert with respect to Ω. Then the circle ΩPQ inverts into $C_1P_1Q_1$, and O into the point O_1. The quadrilateral OO_1P_1P is cyclic, and so is PQQ_1P_1. Hence $O\widehat{O_1}P_1 = Q\widehat{P}P_1 = \Omega\widehat{Q_1}P_1$. Hence $O_1P_1Q_1\Omega$ is cyclic, and $C_1O_1 \,.\, C_1\Omega = C_1P_1 \,.\, C_1Q_1$. But C_1, O_1, Ω are fixed, and therefore $C_1P_1 \,.\, C_1Q_1$ is constant. Hence the locus of P_1 is anallagmatic with respect to C_1.*

6. The distinction between symmetrical curves and anallagmatic curves is more apparent than real, and the latter may be considered as including the former. To see this, let P, Q be inverse points with respect to a centre O, and let PQ cut the circle of inversion in M, M′. Then M and M′ are harmonic conjugates with respect to P and Q. When O is at infinity on OPQ, the point M is the middle point of PQ and the circle of inversion is the line through it perpendicular to PQ. Since the transformation by inversion is a contact transformation which also leaves angles unaltered, the preceding theorems are really included in the following theorems of Moutard.

7. THEOREM III. *Any anallagmatic is the envelope of a series of circles which are orthogonal to a fixed circle (the director circle), while their centres lie on a curve (the first deferente).*

Let P, Q be two infinitely near points on the curve, P′ and Q′ their correspondents; then the quadrilateral PP′Q′Q is cyclic since $OP \,.\, OP' = OQ \,.\, OQ' = k^2$.

Hence the circle through these points touches the anallagmatic at P and P′; and it cuts the circle of inversion orthogonally, for $OP \,.\, OP' = k^2$.

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* This is not the proof given by Maleyx.

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8. **Theorem IV.** Conversely: *If a variable circle be subjected to move orthogonal to a fixed circle, while its centre traces out a given curve, its envelope is anallagmatic.* (Fig. 16.)

Let the fixed circle be

$$x^2+y^2=k^2 \qquad - \quad - \quad - \quad - \quad (1)$$

and the locus of the centre the curve

$$y=f(x) \qquad - \quad - \quad - \quad - \quad (2)$$

Let C be any point $(\alpha, f(\alpha))$ on (2). The circle of the system whose centre is C is the circle

$$x^2-2\alpha x+y^2-2f(\alpha)y+k^2=0 \qquad - \quad - \quad (3)$$

The envelope is found by eliminating α between (3) and

$$x+f'(\alpha)y=0 \qquad - \quad - \quad - \quad (4)$$

The equations (3) and (4) show that there correspond to a given value of α two points P and P′, and (4) shows that they are in a line with the origin O. Hence $OP \cdot OP'=k^2$, and the locus of P (or P′) is anallagmatic. Equation (4) also shows that the tangent to the deferente at C is orthogonal to OPP′, for the latter is inclined at the angle $\tan^{-1}\left(-\dfrac{1}{f'(\alpha)}\right)$ to the x-axis. The tangent at C to the deferente therefore bisects PP′ in the point M.

9. Hence is deduced a means of determining the equation to an anallagmatic. Let M′ be the inverse of M, then the locus of M′ is the polar of the first deferente, and is called the second deferente. Let (ξ, η) be the coordinates of M′. It is not difficult to show that the coordinates of P and P′ are given by the equations

$$\xi=2k^2x/(x^2+y^2+k^2)$$
$$\eta=2k^2y/(x^2+y^2+k^2).$$

Hence, if $f(\xi, \eta)=0$ be any curve, the anallagmatic which has this curve as second deferente is

$$f\left(\frac{2k^2x}{x^2+y^2+k^2},\quad \frac{2k^2y}{x^2+y^2+k^2}\right)=0.$$

In this way the anallagmatic curve is associated with its second deferente, and through the latter with its first deferente.

10. There are other ways of associating with a given curve another which shall be anallagmatic, some of which I wish to indicate.

Let P and P_1 be the inverses of each other with respect to the origin. Let the perpendicular bisector of PP_1 meet the axes in $(\xi, 0)$, $(0, \eta)$. (Fig. 17.)

The circles orthogonal to the circle of inversion whose centres are the points $(\xi, 0)$, $(0, \eta)$ and which pass through P and P_1 are given by

$$x^2 + y^2 - 2x\xi + k^2 = 0 \quad - \quad - \quad - \quad (1)$$

$$x^2 + y^2 - 2y\eta + k^2 = 0 \quad - \quad - \quad - \quad (2).$$

The equations (1) and (2) give

$$x\xi = y\eta \quad - \quad - \quad - \quad - \quad (3).$$

Hence the coordinates of P and P_1 may be denoted by

$$x = \psi(\xi, \eta) \quad + \sqrt{\phi(\xi, \eta)} \quad - \quad - \quad (4)$$

$$y = \frac{\xi}{\eta}\{\psi(\xi, \eta) + \sqrt{\phi(\xi, \eta)}\} \quad - \quad - \quad (5)$$

$$x_1 = \psi(\xi, \eta) \quad - \sqrt{\phi(\xi, \eta)} \quad - \quad - \quad (6)$$

$$y_1 = \frac{\xi}{\eta}\{\psi(\xi, \eta) - \sqrt{\phi(\xi, \eta)}\} \quad - \quad - \quad (7).$$

If P trace out some curve $f(x, y) = 0$, its equation in terms of ξ and η is of the form

$$A(\xi, \eta) + B(\xi, \eta)\sqrt{\phi(\xi, \eta)} = 0,$$

and the locus of P_1 is given by

$$A(\xi, \eta) - B(\xi, \eta)\sqrt{\phi(\xi, \eta)} = 0.$$

Hence if the curve $f(x, y) = 0$ is anallagmatic with respect to the origin, then must $B(\xi, \eta) = 0$, and conversely.

We have here a fresh means of obtaining anallagmatic curves. If a curve is anallagmatic, then the corresponding equation in ξ and η is rational, and when it is not rational the corresponding rationalised equation in ξ and η clearly corresponds to the result obtained by taking the equation

$$f(x, y) \times f(x_1, y_1) = 0.$$

From this point of view curves that are not anallagmatic are in a sense degenerate, as is obvious from geometrical considerations.

More generally, any symmetric function of $f(x, y)$ and $f(x_1, y_1)$ when equated to zero will give rise to a curve which is anallagmatic.

Denote the equivalent of $f(x_1, y_1)$ as a function of x and y by f_1.

The equation $$\Sigma A(f^m f_1^n + f^n f_1^m) = 0$$

can be expressed rationally in terms of ξ and η, and therefore corresponds to an anallagmatic curve. It will not, however, in general be a degenerate curve, when the parent curve is so.

11. In the preceding paragraph the case $\phi(\xi, \eta) = 0$ was neglected. It corresponds to the discriminant of

$$x^2\left(1 + \frac{\xi^2}{\eta^2}\right) - 2x\xi + k^2 = 0$$

and therefore corresponds to

$$\xi^2\eta^2 - k^2(\xi^2 + \eta^2) = 0,$$

a curve in (ξ, η) which is doubly symmetrical.

The corresponding anallagmatic is given by

$$(x^2 + y^2 - k^2)^2 = 0,$$

i.e., is the circle of inversion.

It may also be noted that no finite point (ξ, η) can coincide with the corresponding point (x, y).