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ON A THEOREM OF NIELSEN

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The following theorem proved in this paper is a generalization of a result of Jakob Nielsen. Suppose G is a group of linear fractional transformations acting on the unit disc D in the complex plane; suppose also that each element of G, except the identity, is either a hyperbolic or a parabolic transformation. Then any homeomorphism h of the open disc $\mathring{D}$ onto itself which satisfies the functional equation $hg = g'h$, for some automorphism $g \to g'$ of G, has a unique extension to a homeomorphism of D onto itself.

In this paper we wish to give a topological proof of a theorem of Nielsen. Nielsen in [11] considers a finitely generated group H, acting on the disc $D = \{z \in C : |z| \leq 1, C \text{ the set of complex numbers}\}$, each of whose elements, except the identity e, is a hyperbolic substitution and whose fundamental domain K has the property that $\overline{K} \subset \mathring{D}$, $\mathring{D}$ being the open disc $\{z \in C : |z| < 1\}$. It is easy to see then that $\mathring{D}$ is a covering space of the orbit space $\mathring{D}/H$. Nielsen then proved that any lifting h to $\mathring{D}$, of a homeomorphism of $\mathring{D}/H$ onto itself, has a unique extension to the boundary $S = \{z \in C : |z| \leq 1\}$ of D; h has the further property that it induces an automorphism $g \to g'$ of H onto itself such that $hg = g'h$.

Before we can state our result we need the definitions of homeomorphisms of type 1 and type 2 the "topological analogues" of parabolic and hyperbolic substitutions respectively.

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Let X be a compact metric space and g be a homeomorphism of X onto itself. Then g is said to be of type 2 if there exist two distinct points $a(g)$ and $b(g)$ in X, fixed under g, such that for any compact set $C \subset X - \{b(g)\}$, $\lim_{n\to\infty} g^n(C) = a(g)$, and for any compact set $C \subset X - \{a(g)\}$, $\lim_{n\to -\infty} g^n(C) = b(g)$; $a(g)$ is called the attractive point of g and $b(g)$ the repulsive point of g. We say that g is of type 1 if in the above definition $a(g) = b(g)$. That is, g has only one fixed point and it acts as both the attractive and the repulsive point for g. These homeomorphisms have been studied by Kinoshita [8], [9], [10] and Homma and Kinoshita [2], [3], and Kaul [4], [5].

A group G acting on X is said to be of type 1 (type 2) if each element g of G and $g \neq e$ is of type 1 (respectively type 2). We say that G is a general group if each $g \in G - \{e\}$ is either of type 1 or of type 2.

Let G be a general group acting on X and let $L = \{a(g) : g \in G-\{e\}\}$ and $O = X - \overline{L}$. A homeomorphism h of O onto itself is said to be admissible if it induces an automorphism $g \to g'$ of G onto itself such that $hg = g'h$ on O. For any $a \in L$, let $G_a = \{g \in G : g(a) = a\}$ denote the stabilizer of a. For the definition of minimal set see [1]. We shall prove the following theorems.

THEOREM 1. *Let G be a general group acting on the disc D. Let L be infinite and for any $a \in L$, $G_a \neq G$. If h is an admissible homeomorphism of O onto itself, then h can be extended to a homeomorphism of D onto itself.*

THEOREM 2. *Let G be a general group acting on the disc D. Let L have at most two points. If h is an admissible homeomorphism of O onto itself then h can be extended to a homeomorphism of D onto itself.*

REMARK 1. The problem of generalizing the result of Nielsen mentioned in the opening paragraph above was first proposed by Kinoshita in an unpublished paper [9]. In that paper Kinoshita also announced a theorem similar to Theorem 1; for example:

THEOREM (Kinoshita). *Suppose G is a group of type 2 acting on the*

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disc D . *If* G *satisfies the*

(1) *"continuity" condition,*

(2) *"commutativity" condition, and*

(3) *Sperners condition on* O *and* L *is infinite,*

then any homeomorphism of the orbit space O/G *onto itself has a lifting to a homeomorphism* h *of* O *onto itself which is admissible and* h *has a unique extension to all of* D .

REMARK 2. In Theorem 1 above the three conditions of Kinoshita's theorem have been replaced by the condition that for any $a \in L$, $G_a \neq G$. Furthermore, the group admits elements of both types 1 and 2. In Theorems 1 and 2, in contrast to Nielsen's result, no conditions are imposed on the nature of the fundamental domain or the number of generators of G .

1.

In this section we shall prove some properties of a general group G acting on a compact metric space X that are needed later. Lemma 1.2 *(iii)* is new.

LEMMA 1.1. *Let* g *be a homeomorphism of type* 1 *or* 2 *acting on a compact metric space* X . *Let* f *be any homeomorphism of* X *onto itself. Then* fgf^{-1} *is a homeomorphism of type* 1 *or* 2 *respectively, and* $a\left(fgf^{-1}\right) = f\left(a(g)\right)$ *and* $b\left(fgf^{-1}\right) = f\left(b(g)\right)$.

Proof. Clearly $f\left(a(g)\right)$ and $f\left(b(g)\right)$ are fixed points of fgf^{-1} . Suppose $C \subset X - \left\{f\left(b(g)\right)\right\}$ is compact, then

$$\lim_{n\to\infty} \left(fgf^{-1}\right)^n(C) = \lim_{n\to\infty} \left(fg^nf^{-1}\right)(C) = f \lim_{n\to\infty} g^n\left(f^{-1}(C)\right) = f\left(a(g)\right) ,$$

since $f^{-1}(C) \subset X - \{b(g)\}$. Similarly, we can prove that for any compact $C \subset X - \left\{f\left(a(g)\right)\right\}$, $\lim_{n\to\infty} \left(fg^nf^{-1}\right)(C) = f\left(b(g)\right)$ and the proof is complete.

LEMMA 1.2. *Let* X *be a compact metric space and* G *be a general group acting on* X . *Then the following hold:*

(i) for any $f \in G$, $f(L) = L$, *hence* $f(\overline{L}) = \overline{L}$ *and*

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$f(0) = 0$;

(ii) *if* L *has more than two points then* $\bar{L}$ *is a perfect set; hence* L *is infinite;*

(iii) *if for each* $a \in L$, $G_a \neq G$ *then* $\bar{L}$ *is a minimal set;*

(iv) *if* $X = D$ *then* $L \subset S$.

Proof. *(i)* If $x \in L$ then $x = a(g)$ for some $g \in G$, and by Lemma 1.1, for any $f \in G$, $f(x) = f(a(g)) = a(fgf^{-1})$. Since G is a group, and $f, g \in G$, $fgf^{-1} \in G$ and $f(x) \in L$. Hence $f(L) \subset L$. Applying the same argument to f^{-1} we get that $f^{-1}(L) \subset L$. Hence $f(L) = L$.

(ii) Let $a \in L$. Since L has more than two points there is an $x \in L$ such that $x \neq a(g)$ or $b(g)$, where $a = a(g)$. Since g is a type 1 or 2, $\lim_{n\to\infty} g^n(x) = a$ and by *(i)* each $g^n(x) \in L$. Hence each point of L is a limit point of L . This proves *(ii)*.

(iii) It is enough to show that for any $x \in L$, $L \subset \{g(x) : g \in G\}^- = \overline{G(x)}$. So let $y \in L$ and $f \in G$ be such that $a(f) = y$, and suppose $x \neq y$.

Suppose f is of type 1. Then by definition $\lim_{n\to\infty} f^n(x) = y$ and $y \in \overline{G(x)}$.

Suppose f is of type 2. Two cases arise.

CASE 1. $f(x) \neq x$. Then by definition $\lim_{n\to\infty} f^n(x) = y$ and the proof is complete.

CASE 2. $f(x) = x$. Since $G_x \neq G$ there is a $g \in G$ such that $g(x) \neq x$. Hence $x \neq a(g)$ or $b(g)$ and $g^n(x)$ converges to $a(g)$. If $a(g) = y$, then $y \in \overline{G(x)}$. If not, then, since x, y are fixed points of f , $a(g)$ is not a fixed point of f . Hence $\lim_{n\to\infty} f^n(a(g)) = y$. Finally since $\{f^n\}$ is equicontinuous at $a(g)$ [2], and $\{g^n(x)\}$ converges to $a(g)$, $\{f^n g^n(a(g))\}$ converges to y [6, Lemma (1.1), p. 226].

(iv) For if $a(g) \in \mathring{D}$ for $g \in G-e$, then $\lim_{n\to\infty} g^n(s) = a(g)$, which is impossible because $g(D) = D$.

2.

A Euclidean neighbourhood of a point $x \in D$ is an open set U in D containing x such that $\overline{U}$ is homeomorphic to the disc. We shall denote the boundary of any set A by ∂A. Any homeomorphic image of the closed unit interval is called an arc.

LEMMA 2.1. *Let g be a homeomorphism of type 1 or 2 on D. Then any Euclidean neighbourhood U of $a(g)$ contains an arc β such that*

$$\alpha = \bigcup_{m=0}^{\infty} g^m[\beta] \subset \mathring{D} \text{ and } \bar{\alpha} = \alpha \cup \{a(g)\} \text{ and is an arc in } D .$$

Proof. By a well known result of Kerekjarto any homeomorphism of type 1 or 2 is respectively topologically equivalent to a parabolic or a hyperbolic transformation [7]. Given a parabolic or hyperbolic transformation k and any point $x \in \mathring{D}$ there is an arc β in $\mathring{D}$ from x to $k(x)$ such that $\alpha = \bigcup_{n=0}^{\infty} k^n(\beta)$ is homeomorphic to the half open interval. Now β being a compact subset of $\mathring{D}$, $\{k^n(\beta)\}$ converges only to $a(k)$. Hence $\bar{\alpha} = \alpha \cup a(k)$ is an arc, and the same is true of g.

Now given any U containing $a(g)$ take a point $x \in U$ such that $g(x) \in U$, and construct an arc β as above.

LEMMA 2.2. *Let G be a general group acting on the disc D. Let L be infinite and $\overline{L}$ be minimal. Then any non-empty open set U in S containing a point $a(g)$ of L contains another point $a(f)$ of L distinct from $a(g)$ such that $a(f') \neq a(g')$, where $g \to g'$ is an automorphism of G.*

Proof. Suppose the lemma is not true. Then there exists a non-empty open set U containing an $a(g) \in L$, such that, if $a(f) \in U \cap L$, then $a(f') = a(g')$. By minimality of $\overline{L}$ there exists a finite set $\{p_i : 1 \le i \le n\}$ in G such that $L \subset \bigcup\{p_i U : 1 \le i \le n\}$ [1, Remark (2.12), p. 14]. That is, for any $f \in G$, $a(f) \in p_i U$ for some i,

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$1 \le i \le n$. Hence $p_i^{-1}a(f) \in U$. But $p_i^{-1}a(f) = a\left[p_i^{-1}fp_i\right]$ (Lemma 1.1), and by the above assumption $a\left[\left[p_i^{-1}fp_i\right]'\right] = a(g')$. Now $\left[p_i^{-1}fp_i\right]' = p_i'^{(-1)}f'p_i'$, since $g \to g'$ is a homomorphism, and by Lemma 1.1, $a\left[p_i'^{(-1)}f'p_i'\right] = p_i'^{(-1)}a(f')$. Hence $a(f') = p_i'a(g')$. Now $g \to g'$ being an automorphism of G , $L = \{a(f') : f \in G\} = \{p_i'a(g') : 1 \le i \le n\}$ contradicts the assumption that L is infinite. Hence the lemma is true.

3.

In this section we shall prove Theorem 1. We therefore assume throughout that G is a general group acting on the disc D , that L is infinite and that for each $a \in L$, $G_a \ne G$ so that $\overline{L}$ is a minimal set (Lemma 1.2 *(iii)*), and lastly that h is an admissible homeomorphism of O onto itself inducing an automorphism of G , $g \to g'$, so that $hg = g'h$. For any $A \subset D$ we define $\tilde{A} = \overline{A} \cap \overline{L}$.

LEMMA 3.1. *Let* $a \in \overline{L}$ *and* U *be any Euclidean neighbourhood of* a . *If for any* $g \in G$, $a(g) \notin \overline{U}$, *then* $a(g') \notin \operatorname{Int}_S(\widetilde{hU'})$, *where* $U' = U \cap O$.

Proof. By Lemma 2.2 the open set $V = D - \overline{U}$ containing $a(g) \in L$ contains an $a(f) \in L$, such that $a(f) \ne a(g)$ and $a(f') \ne a(g')$. By Lemma 1.2 *(iv)*, $\mathring{\mathcal{D}} \subset 0$.

It is easy to see that using Lemma 2.1 we can construct an open arc $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$ in $\overline{V}$ where

$$\alpha_1 = \bigcup_{k=0}^{\infty} g^k[\beta_1] \subset \mathring{D} , \quad \alpha_2 = \bigcup_{k=0}^{\infty} g^k[\beta_2] \subset \mathring{\mathcal{D}} ,$$

and α_3 connects the end points of α_1 and α_2 in $\overline{V} \cap \mathring{\mathcal{D}}$ and is otherwise disjoint with them, so that $\alpha = \alpha \cup \{a(f), a(g)\}$ is an arc in D with $\alpha \subset \mathring{\mathcal{D}}$ and its end points $a(f)$ and $a(g)$ in S . Hence $h\alpha = \beta$ is an open arc in $\mathring{\mathcal{D}}$ and $\overline{\beta} = \overline{h\alpha_1} \cup \overline{h\alpha_2} \cup \overline{h\alpha_3}$. But $\overline{h\alpha_3} = \overline{h\alpha_3}$ since α_3 is an arc in $\mathring{D}$. Now

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$$h\alpha_1 = h \bigcup_{k=0}^{\infty} g^k[\beta_1] = \bigcup_{k=0}^{\infty} hg^k[\beta_1] = \bigcup_{k=0}^{\infty} g'^k h[\beta_1] .$$

Since g' is of type 1 or 2, h maps $\mathring{D}$ onto $\mathring{D}$ and $h\beta_1$ is a compact subset of $\mathring{D}$, $h\alpha_1$ has the unique limit point $a(g')$. Similarly, $h\alpha_2$ has the unique limit point $a(f')$. Thus $\overline{\beta} = \beta \cup \{a(f'), a(g')\}$ is a closed arc in D with end points $a(f')$ and $a(g')$. Since $\alpha \subset \mathring{D} - U_1$, $\beta = h\alpha \subset \mathring{D} - hU_1$ as h maps $\mathring{D}$ onto $\mathring{D}$.

Now $\overline{\beta}$ separates D into two components E_1 and E_2. Since U is a Euclidean neighbourhood, $U' = U \cap O$ is a connected set, and $\alpha \cap U' = \emptyset$, hence hU' lies either in E_1 or E_2. Suppose $hU' \subset E_1$. Then $\widetilde{hU'} = \overline{hU'} \cap \overline{L} \subset \overline{E_1} \cap S$ and $a(g')$ being an end point of $\overline{\beta}$ is not an interior point of $\overline{E_1} \cap S$ with respect to S. Hence $a(g')$ is not an interior point of $\widetilde{hU'}$ with respect to S. This proves the lemma.

For any $a \in \overline{L}$ consider a decreasing nested sequence $\{U_n\}$ of Euclidean neighbourhoods of a so that $\bigcap_{n=1}^{\infty} \overline{U_n} = \{a\}$. Then $U'_n = O \cap U_n$ and $O - \overline{U_n}$ are arcwise connected for $n = 1, 2, \ldots$. Let $A(a) = \bigcap_{n=1}^{\infty} \overline{hU'_n}$. Then $A(a)$ being the intersection of compact connected non-empty subsets of D is non-empty compact and connected.

LEMMA 3.2. *For any* $a \in L$, $A(a) = \bigcap_{n=1}^{\infty} \widetilde{hU'_n}$.

Proof. If $x \in A(a) \cap O$ then for all $n \geq 1$, $x \in \overline{hU'_n}$. Since h is a homeomorphism on O, $h^{-1}x \in \overline{U'_n}$. Hence $h^{-1}x \in \bigcap_{n=1}^{\infty} \overline{U'_n} = \{a\} \in \overline{L}$, which is a contradiction, since $O \cap \overline{L} = \emptyset$. Consequently,

$$A(a) = \bigcap_{n=1}^{\infty} \overline{hU'_n} = \left(\bigcap_{n=1}^{\infty} \overline{hU'_n}\right) \cap \overline{L} = \bigcap_{n=1}^{\infty} \left(\overline{hU'_n} \cap \overline{L}\right) = \bigcap_{n=1}^{\infty} \widetilde{hU'_n}$$

and the proof is complete.

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LEMMA 3.3. *If* $a \in \overline{L}$ *then* $A(a)$ *is a singleton; and if* $a = a(f)$ *for some* $f \in G$ *, then* $A(a) = \{a(f')\}$.

Proof. First we claim that $I = \text{Int}_S A(a) = \emptyset$: suppose not. Then by Lemma 2.2 there exist distinct points $a(f)$ and $a(g)$ in I , such that $a(f') \neq a(g')$. Suppose $a(g') \neq a$. Now $a(g) \in I$ implies that $a(g) \in \text{Int}_S|\overline{hU'_n}|$ for each $n = 1, 2, \ldots$. Hence by Lemma 3.1,

$a(g') \in \overline{U_n}$ for each n , that is, $a(g') \in \bigcap_{n=1}^{\infty} \overline{U}_n = \{a\}$, which is a contradiction.

Thus $A(a)$ is a non-empty compact connected subset of S with an empty interior. Hence $A(a)$ is a singleton.

If $a = a(f)$ for some $f \in G$ then f being of type 1 or 2 for any $x \in O$, $\lim_{n\to\infty} f^n(x) = a$; so that the sequence $\{f^n(x)\}$ lies eventually in each $U'_n = U_n \cap O$. Hence $\{hf^n(x)\} = \{f'^n(h(x))\}$ lies eventually in each hU'_n , $n = 1, 2, \ldots$. Since $\bigcap_{n=1}^{\infty} \overline{hU'_n} = A(a)$ is a singleton, $\lim_{n\to\infty} f'^n(h(x)) = A(a)$. But f' being of type 1 or 2, and since $hx \in O$, $\lim_{n\to\infty} f'^n(h(x)) = a(f')$. Therefore $a(f') = A(a)$. This completes the proof of the lemma.

Proof of Theorem 1. Define $h^* : D \to D$ as follows: for $x \in O$, let $h^*(x) = h(x)$ and for $a \in \overline{L}$ let $h^*(a) = A(a)$. We claim that h^* is the required extension.

It is easy to see that for any two defining sequences $\{U_n\}$ and $\{V_n\}$ of Euclidean neighbourhood of $a \in \overline{L}$, $\bigcap_{n=1}^{\infty} \overline{hU'_n} = \bigcap_{n=1}^{\infty} \overline{hV'_n}$, since they both form a neighbourhood base at a . Hence, $A(a)$ being a singleton, h^* is well defined. Since O is open in D , h^* is clearly continuous at each point of O . To see that h^* is continuous at $a \in \overline{L}$, let $\varepsilon > 0$ be given. Since $A(a) = \bigcap_{n=1}^{\infty} \overline{hU'_n} \in U(h^*(a), \varepsilon)$, where $U(h^*(a), \varepsilon)$ is ε-neighbourhood of $h^*(a)$, there exists an integer m , such that $\overline{hU'_n} \subset U(h^*(a), \varepsilon)$ for all $n \geq m$. Recall that $U'_n = U_n \cap O$. Let $x \in \overline{U}_m \cap \overline{L}$ and $\{w_m\}$ be a defining sequence of Euclidean neighbourhood for x . Then $\bigcap_{n=1}^{\infty} w_n = \{x\}$ implies that for some m' , $w_k \subset U_m$ for all $k \geq m'$. Hence $\overline{hw'_m} \subset \overline{hU'_m} \subset U(h^*(a), \varepsilon)$ for all $k \geq m'$ implying that $h^*(x) \in U(h^*(a), \varepsilon)$. Thus $h^*(U_n) \subset U(h^*(a), \varepsilon)$ for some n for which $\overline{U}_n \subset U_m$ and h^* is continuous at a and hence on D .

Now $g \to g'$ being an automorphism of G such that $h^{-1}g' = gh^{-1}$, h^{-1} is also an admissible homeomorphism of O . Working with h^{-1} similarly we have a continuous extension $(h^{-1})^*$ of h^{-1} to D . Hence $h^*(h^{-1})^*$ and $(h^{-1})^*h^*$ are continuous extensions of the identity mappings of O and therefore identity themselves. Thus h is one-to-one and onto and D being compact h is a homeomorphism. This proves the theorem.

4.

Let $U(x, r)$ denote the r-neighbourhood of a point x .

Proof of Theorem 2. CASE 1. Suppose L is a singleton $\{a\}$. Then there is an $f \in G$ of type 1 such that $a = a(f)$. Assume that there is an $\varepsilon > 0$ such that for every positive integer n , $h[U(a, 1/n)-\{a\}] \not\subset U(a, \varepsilon)$. Then there is a sequence $\{x_n\}$ such that $x_n \in U(a, 1/n)$ and $y_n = hx_n \in D - U(a, \varepsilon)$, $n = 1, 2, \ldots$. Clearly $\{h^{-1}y_n\}$ converges to a . But $D - U(a, \varepsilon)$ being compact, assume without loss of generality that $\{y_n\}$ converges to $y \in D$. But then $y \neq a$ implies that $y \in O$, and h^{-1} being continuous at y we have $h^{-1}y = a$. But this is a contradiction since $hy \in O$. Thus for any $\varepsilon > 0$ there is an n such that $h[U(a, 1/n)-\{a\}] \subset U(a, \varepsilon)$.

Define $h^* : D \to D$ by $h^*(x) = h(x)$ if $x \in O$ and $h^*(a) = a$. By

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the last paragraph then h^* is continuous and a homeomorphism.

CASE 2. L has two points $L = \{a, b\}$. Then a, b lies in S. There is a Euclidean neighbourhood U of a such that $\alpha = \partial U$ is an arc in O with end points c, d in $O \cap S$, c, d separate a, b in S, and $b \in D - \overline{U}$. Then $h\alpha$ is an arc in O with end points hc and hd in S such that hc and hd separate a, b in S. Therefore only one of the points a and b lies in $\overline{hU}$, say $a \in \overline{hU}$. Let $h^* = h$ on O and $h^*(a) = a$ and $h^*(b) = b$. Then a similar argument as in Case 1 shows that h^* is the required extension. If $b \in \overline{hU}$, define $h^*(a) = b$ and $h^*(b) = a$. Again an argument as in Case 1 proves h^* to be the required extension.

This completes the proof.

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