TPLP **21** (5): 593–609, 2021.
doi:10.1017/S1471068421000284

Reasoning on Multirelational Contextual Hierarchies via Answer Set Programming with Algebraic Measures

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submitted 09 August 2021; revised 21 August 2021; accepted 22 August 2021

Abstract

Dealing with context-dependent knowledge has led to different formalizations of the notion of context. Among them is the Contextualized Knowledge Repository (CKR) framework, which is rooted in description logics but links on the reasoning side strongly to logic programs and Answer Set Programming (ASP) in particular. The CKR framework caters for reasoning with defeasible axioms and exceptions in contexts, which was extended to knowledge inheritance across contexts in a coverage (specificity) hierarchy. However, the approach supports only this single type of contextual relation and the reasoning procedures work only for restricted hierarchies, due to nontrivial issues with model preference under exceptions. In this paper, we overcome these limitations and present a generalization of CKR hierarchies to multiple contextual relations, along with their interpretation of defeasible axioms and preference. To support reasoning, we use ASP with algebraic measures, which is a recent extension of ASP with weighted formulas over semirings that allows one to associate quantities with interpretations depending on the truth values of propositional atoms. Notably, we show that for a relevant fragment of CKR hierarchies with multiple contextual relations, query answering can be realized with the popular asprin framework. The algebraic measures approach is more powerful and enables, for example, reasoning with epistemic queries over CKRs, which opens interesting perspectives for the use of quantitative ASP extensions in other applications.

KEYWORDS: defeasible knowledge, description logics, ASP, algebraic measures, justifiable exceptions

1 Introduction

Representing and reasoning with context dependent knowledge is a fundamental theme in AI, with proposals dating back to the works of McCarthy (1993) and Giunchiglia and Serafini (1994). It has gained increasing attention for the Semantic Web as knowledge resources must be interpreted with contextual information from their metadata. Several approaches for contextual reasoning, most based on description logics, were developed (Straccia *et al*. 2010; Klarman 2013; Serafini and Homola 2012).

A rich framework among them are *Contextualized Knowledge Repositories (CKR)* (Serafini and Homola 2012): CKR knowledge bases (KBs) are two-layered structures with a *global context*, which contains context-independent *global* knowledge and *meta-knowledge* about the structure of the KB, and *local contexts* containing knowledge about specific situations (e.g. a region in space, a site of an organization). Notably, the global knowledge is propagated to local contexts, where inherited axioms may be *defeasible*, meaning that instances can be "overridden" on an exceptional basis (Bozzato *et al*. 2018a). Reasoning from CKRs strongly links to logic programming, as the KBs are over a Horn-description logic and the working of defeasible axioms was inspired by conflict handling in *inheritance logic programs* (Buccafurri *et al*. 1999). Furthermore, answering instance and conjunctive queries over a CKR is possible via a uniform ASP program that employs a materialization calculus akin to the one by Krötzsch (2010).

For modeling and analyzing complex scenarios where global regulations (e.g. laws, environmental regulations, access control rules) can be refined by more specific situations (e.g. time-bounded events, geographical areas, groups of users), the CKR model was extended (Bozzato *et al*. 2018b) to cater for defeasible axioms in local contexts and knowledge inheritance across hierarchies, based on a *coverage* contextual relation (Serafini and Homola 2012).

This approach, however, is limited to reason only on hierarchies based on this single type of contextual relation. In practice, defeasible inheritance may be necessary under different contextual relations. For example, along a location hierarchy, we may prefer axioms encoding regional laws overriding state-level regulations, while preferring newer rules over older laws along a temporal dimension. A further limitation is that even for a single coverage relation, it is challenging to encode the induced preference relation over CKR interpretation using ASP because the relation may not be transitive and thus not a strict partial order, as assumed, for example, in the popular asprin framework for preferences in ASP (Brewka *et al*. 2015). Instead, a specialized implementation for preferential reasoning was introduced (Bozzato *et al*. 2019), which however needs to consider all answer sets of a program to single out a preferred CKR model.

In this paper, we overcome these limitations and make the following contributions:

(1) We generalize single-relational CKRs to multirelational CKRs, where axioms are not defeasible in general but merely with regard to individual relations of hierarchies. By a combination of preferences over the distinct individual relations, we obtain an overall preference over the models of a CKR. While intuitive, the technical condition has pitfalls and needs care.

(2) We show how to model multirelation CKRs in ASP. Specifically, we use to this end ASP with *algebraic measures* (Eiter and Kiesel 2020), which is a foundation to express many quantitative reasoning problems. Here, *weighted logic* formulas (Droste and Gastin 2005) *measure* values associated with an interpretation $\mathcal{I}$ by performing a computation over a semiring, whose outcome depends on the truth of the propositional variables in $\mathcal{I}$. Such measures can be used for for example, weighted model counting, probabilistic reasoning, and, as in our case, preferential reasoning.

(3) While asprin is a powerful tool for modeling preferences in ASP, it appears to be ill-suited for expressing multirelational CKR. The reasons are *eval*-expressions in CKRs, which propagate predicate extensions from one local context to another. We show, however, that under a well-behaved use of such expressions according to a syntactic dis-

https://doi.org/10.1017/S1471068421000284 Published online by Cambridge University Press

connectedness condition, multirelational CKRs can be expressed in asprin. This enables us to use the asprin solver to evaluate preferences for CKRs, which is showcased in a prototype implementation.

(4) Furthermore, ASP with algebraic measures opens the possibility of reasoning tasks for CKRs beyond asprin's capability, even in absence of *eval*-expression. As examples we consider obtaining preferred CKR models by overall weight queries and epistemic reasoning, which for description logics is specifically needed in aggregate queries (Calvanese *et al*. 2008).

In conclusion, ASP extended with preferences or algebraic computations is a valuable tool to express CKR extensions and reasoning on them, with a promising perspective for further research.

2 Preliminaries

Description logics and $\mathcal{SROIQ}$-RL. We follow the common presentation of *description logics (DLs)* (Baader *et al*. 2003) and the definition of the logic $\mathcal{SROIQ}$ (Horrocks *et al*. 2006).

A *DL vocabulary* Σ consists of the mutually disjoint countably infinite sets NC of *atomic concepts*, NR of *atomic roles*, and NI of *individual constants*. Complex *concepts* are recursively defined as the smallest sets containing all concepts that can be inductively constructed using the operators of the considered DL language $\mathcal{L}_\Sigma$. A DL *knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ consists of: a TBox $\mathcal{T}$ which can contain general concept inclusion axioms $C \sqsubseteq D$, where C and D are concepts; an RBox $\mathcal{R}$ which contains role inclusion axioms $S \sqsubseteq R$, where S and R are roles, and role properties axioms; and an ABox $\mathcal{A}$ which contains assertions of the forms $D(a)$, $R(a, b)$, where a and b are any individual constants.

A *DL interpretation* is a pair $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ where $\Delta^\mathcal{I}$ is a nonempty set called *domain* and $\cdot^\mathcal{I}$ is the *interpretation function* which provides the interpretation for language elements: $a^\mathcal{I} \in \Delta^\mathcal{I}$, for $a \in$ NI; $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, for $A \in$ NC; $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, for $R \in$ NR. The interpretation of complex concepts and roles is defined by the evaluation of their DL operators (see the paper by Horrocks *et al*. (2006) for $\mathcal{SROIQ}$). An interpretation $\mathcal{I}$ *satisfies* an axiom ϕ, denoted $\mathcal{I} \models \phi$, if it verifies the respective semantic condition, in particular: for $\phi = D(a)$, $a^\mathcal{I} \in D^\mathcal{I}$; for $\phi = R(a, b)$, $\langle a^\mathcal{I}, b^\mathcal{I} \rangle \in R^\mathcal{I}$; for $\phi = C \sqsubseteq D$, $C^\mathcal{I} \subseteq D^\mathcal{I}$ (resp. for role inclusions). $\mathcal{I}$ is a *model* of $\mathcal{K}$, denoted $\mathcal{I} \models \mathcal{K}$, if it satisfies all axioms of $\mathcal{K}$. We adopt w.l.o.g. the *standard name assumption (SNA)* in the DL setting, that is, every element in $\mathcal{I}$ is reachable via a distinct constant.We denote by $\mathrm{NI}_S \subseteq$ NI the set of all such constants, called standard names, which are uniform for all interpretations; see the papers by Eiter *et al*. (2008) and de Bruijn *et al*. (2008) for more details.

Most of the following definitions for simple CKR are independent from the DL used as representation language inside contexts: however, as in the paper by Bozzato *et al*. (2018a), we take as reference language a restriction of the $\mathcal{SROIQ}$ syntax called $\mathcal{SROIQ}$-RL which corresponds to OWL-RL. We restrict as follows *left-side concepts* C and *right-side concepts* D:

$$C := A \mid \{a\} \mid C \sqcap C \mid C \sqcup C \mid \exists R.C \mid \exists R.\top$$
$$D := A \mid \neg C \mid D \sqcap D \mid \exists R.\{a\} \mid \forall R.D \mid \leqslant n R.\top,$$

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where $A \in \mathrm{NC}$, $R \in \mathrm{NR}$ and $n \in \{0,1\}$. $\mathcal{SROIQ}$-RL TBox axioms can only take the form $C \sqsubseteq D$, where C is a left-side and D is a right-side or $E \equiv F$, where E and F are both left- and right-side concepts. A $\mathcal{SROIQ}$-RL RBox can contain role inclusions $R \sqsubseteq S$ (with possibly left role composition), role disjointness, irreflexivity, symmetry, asymmetry and transitivity. $\mathcal{SROIQ}$-RL ABox concept assertions can only be of form $D(a)$, where D is a right-side concept. We remark that $\mathcal{SROIQ}$-RL basically defines a restriction of $\mathcal{SROIQ}$ to axioms that are expressible as Horn rules (cf. FO translation provided by Bozzato *et al.* (2018a)).

Normal programs and answer sets. We use *function-free normal (datalog) rules* with *(default) negation* under answer sets semantics (Gelfond and Lifschitz 1991) and gather them in *ASP programs*. A normal (datalog) rule r is an expression of the form:

$$a \leftarrow b_1, \ldots, b_k, \mathtt{not}\, b_{k+1}, \ldots, \mathtt{not}\, b_m, \quad 0 \leq k \leq m, \tag{1}$$

also written $H(r) \leftarrow B(r)$ where $a, b_1, \ldots, b_m$ are function-free FO-atoms and $\mathtt{not}$ is negation as failure (NAF). We allow that a is missing (*constraint*), viewing a as logical constant for falsity. A (datalog) *program* P is a finite set of rules. An atom (rule etc.) is *ground*, if no variables occur in it. A *fact* H is a ground rule r with $m = 0$. The *grounding* of a rule r, $\mathit{grnd}(r)$, is the set of all ground instances of r, and the *grounding* of a program P is $\mathit{grnd}(P) = \bigcup_{r \in P} \mathit{grnd}(r)$.

For any program P, we denote by U_P its Herbrand universe and by B_P its Herbrand base; an *(Herbrand) interpretation* is any subset $I \subseteq B_P$ of B_P. An atom a is *true* in I, denoted $I \models a$, if $a \in I$. Given a rule $r \in \mathit{grnd}(P)$, we say that $B(r)$ is true in I, denoted $I \models B(r)$, if (i) $I \models b$ for each b in $B(r)$ and (ii) $I \not\models b$ for each $\mathtt{not}\, b$ in $B(r)$. A rule r is *satisfied* in I, denoted $I \models r$, if either $I \models H(r)$ or $I \not\models B(r)$. An interpretation I is a *model* of P, denoted $I \models P$, if $I \models r$ for each $r \in \mathit{grnd}(P)$; moreover, I is *minimal*, if $I' \not\models P$ for each subset $I' \subset I$. Furthermore, I is an *answer set* of P, if I is a minimal model of the (Gelfond-Lifschitz) *reduct* $G_I(P)$ of P w.r.t. I, which results from $\mathit{grnd}(P)$ by removing (i) every rule r such that $I \models l$ for some $\mathtt{not}\, l \in B(r)$, and (ii) all formulas $\mathtt{not}\, b$ from the remaining rules. The set of answer sets of P is denoted $\mathcal{AS}(P)$.

Semirings and weighted logic. A *semiring* $\mathcal{R} = (R, \oplus, \otimes, e_\oplus, e_\otimes)$ is a set $R \neq \emptyset$ equipped with binary operations $\oplus$ and $\otimes$, called addition and multiplication, such that (i) $(R, \oplus)$ is a commutative monoid with identity element $e_\oplus$, (ii) $(R, \otimes)$ is a monoid with identity element $e_\otimes$, (iii) multiplication left and right distributes over addition, and (iv) multiplication by $e_\oplus$ annihilates R, that is $\forall r \in R : r \otimes e_\oplus = e_\oplus = e_\oplus \otimes r$. Examples are the natural number semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ with addition and multiplication, the powerset semiring $\mathcal{P}(A) = (2^A, \cup, \cap, \emptyset, A)$, with union and intersection, the Boolean semiring $\mathbb{B} = (\{\mathbf{t}, \mathbf{f}\}, \vee, \wedge, \mathbf{f}, \mathbf{t})$, with disjunction and conjunction, and the tropical semiring $\mathcal{R}_{\mathrm{trop}} = (\mathbb{Q} \cup \{\infty\}, \min, +, \infty, 0)$, with minimum and addition.

Weighted formulas over a semiring $\mathcal{R}$ and an Herbrand base B allow us to assign an interpretation I a semiring value, depending on the truth of propositional variables w.r.t. I. Their syntax is:

$$\alpha ::= k \mid v \mid \neg v \mid \alpha + \alpha \mid \alpha * \alpha,$$

where $k \in R$ and $v \in B$. The semantics $[\![\alpha]\!]_{\mathcal{R}}(I)$ of α over $\mathcal{R}$ w.r.t. I is:

$$[\![k]\!]_{\mathcal{R}}(I) = k \text{ for } k \in R \qquad [\![\ell]\!]_{\mathcal{R}}(I) = \begin{cases} e_\otimes & I \models \ell \\ e_\oplus & \text{otherwise.} \end{cases} \text{ for } \ell \in \{v, \neg v\}$$

$$[\![\alpha_1 + \alpha_2]\!]_{\mathcal{R}}(I) = [\![\alpha_1]\!]_{\mathcal{R}}(I) \oplus [\![\alpha_2]\!]_{\mathcal{R}}(I) \qquad [\![\alpha_1 * \alpha_2]\!]_{\mathcal{R}}(I) = [\![\alpha_1]\!]_{\mathcal{R}}(I) \otimes [\![\alpha_2]\!]_{\mathcal{R}}(I).$$

3 Multirelational simple CKR

We generalize the definition of *simple CKR (sCKR)* introduced by Bozzato *et al.* (2018b; 2019) from single- to multirelational contextual hierarchies. As in the original formulation of CKR by Bozzato *et al.* (2018a; 2013) a simple CKR is still a two layered structure, but the upper layer is simply a poset with multiple orderings, corresponding to different contextual relations. Simple CKRs define a core fragment of CKR allowing us to provide lean definitions on contextual hierarchies: the presented results, however, can be easily generalized to the full CKR.

We provide definitions for multirelational simple CKRs with a general set of context relations and consider the case for 2-relational sCKR based on *temporal* and *coverage* relations.

Syntax. Consider a nonempty set $\mathsf{N} \subseteq \mathrm{NI}$ of *context names*. A *contextual relation* is any strict order $\prec_i \subseteq \mathsf{N} \times \mathsf{N}$ over contexts. We may use the nonstrict relation $\mathsf{c}_1 \preceq_i \mathsf{c}_2$ to indicate that either $\mathsf{c}_1 \prec_i \mathsf{c}_2$ or c_1 and c_2 are the same context. We consider two contextual relations, namely *coverage* $\prec_c$ and *temporal precedence* $\prec_t$. Here, $\mathsf{c}_1 \prec_c \mathsf{c}_2$ (resp. $\mathsf{c}_1 \prec_t \mathsf{c}_2$) means that c_1 is more specific (resp. newer) than c_2. More specific means that c_1 represents a portion of the world covered by the one referred to by c_2, as in the paper by Serafini and Homola (2012). We generalize the definition of defeasible axiom w.r.t. contextual relations:

Definition 1 (*r-defeasible axiom*)
Given a set $\mathcal{R}$ of contextual relations over N and a description language $\mathcal{L}_\Sigma$, an *r-defeasible axiom* is any expression of the form $\mathrm{D}_r(\alpha)$, where α is an axiom of $\mathcal{L}_\Sigma$ and $\prec_r \in \mathcal{R}$.

Thus, we identify coverage-defeasible axioms as $\mathrm{D}_c(\alpha)$ and temporal-defeasible axioms as $\mathrm{D}_t(\alpha)$. We allow for the use of r-defeasible axioms in the local language of contexts:

Definition 2 (*contextual language*)
Given a set of context names N, for every description language $\mathcal{L}_\Sigma$ we define $\mathcal{L}_{\Sigma,\mathsf{N}}$ as the extension of $\mathcal{L}_\Sigma$ where: (i) $\mathcal{L}_{\Sigma,\mathsf{N}}$ contains the set of r-defeasible axioms in $\mathcal{L}_\Sigma$; (ii) $\mathit{eval}(X, \mathsf{c})$ is a concept (resp. role) of $\mathcal{L}_{\Sigma,\mathsf{N}}$ if X is a concept (resp. role) of $\mathcal{L}_\Sigma$ and $\mathsf{c} \in \mathsf{N}$.

Using these definitions, multirelational simple CKRs are defined as follows:

Definition 3 (*multirelational simple CKR*)
A *multirelational simple CKR (sCKR)* over Σ and N is a structure $\mathfrak{K} = \langle \mathfrak{C}, \mathrm{K}_\mathsf{N} \rangle$ where:

– $\mathfrak{C}$ is a structure $(\mathsf{N}, \prec_1, \ldots, \prec_m)$ where each $\prec_i$ is a contextual relation over N, and
– $\mathrm{K}_\mathsf{N} = \{\mathrm{K}_\mathsf{c}\}_{\mathsf{c} \in \mathsf{N}}$ for each context name $\mathsf{c} \in \mathsf{N}$, K_c is a DL knowledge base over $\mathcal{L}_{\Sigma,\mathsf{N}}$.

A sCKR that combines temporal and coverage orderings can be defined by $\mathfrak{C} = (\mathsf{N}, \prec_t, \prec_c)$. For simplicity, we assume that the priority for the combination of orderings is defined by the linear order in which they appear in $\mathfrak{C}$: in the case above, we prioritize $\prec_t$ over $\prec_c$.

Example 1
We consider the following example to explain the expected behavior of defeasible axioms in the case of the combination of coverage and temporal relations. Let us consider $\mathfrak{K}_{org} = \langle \mathfrak{C}, \mathrm{K}_\mathsf{N} \rangle$ with $\mathfrak{C} = (\mathsf{N}, \prec_t, \prec_c)$ describing the organization of a corporation. The corporation has different policies with respect to its local branches, represented by

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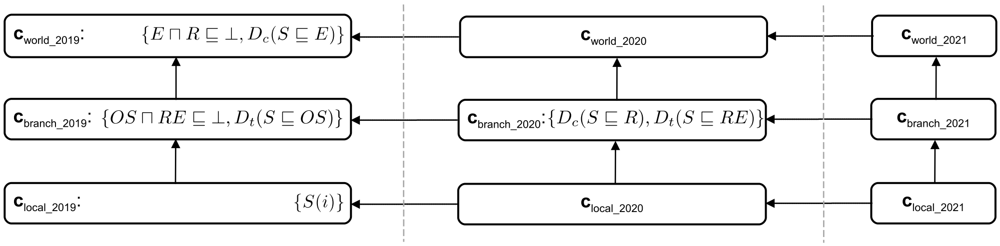

Figure 1. Context hierarchy of multirelational example sCKR, with axioms per context.

coverage, and updates them along the time precedence. The structure of $\mathfrak{C}$, together with the axioms at each context, is shown in Figure 1. We have a chain of three contexts (representing world, branch and local rules) in the direction of the coverage and three "time-slices" (2019, 2020 and 2021) along the time relation: thus, for example, we have $\mathsf{c}_{local\_2021} \prec_c \mathsf{c}_{branch\_2021}$ and $\mathsf{c}_{branch\_2020} \prec_t \mathsf{c}_{branch\_2019}$. The corporation is active in the fields of Electronics (E) and Robotics (R) and employs supervisors (S). In c_{world_2019}, we state that, with respect to coverage, every Supervisor has to be applied by default to Electronics and that Electronics and Robotics are disjoint. In the lower context c_{branch_2019}, we further specify that, with respect to time, Supervisors have to work by default *OnSite* (OS) (where working *OnSite* and *Remote* (RE) are disjoint). In 2019's local context c_{local_2019} we assert that i is a Supervisor. The previous defeasible statements are, however, contradicted by the ones in c_{branch_2020}, where Supervisors are applied to Robotics and work on Remote.

The interpretation of defeasible propagation and preferences, then, should define the interpretation of what is derivable in the local context in the three time-slices. In c_{local_2019} no overriding takes place; then we should derive $E(i), OS(i)$. In c_{local_2020} the more coverage-specific axiom in c_{branch_2020} is preferred, thus we derive $R(i)$; the time-related defeasible axiom $\mathrm{D}_t(S \sqsubseteq RE)$ is applied locally to the 2020 time-slice, thus we derive $RE(i)$. In the 2021 time-slice no new information is provided, thus the overriding preferences should enforce that the more specific and recent information is used: in c_{local_2021} we expect to derive $R(i), RE(i)$. $\Diamond$

Semantics. A sCKR interpretation gathers interpretations for the local contexts as follows.

Definition 4 (sCKR interpretation)
An interpretation for $\mathcal{L}_{\Sigma,\mathbf{N}}$ is a family $\mathfrak{I} = \{\mathcal{I}(\mathsf{c})\}_{\mathsf{c}\in\mathbf{N}}$ of $\mathcal{L}_\Sigma$ interpretations, such that $\Delta^{\mathcal{I}(\mathsf{c})} = \Delta^{\mathcal{I}(\mathsf{c}')}$ and $a^{\mathcal{I}(\mathsf{c})} = a^{\mathcal{I}(\mathsf{c}')}$, for every $a \in \mathrm{NI}$ and $\mathsf{c}, \mathsf{c}' \in \mathbf{N}$.

The interpretation of concepts and role expressions in $\mathcal{L}_{\Sigma,\mathbf{N}}$ is obtained by extending the standard interpretation to *eval* expressions: for every $\mathsf{c} \in \mathbf{N}$, $eval(X, \mathsf{c}')^{\mathcal{I}(\mathsf{c})} = X^{\mathcal{I}(\mathsf{c}')}$. We consider the definition of axiom instantiation provided by Bozzato *et al.* (2018a): given an axiom $\alpha \in \mathcal{L}_\Sigma$ with FO-translation $\forall \mathbf{x}.\phi_\alpha(\mathbf{x})$, the *instantiation* of α with a tuple $\mathbf{e}$ of individuals in NI, written $\alpha(\mathbf{e})$, is the specialization of α to $\mathbf{e}$, that is, $\phi_\alpha(\mathbf{e})$, depending on the type of α.

For a structure $\mathfrak{C} = (\mathbf{N}, \prec_1, \ldots, \prec_m)$ and $1 \leq i \leq m$, we denote by $\preceq_{-i}$ the order obtained as the reflexive and transitive closure of $\bigcup_{j \neq i} \prec_j$, that is, the union of all orders $\prec_j$ except for $\prec_i$. We denote by $\preceq_*$ the order obtained as the reflexive and transitive closure of the union of all $\prec_j$.

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Definition 5 (*clashing assumptions and sets*)
A *clashing assumption* for a context c and contextual relation r is a pair $\langle \alpha, \mathbf{e} \rangle$ such that $\alpha(\mathbf{e})$ is an axiom instantiation of α, and $\mathrm{D}_r(\alpha) \in \mathrm{K}_{\mathsf{c}'}$ is a defeasible axiom of some $\mathsf{c}' \succeq_{-r} \mathsf{c}'' \succ_r \mathsf{c}$. A *clashing set* for $\langle \alpha, \mathbf{e} \rangle$ is a satisfiable set S of ABox assertions s.t. $S \cup \{\alpha(\mathbf{e})\}$ is unsatisfiable.

A clashing assumption $\langle \alpha, \mathbf{e} \rangle$ represents that $\alpha(\mathbf{e})$ is not satisfiable in context c, and a clashing set S provides a "justification" for the local assumption of overriding of α on $\mathbf{e}$. CAS-interpretations include a set of clashing assumptions for each context and contextual relation:

Definition 6 (*CAS-interpretation*)
A *CAS-interpretation* is a structure $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$ where $\mathfrak{I}$ is an interpretation and $\overline{\chi} = \{\chi_1, \dots, \chi_m\}$ such that each χ_i, for $i \in \{1, \dots, m\}$, maps every $\mathsf{c} \in \mathbf{N}$ to a set $\chi_i(\mathsf{c})$ of clashing assumptions for context c and context relation $\prec_i$.

Satisfaction of a sCKR $\mathfrak{K}$ needs to consider the effect of the different relations:

Definition 7 (*CAS-model*)
Given a multirelation sCKR $\mathfrak{K}$, a CAS-interpretation $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$ is a *CAS-model* for $\mathfrak{K}$ (denoted $\mathfrak{I}_{CAS} \models \mathfrak{K}$), if the following holds[1]:

(i) for every $\alpha \in \mathrm{K}_{\mathsf{c}}$ (strict axiom), and $\mathsf{c}' \preceq_* \mathsf{c}$, $\mathcal{I}(\mathsf{c}') \models \alpha$;
(ii) for every $\mathrm{D}_i(\alpha) \in \mathrm{K}_{\mathsf{c}}$ and $\mathsf{c}' \preceq_{-i} \mathsf{c}$, $\mathcal{I}(\mathsf{c}') \models \alpha$;
(iii) for every $\mathrm{D}_i(\alpha) \in \mathrm{K}_{\mathsf{c}}$ and $\mathsf{c}'' \prec_i \mathsf{c}' \preceq_{-i} \mathsf{c}$, if $\langle \alpha, \mathbf{d} \rangle \notin \chi_i(\mathsf{c}'')$, then $\mathcal{I}(\mathsf{c}'') \models \phi_\alpha(\mathbf{d})$.

Intuitively (i) strict axioms are propagated across the hierarchy structures over $\preceq_*$ from higher to lower contexts; (ii) considering contexts that are related by relations other than $\prec_i$ (including the context in which axioms are declared), defeasible axioms $\mathrm{D}_i(\alpha)$ are interpreted as strict axioms; and (iii) over relation $\prec_i$, axioms $\mathrm{D}_i(\alpha)$ are verified in context c'' only if applied to instances $\mathbf{d}$ that are not in the clashing assumptions for c'' and relation $\prec_i$. Note that these propagation rules are applied for every contextual relation: however, the definition can be easily extended to assign different conditions for propagation and overriding for each of the orderings.

We provide a *local preference* on clashing assumption sets for each of the relations:

(LP). $\chi_i^1(\mathsf{c}) > \chi_i^2(\mathsf{c})$, if for every $\langle \alpha_1, \mathbf{e} \rangle \in \chi_i^1(\mathsf{c}) \setminus \chi_i^2(\mathsf{c})$ with $\mathrm{D}_i(\alpha_1)$ at a context $\mathsf{c}_1 \succeq_{-i} \mathsf{c}_{1b} \succ_i \mathsf{c}$, some $\langle \alpha_2, \mathbf{f} \rangle \in \chi_i^2(\mathsf{c}) \setminus \chi_i^1(\mathsf{c})$ exists with $\mathrm{D}_i(\alpha_2)$ at context $\mathsf{c}_2 \succeq_{-i} \mathsf{c}_{2b} \succ_i \mathsf{c}$ s.t. $\mathsf{c}_{1b} \succ_i \mathsf{c}_{2b}$.

Intuitively, $\chi_i^1(\mathsf{c})$ is preferred to $\chi_i^2(\mathsf{c})$ if $\chi_i^1(\mathsf{c})$ exchanges the "more costly" exceptions of $\chi_i^2(\mathsf{c})$ at more specialized contexts with "cheaper" ones at more general contexts. As above, multiple options for local preference can be adopted, cf. the work of Bozzato *et al.* (2018b) for ranked hierachies.

Two DL interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ are NI-*congruent*, if $c^{\mathcal{I}_1} = c^{\mathcal{I}_2}$ holds for every $c \in \mathrm{NI}$. This extends to CAS interpretations $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$ by considering all context interpretations $\mathcal{I}(\mathsf{c}) \in \mathfrak{I}$.

[1] Here, it is important to ensure that (defeasible) axioms are correctly propagated w.r.t. *any* context relation $\prec_i$.

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Definition 8 (*justification*)
We say that $\langle \alpha, \mathbf{e} \rangle \in \chi_i(\mathsf{c})$ is *justified* for a *CAS* model $\mathfrak{I}_{CAS}$, if some clashing set $S_{\langle \alpha, \mathbf{e} \rangle, \mathsf{c}}$ exists such that, for every $\mathfrak{I}'_{CAS} = \langle \mathfrak{I}', \overline{\chi} \rangle$ of $\mathfrak{K}$ that is NI-congruent with $\mathfrak{I}_{CAS}$, it holds that $\mathcal{I}'(\mathsf{c}) \models S_{\langle \alpha, \mathbf{e} \rangle, \mathsf{c}}$. A *CAS* model $\mathfrak{I}_{CAS}$ of a sCKR $\mathfrak{K}$ is *justified*, if every $\langle \alpha, \mathbf{e} \rangle \in \overline{\chi}$ is justified in $\mathfrak{K}$.

We define a *model preference* by combining the preferences of the relations: it is a global lexicographical ordering on models where each $\prec_i$ defines the ordering at the ith position.
(MP). $\mathfrak{I}^1_{CAS} = \langle \mathfrak{I}^1, \chi^1_1, \ldots, \chi^1_m \rangle$ is preferred to $\mathfrak{I}^2_{CAS} = \langle \mathfrak{I}^2, \chi^2_1, \ldots, \chi^2_m \rangle$ if

(i) there exists $i \in \{1, \ldots, m\}$ and some $\mathsf{c} \in \mathbf{N}$ s.t. $\chi^1_i(\mathsf{c}) > \chi^2_i(\mathsf{c})$ and not $\chi^2_i(\mathsf{c}) > \chi^1_i(\mathsf{c})$, and for no context $\mathsf{c}' \neq \mathsf{c} \in \mathbf{N}$ it holds that $\chi^1_i(\mathsf{c}') < \chi^2_i(\mathsf{c}')$ and not $\chi^2_i(\mathsf{c}') < \chi^1_i(\mathsf{c}')$.
(ii) for every $j < i \in \{1, \ldots, m\}$, it holds $\chi^1_j \approx \chi^2_j$ (i.e. (i) or its converse do not hold for $\prec_j$).

Then, CKR models are defined by taking into account justification and model preference.

Definition 9 (*CKR model*)
An interpretation $\mathfrak{I}$ is a *CKR model* of a sCKR $\mathfrak{K}$ (in symbols, $\mathfrak{I} \models \mathfrak{K}$) if: (i) $\mathfrak{K}$ has some justified CAS model $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$; (ii) there exists no justified $\mathfrak{I}'_{CAS} = \langle \mathfrak{I}', \overline{\chi}' \rangle$ that is preferred to $\mathfrak{I}_{CAS}$.

Example 2
By considering the sCKR of Example 1, we can show how the preference for different relations influences the global model preference. In the case of $\mathfrak{K}_{org}$, we have eight justified interpretations that are based on combinations of the following clashing assumption sets (for both relations) on contexts c_{local_2020} and c_{local_2021}. For any CAS model $\chi_t(\mathsf{c}_{local\_2020}) = \chi^0_t(\mathsf{c}_{local\_2020}) = \{\langle S \sqsubseteq OS, i \rangle\}$ and $\chi_c(\mathsf{c}_{local\_2020})$ is either $\chi^0_c(\mathsf{c}_{local\_2020}) = \{\langle S \sqsubseteq E, i \rangle\}$ or $\chi^1_c(\mathsf{c}_{local\_2020}) = \{\langle S \sqsubseteq R, i \rangle\}$.

For c_{local_2021} we have that $\chi_t(\mathsf{c}_{local\_2021})$ is either $\chi^1_t(\mathsf{c}_{local\_2021}) = \{\langle S \sqsubseteq OS, i \rangle\}$ or $\chi^2_t(\mathsf{c}_{local\_2021}) = \{\langle S \sqsubseteq RE, i \rangle\}$. For $\chi_c(\mathsf{c}_{local\_2021})$, we have the same choices as for $\chi_c(\mathsf{c}_{local\_2020})$.

According to the (LP) definition, $\chi^0_c(\mathsf{c}_{local\_2020}) > \chi^1_c(\mathsf{c}_{local\_2020})$ since $D_c(S \sqsubseteq E)$ occurs at a less specific context w.r.t. $\prec_c$ than $D_c(S \sqsubseteq R)$. Similarly, $\chi^1_t(\mathsf{c}_{local\_2021}) > \chi^2_t(\mathsf{c}_{local\_2021})$.

Since we can choose the clashing assumptions per context independently, the clashing assumption map of CKR models is uniquely determined by (MP) as $\overline{\chi} = \langle \chi_t, \chi_c \rangle$ where $\chi_t = \chi^0_t \cup \chi^1_t$ and $\chi_c = \chi^0_c \cup \chi^2_c$. Indeed, this corresponds to the intuitive model where overridings over temporal relation occur on defeasible axioms in the "older" contexts and in the "higher" contexts with respect to the coverage relation. $\diamondsuit$

Reasoning and complexity. We consider the following *reasoning tasks* for sCKR:

– c-*entailment* $\mathfrak{K} \models \mathsf{c} : \alpha$, denoting that axiom α is entailed in each CKR-model of $\mathfrak{K}$ at context c.
– *Boolean conjunctive query (BCQ) answering* $\mathfrak{K} \models \exists \mathbf{y} \gamma(\mathbf{y})$, where $\gamma(\mathbf{y}) = \gamma_1 \wedge \cdots \wedge \gamma_m$ is an existentially closed conjunction of atoms $\gamma_i = \mathsf{c}_i : \alpha_i(\mathbf{t}_i)$ with context name c_i and assertion $\alpha_i(\mathbf{t}_i)$.

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The complexity of reasoning with contextual hierarchies in sCKR was studied by Bozzato *et al.* (2018b; 2019); in particular, CKR satisfiability is NP-complete while CKR model checking is coNP-complete already for ranked hierarchies. This causes the complexity of c-entailment to increase in presence of hierarchies: for polynomial-time local preferences on overridings, c-entailment is Π_2^p-complete. In contrast, BCQ answering remains Π_2^p-complete as verifying a guess for a countermodel to the query remains in coNP. These results would carry over to multirelational hierarchies: for combinations of polynomial-time preferences (like the global preference we considered), c-entailment and similarly BCQ answering would still be Π_2^p-complete.

4 Preferences with algebraic measures

The question rises how the reasoning problems above can be expressed and solved. Previously, in the case of sCKRs with a single relation the strategy was to encode the problem in ASP using a program whose stable models correspond to the least justified models of the sCKR. The preferred models, that is, sCKR models, were then selected using weight constraints in the restricted case of ranked hierarchies (Bozzato *et al.* 2018a) or by using a dedicated algorithm for general hierarchies (Bozzato *et al.* 2019). The preference over models for multirelational sCKRs is more complicated and thus not easily expressed with weight constraints: we can leverage the power of quantitative extensions of ASP to express model preferences induced by multirelational sCKRs.

The recently introduced algebraic measures for ASP, which connect ASP with weighted formulas, were shown to be a general framework for specifying quantitative reasoning problems (Eiter and Kiesel 2020). Also preferential reasoning falls into this category, thus allowing us to use algebraic measures to specify a preference on the answer sets in such a way that the preferred answer sets correspond to the preferred least justified models. The concept is as follows.

Definition 10 (*Algebraic Measure*)
An *algebraic measure* $\mu = \langle \Pi, \alpha, \mathcal{R} \rangle$ consists of an answer set program Π, a weighted formula α, and a semiring $\mathcal{R}$. The weight of an answer set $S \in \mathcal{AS}(\Pi)$ is $\mu(S) = [\![\alpha]\!]_{\mathcal{R}}(S)$. And the *overall weight* of μ is defined as $\mu(\Pi) = \bigoplus_{S \in \mathcal{AS}(\Pi)} \mu(S)$.

Intuitively, given $\mu = \langle \Pi, \alpha, \mathcal{R} \rangle$, the program Π specifies which interpretations are accepted and the weighted formula α measures some value associated with them. Using algebraic measures, we can not only assign answer sets a weight but also obtain some information from all answer sets by considering the overall weight $\mu(\Pi)$.

Example 3
Let Π be some answer set program. Then, for example, for $\mu_1 = \langle \Pi, 1, \mathbb{N} \rangle$ the overall weight $\mu(\Pi)$ is the number of answer sets of Π. For $\mu_2 = \langle \Pi, (a_1 * 1 + \neg a_1) * \ldots * (a_n * 1 + \neg a_n), \mathcal{R}_{\max} \rangle$, where $\mathcal{R}_{\max} = \langle \mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$, the weight $\mu(S)$ of an answer set S is the number of atoms $a_1, \ldots, a_n$ it satisfies. We need the additional term $\neg a_i$, since $a_i * 1$ evaluates to $e_\oplus = -\infty$ when a_i is false and not to the desired value $e_\otimes = 0$. Due to the usage of the semiring $\mathcal{R}_{\max}$, the overall weight $\mu(\Pi)$ is the maximum number of atoms from $a_1, \ldots, a_n$ that are satisfied in any answer set of Π.

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A natural use case of algebraic measures is preferential reasoning. In the sequel, a *preference relation* is any asymmetric relation.

Definition 11 (*Preferred Answer Set*)
Given a measure $\mu = \langle \Pi, \alpha, \mathcal{R} \rangle$ and a preference relation $>$ on R, an answer set $S \in \mathcal{AS}(\Pi)$ is *preferred* w.r.t. μ and $>$ if no $S' \in \mathcal{AS}(\Pi)$ exists such that $\mu(S') > \mu(S)$.

Intuitively, we use μ as an optimization function and take the preferred answer sets as those that achieve an optimal value.

Example 4
Reconsider the measure μ_2 from Example 3. If $a_1, \ldots, a_n$ are desired to be true, then we only want to consider those answer sets for which a maximal number of them is true. These are exactly the preferred answer sets with respect to the measure μ_2 and the usual order over the reals.

We assume a program $PK(\mathfrak{K})$ (see Section 5 for more), which intuitively guesses a set of atoms $\mathtt{ovr}(\phi, \mathbf{e}, \mathsf{c}, i)$, each corresponding to a clashing assumption $\langle \phi, \mathbf{e} \rangle$ in $\chi_i(\mathsf{c})$, and checks whether there is an CAS model $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$. The answer sets I corresponds to the least CAS models with that property. Then we can introduce a measure μ_{opt} and order $>_{opt}$ to obtain those answer sets of $PK(\mathfrak{K})$ as preferred answer sets w.r.t. μ_{opt} and $>_{opt}$ that correspond to the preferred least justified models of $\mathfrak{K}$. Here, we do not require any restrictions on the $\mathfrak{K}$ at all.

We use the powerset semiring $\mathcal{P}(CA)$ over the set CA, which contains the tuple $\langle \phi, \mathbf{e}, \mathsf{c}, i \rangle$ for each possible clashing assumption $\langle \phi, \mathbf{e} \rangle$ that can occur at context c w.r.t. relation i. The weighted formula of $\mu_{opt} = \langle PK(\mathfrak{K}), \alpha, \mathcal{P}(CA) \rangle$ is given by $\alpha = \Sigma_{\langle \phi, \mathbf{e}, \mathsf{c}, i \rangle \in CA} \mathtt{ovr}(\phi, \mathbf{e}, \mathsf{c}, i) * \{\langle \phi, \mathbf{e}, \mathsf{c}, i \rangle\}$. It is easy to see that for each answer set I of $PK(\mathfrak{K})$ it holds that $\langle \phi, \mathbf{e}, \mathsf{c}, i \rangle$ is in $\mu_{opt}(I)$ iff $\mathtt{ovr}(\phi, \mathbf{e}, \mathsf{c}, i)$ is in I. Thus, we only need to define the order $>_{opt}$ on the semiring values $S \subseteq CA$ that correctly captures the ordering on the justified models. For this, we let $S \subseteq CA$ and define $(\chi_i^{(S)})_{i \in [m]}$, the clashing assumption maps corresponding to S, by setting

$$\chi_i^{(S)}(\mathsf{c}) = \{\langle \phi, \mathbf{e} \rangle \mid \langle \phi, \mathbf{e}, \mathsf{c}, i \rangle \in S\}.$$

Then for $S, S' \subseteq CA$, we define $S >_{opt} S'$ iff

(i) there exists $i \in \{1, \ldots, m\}$ and some $\mathsf{c} \in \mathbf{N}$ s.t. $\chi_i^{(S)}(\mathsf{c}) > \chi_i^{(S')}(\mathsf{c})$ and not $\chi_i^{(S')}(\mathsf{c}) > \chi_i^{(S)}(\mathsf{c})$, and for no context $\mathsf{c}' \neq \mathsf{c} \in \mathbf{N}$ it holds that $\chi_i^{(S)}(\mathsf{c}') < \chi_i^{(S')}(\mathsf{c}')$ and not $\chi_i^{(S')}(\mathsf{c}') < \chi_i^{(S)}(\mathsf{c}')$.

(ii) for every $1 \leq j < i \leq m$, we have $\chi_j^{(S)} \approx \chi_j^{(S')}$ (i.e. (i) or its converse is unprovable for $\prec_j$).

Theorem 1
Let $\mathfrak{K}$ be an sCKR and $PK(\mathfrak{K})$ as described above. Then the preferred answer sets w.r.t. μ_{opt} and $>_{opt}$ correspond to the least CKR models $\langle \mathfrak{I}, \overline{\chi} \rangle$ of $\mathfrak{K}$, that is, those where $\mathfrak{I}$ is the $\subseteq$-minimal interpretation such that $\langle \mathfrak{I}, \overline{\chi} \rangle$ is a CKR model.

In the following, we outline how such an ASP program $PK(\mathfrak{K})$ can be constructed. Furthermore, we show that for suitably restricted $\mathfrak{K}$, we can also express algebraic measures and preferential answer sets using asprin.

https://doi.org/10.1017/S1471068421000284 Published online by Cambridge University Press

5 ASP encoding of reasoning problems

ASP translation process. The ASP translation by Bozzato *et al.* (2018a) for instance checking (w.r.t. c-entailment, under UNA) in a $\mathcal{SROIQ}$-RL CKR can be extended to multirelational sCKRs $\mathfrak{K} = \langle \mathfrak{C}, \mathrm{K}_{\mathbf{N}} \rangle$, such that (1) a set of *input rules* I encode the contextual structure and local contents of contexts in $\mathfrak{K}$ as facts and (2) uniform *deduction rules* P encode the interpretation of axioms; and (3) the instance query is encoded by *output rules* O as ground facts.

Formally, the *CKR program* $PK(\mathfrak{K}) = PG(\mathfrak{C}) \cup \bigcup_{\mathsf{c} \in \mathbf{N}} PC(\mathsf{c}, \mathfrak{K})$ encodes the whole sCKR, where $PG(\mathfrak{C}) = I_{glob}(\mathfrak{C}) \cup P_{glob}$ is the *global program* for $\mathfrak{C}$ and $PC(\mathsf{c}, \mathfrak{K}) = I_{loc}(\mathrm{K}_{\mathsf{c}}, \mathsf{c}) \cup P_{loc}$ is the *local program* for $\mathsf{c} \in \mathbf{N}$. Query answering $\mathfrak{K} \models \mathsf{c} : \alpha$ is then achieved by testing whether the instance query, translated to $O(\alpha, \mathsf{c})$, is a consequence of the *preferred* models of $PK(\mathfrak{K})$, that is, whether $PK(\mathfrak{K}) \cup P_{pref} \models O(\alpha, \mathsf{c})$ holds, where P_{pref} are the newly added rules for selection of preferred models. Analogously, this can be extended to conjunctive queries as shown by Bozzato *et al.* (2018a). The details of the translation rules are in the Appendix; in the following, we further discuss P_{pref}.

Asprin-based model selection. From the translation $PK(\mathfrak{K})$ we obtain the least justified models of $\mathfrak{K}$ as answer sets of an ASP program. In Section 4, we showed how to use algebraic measures for describing which answer sets correspond to preferred models. By suitably restricting the input CKR $\mathfrak{K}$, we show that we can implement the preference already in the *asprin* framework (Brewka *et al.* 2015). The latter can not express sCKR preference relations in general as *eval*-expressions may cause nontransitive and even cyclic preference relations. We thus restrict the use of *eval*-expressions such that we can define an asprin preference relation $>$ that has the same preferred answer sets as μ_{opt} but is a strict partial order. For this, we consider a dependency graph.

Definition 12 (*Dependency Graph*)
The *dependency graph* of an sCKR $\mathfrak{K}$ is the directed graph $DEP(\mathfrak{K}) = (V, E)$ is $\mathfrak{K}$, where:

- $V = \{X_{\mathsf{c}} \mid X$ is a concept or role that occurs in $\mathrm{K}_{\mathsf{c}}\}$, that is, we have a vertex X_{c} for every combination of a concept or role X that occurs in $\mathfrak{K}$ and context $\mathsf{c} \in \mathbf{N}$.
- $(X_{\mathsf{c}}, X'_{\mathsf{c}'}) \in E$ if either: (i) $\mathsf{c} = \mathsf{c}'$, X is a complex concept or role and X' is a subexpression of X; (ii) $\mathsf{c} = \mathsf{c}'$ and X, X' co-occur in some (possibly defeasible) axiom; or (iii) $X = \mathit{eval}(X', \mathsf{c}')$.

Intuitively, a path connects two concepts/roles $X_{\mathsf{c}}, X'_{\mathsf{c}'}$ in $DEP(\mathfrak{K})$ if the interpretations of X, X' at contexts c, c', respectively, may depend on each other. If there are no *eval*-expressions, then clearly there is no path between $X_{\mathsf{c}}, X'_{\mathsf{c}'}$ when $\mathsf{c} \neq \mathsf{c}'$. In this case, we can choose the interpretations per context independently, which simplifies the choosing of preferred interpretations significantly. However, as the preference only refers to clashing assumptions caused by defaults, we can also use a weaker condition to a similar effect:

Definition 13 (*eval-Disconnectedness*)
Let $\mathfrak{K}$ be an sCKR and X, X' two concepts or roles that occur in default axioms. Then X, X' are *eval-disconnected* if there is no path between $X_{\mathsf{c}}, X'_{\mathsf{c}'}$ in $DEP(\mathfrak{K})$ for every $\mathsf{c} \neq \mathsf{c}'$. Furthermore, $\mathfrak{K}$ is *eval*-disconnected if every such X, X' are *eval*-disconnected.

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In the following, we confine to *eval*-disconnected sCKR's and define the preference in asprin as follows. We use so called "poset" preferences, which are specified using statements of the form:

$$\texttt{\#preference}(p,\texttt{poset})\{\ F_1\ \texttt{>>}\ F_2;\ F_3\ \texttt{>>}\ F_4;\ \ldots\ ;\ F_{2n-1}\ \texttt{>>}\ F_{2n}\ \}.$$

Here each F_i is a Boolean formula, and a partial order $>$ on such formulas is defined by the transitive closure of `>>`. An interpretation X is preferred over interpretation Y w.r.t. p (written $X >_p Y$) if (i) for some i, $X \models F_i$ and $Y \not\models F_i$, and (ii) for every i s.t. $Y \models F_i$ and $X \not\models F_i$, some j exists s.t. $F_j > F_i$ and $X \models F_j$ and $Y \not\models F_j$.

We then define the local preference w.r.t. context $\mathtt{c}$ and relation i by

$$
\begin{array}{l}
\texttt{\#preference(LocPref(c,}i\texttt{),poset)\{}\\
\quad \neg\mathtt{ovr}(\alpha, X, \mathtt{c}, i)\ \texttt{>>}\ \mathtt{ovr}(\alpha, X, \mathtt{c}, i);\\
\quad \neg\mathtt{ovr}(\alpha_2, Y, \mathtt{c}, i)\ \texttt{>>}\ \neg\mathtt{ovr}(\alpha_1, X, \mathtt{c}, i);\ \texttt{for}\ \mathtt{c}_1 \succeq_{-i} \mathtt{c}_{1b} \succ_i \mathtt{c}\ \texttt{and}\\
\qquad \mathtt{c}_2 \succeq_{-i} \mathtt{c}_{2b} \succ_i \mathtt{c}\ \texttt{and}\ \mathtt{c}_{1b} \succ_i \mathtt{c}_{2b}\ \texttt{and}\ D_i(\alpha_i)\ \texttt{in}\ \mathrm{K}_{\mathtt{c}_i}.\ \texttt{\}.}
\end{array}
$$

This encodes that, whenever possible, we prefer not to override a defeasible axiom $D_i(\alpha)$ (line 2); further, if we have to override some defeasible axiom, then we prefer to override the least specific one possible (line 3). Next, we emulate the preference definition (MP), where item (i) combines the local preferences into a preference per defeasibility relation and item and (ii) states that the global ordering is the lexicographical combination of the preferences per relation.

Using asprin, we can combine existing preference orders into a new one. This is where *eval*-disconnectedness comes into play. While for general sCKRs this is not the case, for *eval*-disconnected sCKRs, the preferred models w.r.t. (i) are the *pareto optimal* models X, that is, no model Y exists that is strictly better than X on one of the local preferences `LocPref(c,`i`)` and at least as good on all the others. Thus, we use the `pareto` type to define the preference per relation i:

$$\texttt{\#preference(RelPref(}i\texttt{),pareto)\{**LocPref(C,}i\texttt{)\ :\ context(C)\}.}$$

Here, the condition `context(C)` enforces that we take the pareto order over the orders `LocPref(C,`i`)` for every context `C`. Finally, for (ii), we use asprin's *lexicographical preference* over orders $(p_i)_{i\in[n]}$with weights $(w_i)_{i\in[n]}$. When $w_i > w_j$ we may worsen p_j to improve p_i.

```
#preference(GlobPref,lexico){W::**RelPref(I) : rel_w(I,W)}.
```

Similar to above, the condition `rel_w(I,W)` ensures that we obtain the lexicographical order over all preferences `RelPref(I)`, where `I` is a relation with weight `W`; in our case, `W` is its index.

Correctness. The presented encoding yields a sound and complete reasoning method for multirelational sCKRs in $\mathcal{SROIQ}$-RLD *normal form*, on time and coverage relations. $\mathcal{SROIQ}$-RLD disallows defeasible $\mathcal{SROIQ}$-RL-axioms that introduce disjunctive information. The normal form of $\mathcal{SROIQ}$-RLD due to Bozzato *et al.* (2018a) is summarized in the Appendix. Formally,

Theorem 2
Let $\mathfrak{K}$ be a multirelational sCKR that is *eval*-disconnected and in $\mathcal{SROIQ}$-RLD normal form. Then under the unique name assumption (UNA),

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(i) for every α and c such that $O(\alpha, \mathsf{c})$ is defined, $\mathfrak{K} \models \mathsf{c} : \alpha$ iff $PK(\mathfrak{K}) \cup P_{pref} \models O(\alpha, \mathsf{c})$;
(ii) for every BCQ $Q = \exists \mathbf{y}\gamma(\mathbf{y})$ on $\mathfrak{K}$, $\mathfrak{K} \models Q$ iff $PK(\mathfrak{K}) \cup P_{pref} \models O(Q)$.

Similarly to what was done by Bozzato *et al.* (2019; 2018b), the result is shown by proving a correspondence between the least CAS models of $\mathfrak{K}$ and the answer sets of $PK(\mathfrak{K})$, and then between preferred CAS models and answer sets, which are here selected by our asprin preference. For space reasons, we confine to a proof outline; more details are given in the Appendix.

Without loss of generality, we can restrict to *named models*, that is, models $\mathcal{I}$ s.t. the interpretation of atomic concepts and roles belongs to $N^{\mathcal{I}}$ for some $N \subseteq \mathrm{NI} \setminus \mathrm{NI}_S$. This allows us to concentrate on Herbrand models for $\mathfrak{K}$; in particular, w.r.t. a clashing assumption $\overline{\chi} = (\chi_t, \chi_c)$, we have a least Herbrand model which we denote as $\hat{\mathfrak{I}}(\overline{\chi})$.

Suppose $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$ is a justified named CAS-model. We can build from $\mathfrak{I}_{CAS}$ a corresponding Herbrand interpretation $I(\mathfrak{I}_{CAS})$ for the program $PK(\mathfrak{K})$. Along the lines of the result of Bozzato *et al.* (2018a, Lemma 6), we can then show that the answer sets of $PK(\mathfrak{K})$ coincide with the sets $I(\hat{\mathfrak{I}}(\overline{\chi}))$ where $\overline{\chi}$ is the clashing assumption of a named CAS model of $\mathfrak{K}$. With this in place, we show that in case of a multirelational hierarchy, the answer sets of $PK(\mathfrak{K}) \cup P_{pref}$ found optimal by the asprin preference `GlobPref` (implementing P_{pref}) coincide with the sets $I(\hat{\mathfrak{I}}(\overline{\chi}))$ where $\overline{\chi}$ is the clashing assumption of a named preferred CAS model (i.e. CKR model) of $\mathfrak{K}$.

Prototype implementation. The ASP translation presented above is implemented as a proof-of-concept in the CKR*ew (CKR datalog rewriter)* prototype (Bozzato *et al.* 2018a). CKR*ew* is a Java-based command line application that builds on dlv. It accepts as input RDF files representing the contextual structure and local KBs and produces as output a single `.dlv` text file with the ASP rewriting for the input CKR. The latest version of CKR*ew* is available at github.com/dkmfbk/ckrew/releases and includes sample RDF files for $\mathfrak{K}_{org}$ of Example 1.

6 Additional possibilities with algebraic measures

We highlight further fruitful usages of algebraic measures for reasoning with sCKRs.

Preferred model as an overall weight. First, we show another alternative way of obtaining a preferred model as the result of an overall weight query. Formally, we have the following:

Theorem 3
Let $\mathfrak{K}$ be a single-relational, *eval*-free sCKR. Then there exist a semiring $\mathcal{R}_{\text{one}}(\mathfrak{K})$ and weighted formula α_{one} such that the overall weight of $\mu_{\text{one}} = \langle PK(\mathfrak{K}), \alpha_{\text{one}}, \mathcal{R}_{\text{one}}(\mathfrak{K}) \rangle$ is either (I, χ), where I is the minimum lexicographical preferred answer set of $PK(\mathfrak{K})$ and χ is the corresponding clashing assumption map, or $\mathbf{0}$ if there is no preferred answer set.

Here, the lexicographical order $>_{lex}$ over answer sets is given by $I >_{lex} I'$ iff there exists some $b \in B_{PK(\mathfrak{K})}$ such that $b \in I \setminus I'$ and for all $b' <_{var} b$ it holds that $b' \in I$ iff $b' \in I'$, where $<_{var}$ is an arbitrary but fixed total order on $B_{PK(\mathfrak{K})}$.

Intuitively, we define $\mathcal{R}_{\text{one}}(\mathfrak{K})$ by the following strategy. The domain R is the set of all pairs (I, χ), where I is an interpretation of $PK(\mathfrak{K})$ and χ a possible clashing assumption map, and two constants $\mathbf{0}, \mathbf{1}$, which act as the zero and one of the semiring. The multipli-

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cation $\otimes$ of $\mathcal{R}_{\text{one}}(\mathfrak{K})$ is (pointwise) union and can thus be used to build a representation of the interpretation I and its clashing assumption map χ. The addition $\oplus$ corresponds to taking the "more preferred" interpretation or the one which is lexicographically smaller, in case of a tie.

Note that the restriction to *eval*-free sCKRs (or a similar fragment) is necessary: the strategy explained above is only viable if the preference relation over the models is transitive.

Epistemic reasoning using overall weight queries. Using asprin, we can enumerate preferred models. For obtaining all of them at once, we can use an overall weight query.

Theorem 4
Let $\mathfrak{K}$ be a single-relational, *eval*-free sCKR. Then there exists a semiring $\mathcal{R}_{\text{all}}(\mathfrak{K})$ and weighted formula α_{all} such that the overall weight of $\mu_{\text{all}} = \langle PK(\mathfrak{K}), \alpha_{\text{all}}, \mathcal{R}_{\text{all}}(\mathfrak{K})\rangle$ is $(A_{\mathsf{c}})_{\mathsf{c}\in\mathbf{N}}$ and the set of CKR models corresponds to $\{(\mathcal{I}(\mathsf{c}))_{\mathsf{c}\in\mathbf{N}} \mid \text{ for each } c \in \mathbf{N} : (\mathcal{I}(\mathsf{c}), \chi(\mathsf{c})) \in A_{\mathsf{c}}\}$.

The definition of $\mathcal{R}_{\text{all}}(\mathfrak{K})$ is similar to that of $\mathcal{R}_{\text{one}}(\mathfrak{K})$. However, instead of pairs (I, χ) the semiring values here are sets of pairs (I, χ). Given such sets A, B, addition and multiplication select the preferred pairs in the result of the union $A \cup B$ and the "Cartesian" union $\{(S_1 \cup S_2, \chi_1 \cup \chi_2) \mid (S_1, \chi_1) \in A, (S_2, \chi_2) \in B\}$, respectively.

We can use the overall weight $\mu_{\text{all}}(PK(\mathfrak{K}))$ not only to single out all preferred models but also for further advanced tasks. For example, the cautious and brave consequences at context c are obtained by

$$\bigcap\{\mathcal{I}(\mathsf{c}) \mid \mathcal{I}(\mathsf{c}) \in \mu_{\text{all}}(PK(\mathfrak{K}))_{\mathsf{c}}\} \quad \text{respectively} \quad \bigcup\{\mathcal{I}(\mathsf{c}) \mid \mathcal{I}(\mathsf{c}) \in \mu_{\text{all}}(PK(\mathfrak{K}))_{\mathsf{c}}\}.$$

Apart from this, we can also use the result to evaluate epistemic aggregate queries, akin to the ones defined by Calvanese *et al.* (2008), of the form

$$q(\overline{x}, \alpha(\overline{y})) \leftarrow \mathbf{K}\, \overline{x}, \overline{y}, \overline{z}.\phi, [\psi],$$

where ϕ and ψ are conjunctions of possibly nonground atoms and $\overline{x}, \overline{y}, \overline{z}$ are sequences of variables that occur in ϕ, such that $\overline{z}$ is distinct from $\overline{x}$ and $\overline{y}$. Furthermore, α is an aggregation function. The meaning of this expression given a knowledge base $\mathcal{KB}$ is intuitively as follows. For each assignment to $\overline{x}$, we aggregate over all values $\overline{y}$ using α, subject to the constraint that for every model D of $\mathcal{KB}$ the assignment to $\overline{x}, \overline{y}$ can be completed to an assignment γ to all the variables in ϕ and ψ such that (i) ϕ and ψ are satisfied by D w.r.t. γ and (ii) for every model D' of $\mathcal{KB}$ it holds that γ restricted to $\overline{x}, \overline{y}, \overline{z}$ is a certain answer for the query $(*)$ $aux_q(\overline{x}, \overline{y}, \overline{z}) \leftarrow \phi, \psi$. Then, $q(t, z)$ is an answer of the above epistemic aggregate query if it is the result of the query in every model D. For formal details, we refer to the paper by Calvanese *et al.* (2008).

Calvanese *et al.* showed that for "restricted" queries, the value of the aggregate is obtained by

$$q_0(\overline{x}, \overline{y}, \overline{z}^{\phi}) \leftarrow \mathrm{Cert}(aux_q, \mathbf{K})(\overline{x}, \overline{y}, \overline{z}). \qquad q_1(\overline{x}, \alpha(\overline{y})) \leftarrow q_0(\overline{x}, \overline{y}, \overline{z}^{\phi}).$$

Here $\overline{z}^{\phi}$ are the variables of $\overline{z}$ that occur in ϕ and $\mathrm{Cert}(aux_q, \mathbf{K})(\overline{x}, \overline{y}, \overline{z})$ refers to the certain answers of the query $(*)$. Unfortunately, we cannot use ASP alone to compute the certain answers in the presence of defeasible axioms and preferences in sCKRs. However, the overall weight $\mu^*(PK(\mathfrak{K}))$ contains the information necessary to conclude what the certain answers are. These in turn can then be used to evaluate epistemic aggregates over sCKRs.

7 Discussion and conclusions

We considered the application of ASP with algebraic measures for expressing preferences of defeasibility in multirelational CKRs. The problem of representing notions of defeasibility in DLs has led to many proposals and is still an active area of research (Giordano *et al.* 2011; Bonatti *et al.* 2015; Pensel and Turhan 2018; Britz *et al.* 2021). A detailed comparison of justifiable exceptions with other definitions of nonmonotonicity in DLs and contextual systems can be found in the papers by Bozzato *et al.* (2018a; 2019). Our work on CKRs with multiple contextual relations was influenced by approaches dealing with exceptions under different relations or diverse definitions of normality. One of the latest in this direction is the work by Giordano and Dupré (2020), where the notion of typicality in DLs is extended to a "concept-aware multipreference semantics": the domain elements are organized in multiple preference orderings $\leq_C$ to represent their typicality w.r.t. a concept C; models are then ordered by a global preference combining the concept-related preferences. Similar to our approach, entailment is encoded in ASP using a fragment of Krötzsch's (2010) materialization calculus and representing combination of preferences in asprin. Gil (2014) earlier studied the effects of adding multiple preferences to a typicality extension of $\mathcal{ALC}$.

Concerning semirings for general quantitative specifications, several works used semirings to define quantitative generalisations of well-known qualitative problems. For example, *Semiring-based Constraint Satisfaction Problems (SCSP)* (Bistarelli *et al.* 1999) allow for quantitative semantics of CSP's and capture other quantitative extensions of CSP's (weighted CSP) as special cases for some specific semiring. Semiring Provenance (Green *et al.* 2007), generalizes the bag semantics and other definitions of provenance for relational algebra to semirings: this allows one to capture existing quantitative semantics, but also to introduce additional novel capabilities to obtain the provenance lineage of a query. Moreover, algebraic ProbLog (Kimmig *et al.* 2011) introduced an algebraic semantics of logic programs by facilitating semirings. Intuitively, their approach can be seen as a fragment of ASP with algebraic measures allowing only a restricted use of negation in programs and no arbitrary recursive sums and products in the weighted formulas.

The parametrization of semantics with a semiring allows for flexible and highly general quantitative frameworks: in particular, algebraic measures allow for an intuitive specification of computations depending on the truth of propositional variables. Building on ASP, they offer an appealing specification language for quantitative reasoning problems like preferential reasoning.

Outlook. In the direction of using the capabilities of algebraic measures for comparing models, we plan to further study the possibilities for epistemic reasoning on DLs as introduced in previous sections. With respect to contextual reasoning, a possible continuation of this work can consider a refinement of the definitions of preference and knowledge propagation across different contextual relations, possibly by considering a motivating real-world application.

Acknowledgments. This work was partially supported by the European Commission funded project "Humane AI: Toward AI Systems That Augment and Empower Humans by Understanding Us, our Society and the World Around Us" (grant #820437) and the

Austrian Science Fund (FWF) project W1255-N23. The support is gratefully acknowledged.

Supplementary material

To view supplementary material for this article, please visit http://dx.doi.org/10.1017/S1471068421000284.

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