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Exotic Torsion, Frobenius Splitting and the Slope Spectral Sequence

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Abstract. In this paper we show that any Frobenius split, smooth, projective threefold over a perfect field of characteristic $p > 0$ is Hodge–Witt. This is proved by generalizing to the case of threefolds a well-known criterion due to N. Nygaard for surfaces to be Hodge-Witt. We also show that the second crystalline cohomology of any smooth, projective Frobenius split variety does not have any exotic torsion. In the last two sections we include some applications.

1 Introduction

In response to a question by V. B. Mehta, it was shown that there exist examples of Frobenius split varieties of dimension at least three which are not ordinary (see [12]). And if the dimension is bigger than four, then these examples are not even Hodge–Witt. It was also shown in [12] that Frobenius split varieties of dimension at most two are ordinary (and hence Hodge–Witt).

The purpose of the present note is to complement these results by proving that any F-split smooth projective threefold is Hodge–Witt (see Theorem 6.2). Thus our Theorem 6.2 on the Hodge–Witt property of Frobenius split threefolds is the best possible.

Theorem 6.2 is deduced as a consequence of an explicit criterion (see Theorem 6.1) for the degeneration of the slope spectral sequence of any smooth projective threefold, which is of independent interest. This criterion is the generalization to threefolds of the well-known criterion for surfaces due to Nygaard (see [18]). Our criterion together with a finiteness result [12] leads to the proof of Theorem 6.2. This criterion also applies to any unirational or Fano threefolds and we deduce that these are Hodge–Witt (see Theorem 10.1).

Mehta has also raised the following question.

Question 1.1 Suppose X is a smooth, projective and F-split variety, does there then exist a finite étale cover $X' \to X$ such that $H^2_{\mathrm{cris}}(X'/W)$ is torsion free?

In an attempt to answer Mehta's question, we show that the second crystalline cohomology of a Frobenius split variety does not have exotic torsion (see Theorem 7.1). Our result on exotic torsion together with the fact that an étale cover of a Frobenius split variety is again Frobenius split shows that we may ignore exotic torsion in

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Mehta's question. The reader should recall that exotic torsion is the least understood part of crystalline torsion. Our result is the best possible, as a variant of an example of Igusa shows [9] . Igusa's construction gives an example of an F-split variety whose Picard scheme is not reduced, indicating the presence of torsion in the second crystalline cohomology. Our method of proof also shows, more generally, that the vanishing of $H^1(X, B_1\Omega^1_X)$ is sufficient to ensure the vanishing of exotic torsion in the second crystalline cohomology of X. In particular, this applies to ordinary varieties as well, and one deduces that varieties which are ordinary do not have exotic torsion in their second crystalline cohomology.

Section 11 consists of several remarks which provide applications of Theorem 6.1 to algebraic cycles. The remarks of primary interest are 11.3–11.5.

We also include here an unpublished result of Mehta that Frobenius split varieties lift to W_2.

Throughout this paper the following notations will be in force. Let k be an algebraically closed field of characteristic $p > 0$; $W = W(k)$ is the ring of Witt vectors of k; and for all $n \geq 1$, we write $W_n(k) = W/p^n$. Let K be the quotient field of $W(k)$.

2 Frobenius Split Varieties

Let X/k be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. Let $F\colon X \to X$ be the absolute Frobenius morphism of X. Following Mehta and Ramanathan (see [14]), we say that X is Frobenius split (or simply F-split) if the exact sequence

$$0 \to \mathcal{O}_X \to F_*(\mathcal{O}_X) \to B_1\Omega^1_X \to 0, \tag{2.1}$$

splits as a sequence of locally free $\mathcal{O}_X$ modules.

This notion was defined in [14], where a number of principal properties of such varieties were proved. The notion of Frobenius splitting has played an important role in the study of Schubert varieties and algebraic groups since its inception in [14]. In [15], it was shown that an abelian variety is F-split if and only if it is ordinary. In [12], it was shown that any smooth, projective F-split surface is Block–Kato ordinary (for this notion see [3, 11]).

We will say that a smooth projective variety X is a Calabi–Yau variety if the canonical bundle of X is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$. From [14] it follows that a Calabi–Yau variety of dimension $n = \dim(X)$ is Frobenius split if and only if $F\colon H^n(X, \mathcal{O}_X) \to H^n(X, \mathcal{O}_X)$ is injective.

3 The Slope Spectral Sequence

We recall the formalism of de Rham–Witt complexes as developed in [3, 10, 11]. Specifically, let R be the Cartier–Dieudonné–Raynaud ring over k. Recall that this is a graded W-algebra generated by symbols F, V, d that is generated in degree zero by F, V with relations $FV = VF = p$ and $Fa = \sigma(a)F$ for all $a \in W$, and $aV = V\sigma^{-1}(a)$ for all $a \in W$; and R is generated in degree 1 by d with properties $FdV = d$ and

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$d^2 = 0$, $da = ad$ for all $a \in W$. Thus $R = R^0 \oplus R^1$ where R^0 is the usual Cartier–Dieudonné ring and R^1 is an R^0-bimodule generated by d. A graded module over R is a complex of R^0-modules $\cdots \to M^i \to M^{i+1} \to \cdots$ where the differentials satisfy $FdV = d$.

The de Rham–Witt complex on X is a sheaf of graded R-modules on X, denoted by $W\Omega_X^\bullet$. Furthermore, $W\Omega_X^0 = W(\mathcal{O}_X)$ is the sheaf of Witt vectors on X constructed by Serre (see [21]). The cohomology of the de Rham–Witt complex, denoted $H^j(X, W\Omega_X^\bullet)$ in the sequel, inherits the structure of a graded R-module. As was shown in [3, 10], the de Rham–Witt complex computes the crystalline cohomology of X and we also have a spectral sequence (the slope spectral sequence)

$$E_1^{i,j} = H^j(X, W\Omega_X^i) \implies H^{i+j}_{\mathrm{cris}}(X/W).$$

The groups $H^j(X, W\Omega_X^i)$ are not always of finite type over W. We say that X is Hodge-Witt if for all i, j, the groups $H^j(X, W\Omega_X^i)$ are finite type modules over W.

4 Dominoes and Ekedahl's duality

We recall briefly the formalism of dominoes developed in [6, 11]. Let M be a graded R-module over the Cartier–Dieudonné–Raynaud ring; fix $i \in \mathbb{Z}$ and let $V^{-r}Z^i = \{x \in M^i \mid V^r(x) \in Z^i\}$ for any $r \geq 0$, where $Z^i = \ker(d\colon M^i \to M^{i+1})$. These form a decreasing sequence of W-submodules of M^i and we let $V^{-\infty}Z^i = \bigcap_{r\geq 0} V^{-r}Z^i$. Similarly, let $F^\infty B^{i+1} = \bigcup_{s\geq 0} F^s(B^{i+1})$ where $B^{i+1} = \mathrm{Image}(d\colon M^i \to M^{i+1})$. The differential $d\colon M^i \to M^{i+1}$ factors canonically as

$$M^i \to M^i/V^{-\infty}Z^i \to F^\infty B^{i+1} \to M^{i+1}.$$

Let $M = M^0 \to M^1$ be a graded R-module concentrated in two degrees. We say M is a domino if $V^{-\infty}Z^0 = M^0$ and $F^\infty B^1 = M^1$. Any domino has a filtration by sub-R-modules (graded) such that graded pieces are certain standard dominoes (see [11]). The length of the filtration is an invariant of the domino, is equal to $\dim_k M^0/VM^0$, and is called the dimension of the domino M.

Let M be a graded R-module. For all i, the standard factorization of the differential $M^i \to M^i/V^{-\infty}Z^i \to F^\infty B^{i+1} \to M^{i+1}$ gives an associated domino $M^i/V^{-\infty}Z^i \to F^\infty B^{i+1}$. By [11, Proposition 2.18] we have that the differential $d\colon M^i \to M^{i+1}$ is zero if and only if the associated domino is zero dimensional.

In particular, we will write $T^{i,j}$ for the dimension of the domino associated with the differential $d\colon H^j(X, W\Omega_X^i) \to H^j(X, W\Omega_X^{i+1})$.

The key input for our result is the following duality theorem due to T. Ekedahl [6].

Theorem 4.1 (Ekedahl's duality for dominoes) *Let X be a smooth, projective variety of dimension n over a perfect field k and let $T^{i,j}$ be the dimension of the domino associated to the differential $H^j(W\Omega_X^i) \to H^j(W\Omega_X^{i+1})$. Then we have $T^{i,j} = T^{n-i-2,n-j+2}$.*

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5 A Finiteness Result

Let X/k be a smooth projective, Frobenius split variety over k. The following theorem was proved in [12]. Here we give a proof of this fact from a slightly different point of view. We will need this result throughout this paper.

Theorem 5.1 *Let X be as above. Then for all $i \geq 0$ the Hodge–Witt groups $H^i(X, W(\mathcal{O}_X))$ are of finite type W-modules.*

Proof We first show that the differential $H^i(X, W(\mathcal{O}_X)) \to H^i(X, W\Omega^1_X)$ is zero. To prove this, we observe that by [11, Corollary 3.8] it suffices to show $H^i(X, W(\mathcal{O}_X))$ is F-finite. Observe that to prove F-finiteness it suffices to prove $H^i(X, B_n\Omega^1_X)$ is of bounded dimension as n varies. This is where we use the fact that X is F-split. It is immediate (see [12]) from the definition of F-splitting that $H^i(X, B_1\Omega^1_X) = 0$ for all $i \geq 0$. By [10] we have an exact sequence

$$0 \to B_1\Omega^1_X \to B_{n+1}\Omega^1_X \to B_n\Omega^1_X \to 0, \tag{5.1}$$

so the required boundedness follows from this by induction on n.

As there are no de Rham–Witt differentials which map to $H^i(X, W(\mathcal{O}_X))$, we see that $E_2^{1,i} = H^i(X, W(\mathcal{O}_X))$. But by [11] we know that the E_2 terms of the slope spectral sequence are all of finite type W-modules. Hence we see that $H^i(X, W(\mathcal{O}_X))$ is of finite type. ■

6 On the Slope Spectral Sequence of Threefolds

Throughout this section X/k is a smooth projective variety of dimension three. The main aim of this section is to give a fairly explicit criterion, which is the analogue for threefolds of [18], for the degeneration of the slope spectral sequence at E_1.

Theorem 6.1 *Let X/k be a smooth projective threefold. Then the slope spectral sequence of X degenerates at E_1 if and only if for all $i \geq 0$, the groups $H^i(X, W(\mathcal{O}_X))$ are finite type modules over W. In other words, X is Hodge–Witt if and only if $H^i(X, W(\mathcal{O}_X))$ are of finite type for $i \geq 0$.*

Proof By definition we know that when X is Hodge–Witt, the relevant cohomology groups are of finite type over W. So it suffices to prove that X is Hodge–Witt under the assumption that the cohomology groups $H^i(X, W(\mathcal{O}_X))$ are of finite type over W.

We note a number of reductions. To prove that these cohomology groups are of finite type over W, it suffices to show that all the differentials in the slope spectral sequence are zero, because all the terms of the slope spectral sequence from E_2 onwards are all of finite type over W. Next we note that by [10, Corollary 2.17], $H^0(X, W\Omega^i_X)$ is finite type for all $i \geq 0$. By [10, Corollary 2.18], we know that $H^j(X, W\Omega^n_X)$ is of finite type for all $j \geq 0$. By [11, Corollary 3.11], we know that $H^1(X, W\Omega^i_X)$ is of finite

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type over W. Thus we are left with the following cohomologies and differentials:

$$H^2(X, W(\mathcal{O}_X)) \to H^2(X, W\Omega^1_X) \to H^2(X, W\Omega^2_X),$$
$$H^3(X, W(\mathcal{O}_X)) \to H^3(X, W\Omega^1_X) \to H^3(X, W\Omega^2_X).$$

Now by Theorem 5.1, the differentials in the left column are zero. By Ekedahl's duality (see Theorem 4.1) $T^{1,3} = T^{3-1-2,3-3+2} = T^{0,2}$ and $T^{1,2} = T^{3-1-2,3-2+2} = T^{0,3}$, and as the left column of differentials is zero, we have $T^{0,3} = T^{0,2} = 0$. This completes the proof. ■

Corollary 6.2 *Let X be a smooth, projective and F-split threefold. Then X is Hodge–Witt.*

Proof This is immediate from Theorem 5.1 and Theorem 6.1. ■

The structure of crystalline cohomology of Hodge–Witt varieties has been studied in detail in [11], and we recall the following here for the readers convenience.

Theorem 6.3 *Let X be a smooth, projective, Hodge–Witt threefold. Then the crystalline cohomology of X admits a Newton–Hodge decomposition*

$$H^n_{cris}(X/W) = \bigoplus_{i+j=n} H^j(X, W\Omega^i_X)$$

and one has further decomposition into pure and fractional slope parts

$$H^j(X, W\Omega^i_X) = H^{i+j}_{[i]} \oplus H^{i+j}_{]i,i+1[},$$

where the first term has pure slope i and the other term consists of slopes strictly between i and i + 1.

7 Exotic Torsion

In this section we prove that any smooth, projective, Frobenius split variety does not have exotic torsion in its second crystalline cohomology (for the definition of exotic torsion see [10]).

Illusie [10] gave the following devisage of torsion in $H^2_{\text{cris}}(X/W)$ for any smooth projective variety X/k. There is a W-submodule of torsion in $H^2_{\text{cris}}(X/W)$ which is called the divisorial torsion and the quotient of torsion in $H^2_{\text{cris}}(X/W)$ by the divisorial torsion is the exotic torsion. Examples of exotic torsion, while hard to construct, do exist. In some sense, exotic torsion is the geometrically least understood part of torsion in second crystalline cohomology. Divisorial torsion, on the other hand, has geometric manifestation: for instance, part of divisorial torsion (the V-torsion) is zero if and only if the Picard scheme is reduced, while the remaining part of divisorial torsion is the torsion arising from the Neron–Severi group of X via the crystalline cycle class map (see [10]).

It was shown [15] that any smooth, projective, F-split variety with trivial tangent bundle has a finite Galois étale cover which is an abelian variety, and it is standard (see for instance [10]) that the crystalline cohomology of an abelian variety is torsion free.

The main results of this section are Theorem 7.1 and Theorem 7.3.

Theorem 7.1 *Let X/k be a smooth, projective, F-split variety over an algebraically closed field of characteristic $p > 0$. Then the second crystalline cohomology of X does not have exotic torsion.*

Proof We first show that for all $i \geq 0$ we have $F\colon H^i(X, W(\mathcal{O}_X)) \to H^i(X, W(\mathcal{O}_X))$, is an isomorphism. To see this we proceed as follows. We have an exact sequence (see [10])

$$(7.1) \qquad \cdots \to H^{i-1}(X, W(\mathcal{O}_X)/FW(\mathcal{O}_X)) \to H^i(X, W(\mathcal{O}_X)) \to \xrightarrow{F} H^i(X, W(\mathcal{O}_X)) \to H^i(X, W(\mathcal{O}_X)/FW(\mathcal{O}_X)) \to \cdots .$$

So it suffices to show that $H^i(X, W(\mathcal{O}_X)/FW(\mathcal{O}_X))$ is zero. But

$$H^i(X, W(O_X)/FW(\mathcal{O}_X)) = \varprojlim_n H^i(X, B_n\Omega^1_X),$$

by [10, 2.2.2, p. 609]. But as X is F-split by the proof Theorem 5.1, we know that the groups on the right are all zero.

We are now ready to prove Theorem 7.1. Recall that by [10, 6.7.3, p. 643] exotic torsion in $H^2_{\mathrm{cris}}(X/W)$, denoted $H^2_{\mathrm{cris}}(X/W)_e$, is the quotient

$$H^2_{\mathrm{cris}}(X/W)_e = \frac{Q^2}{H^2(X, W(\mathcal{O}_X))_{V\text{-}\mathrm{tor}}},$$

where $Q^2 = \mathrm{Image}(H^2_{\mathrm{cris}}(X/W)_{\mathrm{Tor}} \to H^2(X, W(\mathcal{O}_X))$.

Thus to prove our result it will suffice to show that all the p-torsion in $H^2(X, W(\mathcal{O}_X))$ is also V-torsion. But we have seen that F is an automorphism of $H^2(X, W(\mathcal{O}_X))$ and the relation $FV = p$ then shows that $V = F^{-1}p$. So all the p-torsion is V-torsion. This finishes the proof. ■

Corollary 7.2 *Let X be a smooth, projective Frobenius split variety over a perfect field k. Then* $\mathrm{Pic}(X)$ *is reduced if and only if* $H^2(X, W(\mathcal{O}_X))_{\mathrm{Tor}} = 0$

Proof It is clear from [10] that $\mathrm{Pic}(X)$ is reduced if and only if $H^2(X, W(\mathcal{O}_X))$ does not contain V-torsion. But by the proof of Theorem 7.1, we see that all the p-torsion of $H^2(X, W(\mathcal{O}_X))$ is V-torsion, hence the assertion. ■

The method of proof of Theorem 7.1 can be distilled to provide the following sufficient condition for the vanishing of exotic torsion in any smooth projective variety.

Theorem 7.3 *Let X be a smooth, projective variety over a perfect field k of characteristic $p > 0$. Suppose that $H^1(X, B_1\Omega^1_X) = 0$. Then $H^2_{\mathrm{cris}}(X/W)$ does not have exotic torsion.*

Proof From the proof of Theorem 7.1, it is clear that if F is injective on $H^2(X, W(\mathcal{O}_X))$, then X has no exotic torsion. So it suffices to verify that the F-torsion is zero under our hypothesis. By [10, Corollary 3.5; Corollary 3.19], we get

$$0 \to H^1(W(\mathcal{O}_X))/F \to H^1(W(\mathcal{O}_X)/F) \to {}_F H^2(W(\mathcal{O}_X)) \to 0,$$

where the F-torsion of $H^2(X, W(\mathcal{O}_X))$ (the term on the extreme right in the above equation) is the image of $H^1\big(X, W(\mathcal{O}_X)/FW(\mathcal{O}_X)\big)$. But by [10, Corollary 2.2], we know that $H^1(X, W(\mathcal{O}_X)/FW(\mathcal{O}_X)) = \varprojlim_n H^1(X, B_n\Omega^1_X)$. By our hypothesis $H^1(X, B_1\Omega^1_X) = 0$, by induction on n using (5.1), one sees that $H^1(X, B_n\Omega^1) = 0$ for all n so that the inverse limit is zero and hence the F-torsion of $H^2(X, W(\mathcal{O}_X))$ is zero. ∎

Remark 7.4 It is easy to see that the exotic torsion may be zero while

$$H^1(X, B_1\Omega^1_X) \neq 0.$$

The standard example of this phenomenon is when X is a supersingular $K3$ surface, where one knows that $H^1(X, B_n\Omega^1_X)$ is n-dimensional for $n \geq 1$, but $H^2_{\mathrm{cris}}(X/W)$ is torsion free while $H^2(X, W(\mathcal{O}_X))$ is F-torsion.

8 Ordinariness of the Albanese Scheme

In this section we prove that the reduced Picard scheme of a Frobenius split variety is ordinary. This result can be proved by other methods as well, but we present a proof here using the methods of the previous sections. As the Albanese variety is the dual of the reduced Picard scheme of X, we state the results in terms of $\mathrm{Pic}^0(X)_{\mathrm{red}}$.

Theorem 8.1 *Let X/k be a smooth, projective, F-split variety and assume that k is algebraically closed. Then for all $i \geq 0$, the natural map*

$$H^i_{et}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \to H^i(X, W(\mathcal{O}_X)) \tag{8.1}$$

is an isomorphism.

Proof By [11, Corollary 3.5] we have the following exact sequence

$$0 \to H^i(X, W\Omega^0_{\log}) \to H^i(X, W(\mathcal{O}_X)) \xrightarrow{1-F} H^i(X, W(\mathcal{O}_X)) \to 0,$$

and by definition (see [11, p. 119]), we know that $H^i(X, W\Omega^0_{\log}) \simeq H^i_{et}(X, \mathbb{Z}_p)$.

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Next we know by Theorem 5.1 (see [12]), that $H^i(X, W(\mathcal{O}_X))$ are all finite type W-modules. Now from [10, Lemma 6.8.4, p. 643] and the fact that F is an automorphism on $H^i(X, W(\mathcal{O}_X))$, we see that the natural map

$$\ker(1 - F) \otimes_{\mathbb{Z}_p} W(k) = H^i(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \to H^i(X, W(\mathcal{O}_X))$$

is an isomorphism, and this finishes the proof. ■

Theorem 8.2 *Let X be any smooth, projective, F-split variety and let $A = \mathrm{Pic}(X)^0_{\mathrm{red}}$. Then A is an ordinary abelian variety.*

Proof If $A = 0$, then there is nothing to prove, so we will assume that A is non-zero. We show that the F-crystal $(H^1_{\mathrm{cris}}(X/W) \otimes K, F)$, which is the Dieudonné module of the reduced Picard variety of X, has only two slopes $0, 1$. This together with duality shows that the abelian variety A is ordinary. So it remains to check the statement about slopes. From Theorem 8.1 we know that

$$\ker(1 - F) \otimes_{\mathbb{Z}_p} W(k) = H^1(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \to H^1(X, W(\mathcal{O}_X))$$

is an isomorphism. The space on the right is the part of $H^1_{\mathrm{cris}}(X/W)$ on which Frobenius acts through slopes $[0, 1[$, and the cohomology on the left is the unit root sub crystal (*i.e.*, part of the cohomology on which Frobenius operates via slopes zero) of $H^1_{\mathrm{cris}}(X/W)$ (see [10, p. 627, §5.4]). Thus we see that the slope zero part of the Frobenius is isomorphic to the part with slopes between $[0, 1[$. So the only possible slopes for the F-crystal $(H^1_{\mathrm{cris}}(X/W), F)$ are zero and one, which proves the assertion. ■

9 Liftings to W_2; Hodge–de Rham Spectral Sequence

We collect together a few facts about F-split varieties. The following consequence of F-splitting and Deligne–Illusie technique (see [5]) is due to unpublished work of V. B. Mehta.

Theorem 9.1 *Let X be a smooth, projective, F-split variety over an algebraically closed field k of characteristic $p > 0$. Then for all $i + j < p$, the Hodge to de Rham spectral sequence degenerates at $E_1^{i,j}$. In particular, for $i + j = 1$ we have the following exact sequence*

$$0 \to H^0(X, \Omega^1_X) \to H^1_{\mathrm{DR}}(X/k) \to H^1(X, \mathcal{O}_X) \to 0. \tag{9.1}$$

Moreover, any F-split variety with $\dim(X) < p$ satisfies Kodaira–Akizuki–Nakano vanishing.

Proof After [5] we only need to check that X admits a proper flat lifting to $W_2(k)$. As was noted in [23], the obstruction to lifting a variety to $W_2(k)$ is the image of the obstruction to lifting the pair (X, F) to $W_2(k)$, under the connecting homomorphism

$$\mathrm{Ext}^1(\Omega^1_X, F_*(\mathcal{O}_X)) \to \mathrm{Ext}^1(\Omega^1_X, B_1\Omega^1_X) \to \mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X).$$

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But F-splitting implies that $\mathrm{Ext}^1(\Omega^1_X, B_1\Omega^1_X)$ is a direct summand of $\mathrm{Ext}^1(\Omega^1_X, F_*(\mathcal{O}_X))$. This implies that the obstruction to lifting X to a flat scheme over $W_2(k)$, which is an element of $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$, is zero. ■

Corollary 9.2 *Let X/k be a smooth, projective, Frobenius split variety. Then X admits a flat lifting to W_2.*

Remark 9.3 I am indebted to John Millson for pointing out that Bogomolov has proved that the deformation theory of Calabi–Yau varieties over complex numbers is unobstructed. The above result implies that any smooth projective, Frobenius split, Calabi–Yau variety lifts to W_2.

10 Unirational and Fano Threefolds

In this section we give an application of our criterion for the degeneration of the slope spectral sequence to smooth, projective unirational or Fano threefolds.

Let X be a smooth projective variety of dimension n. We say X is Fano if Ω^n_X is the inverse of an ample line bundle. In characteristic zero, if X is a Fano variety of dimension n, then the Kodaira vanishing theorem gives $H^i(X, \mathcal{O}_X) = 0$ for $0 < i \leq n$. This is not known in positive characteristic without some additional hypothesis.

Let X be a smooth projective variety. We will say that X is unirational if there exists a dominant surjective, generically finite and separable morphism $\mathbb{P}^n \to X$.

Theorem 10.1 *Let X be a smooth, projective, unirational or Fano threefold over a perfect field k. Then X is Hodge–Witt.*

Proof The proof is inspired by [17]. By Theorem 6.1 we need to check that for $i \geq 0$, $H^i(X, W(\mathcal{O}_X))$ are of finite type when X is Fano or a unirational variety. In both the cases we are reduced, by induction, to using the exact sequence

$$0 \to W_n(\mathcal{O}_X) \to W_{n+1}(\mathcal{O}_X) \to \mathcal{O}_X \to 0 \tag{10.1}$$

to proving that $H^i(X, W_1(\mathcal{O}_X)) = H^i(X, O_X) = 0$ for $i > 0$.

In [17] it was shown that $H^i(X, \mathcal{O}_X)$ is zero for unirational threefolds. In the case when X is a Fano threefold, it was proved in [22] that for any smooth Fano threefold, we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. ■

Remark 10.2 Using the universal coefficient theorem and Nygaard's result [17], it is easy to see that $H^2_{\mathrm{cris}}(X/W)$ is torsion free if X is a smooth, projective, unirational variety with $\pi_1^{\mathrm{alg}}(X) = 0$. When X is a unirational threefold, we can drop the assumption that X is simply connected, as Nygaard has proved that unirational threefolds are simply connected. Hence one deduces (via [10]) that the Neron–Severi group of X has no p-torsion if X is a unirational threefold. This is the characteristic $p > 0$ variant of the corresponding result of J.-P. Serre in characteristic zero (see [21]).

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11 Some Remarks and Applications

The purpose of this section is to provide some applications of a different nature of the results of Section 6 in conjunction with the results of Gros and Suwa (see [7, 8, 24]). The nature of these remarks is, no doubt, elementary but it is the author's belief that they are not altogether content free.

We need to recall a few facts about Abel–Jacobi mappings of Bloch and Gros–Suwa (see [1,7]). First suppose that $\ell \neq p$ is a prime. Let $\mathrm{CH}^i(X_{\bar{k}})$ be the group of algebraic cycles of codimension i on $X_{\bar{k}}$ modulo rational equivalence and let $\mathrm{CH}^i(X_{\bar{k}})_{\mathrm{alg}}$ be the group subgroup of cycles which are algebraically equivalent to zero. Let

$$\lambda_i\colon\ \mathrm{CH}^i(X_{\bar{k}})_{\ell\text{-}\mathrm{tor}} \to H^{2i-1}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

be the ℓ-adic Abel–Jacobi mapping defined by Bloch (see [1]).

When $\ell = p$, we will use a variant of this map

$$\lambda_i'\colon\ \mathrm{CH}^i(X_{\bar{k}})_{p\text{-}\mathrm{tor}} \to H^{2i-1}(X_{\bar{k}}, \mathbb{Q}_p/\mathbb{Z}_p(i))$$

which was constructed by Gros and Suwa in [7,8]. The target of λ_i' is defined in terms of logarithmic cohomology groups of X.

For $\ell \neq p$ it is known that λ_2 is injective (see [4, 16]). This injectivity assertion is also valid for the p-adic Abel–Jacobi mapping λ_2' and is proved in [7].

Remark 11.1 Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$ and let $\ell \neq p$ be a prime. Suppose that for some $i \geq 1$, the cohomology group $H^{2i-1}(X, W(\mathcal{O}_X)) \otimes K \neq 0$. Then the mapping

$$\lambda_i\colon\ \mathrm{CH}^i(X_{\bar{k}})_{\mathrm{alg},\ell\text{-}\mathrm{tor}} \to H^{2i-1}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

is not surjective. To prove this we proceed as follows. Suwa [24, Corollary 3.4] showed that if $H^{2i-1}_{cris}(X/W)$ has slopes outside the interval $[i-1, i]$, then the mapping λ_i is not surjective. Our hypothesis shows that X has a nonzero slope zero part in $H^{2i-1}_{\mathrm{cris}}(X/W)$.

Remark 11.2 Let X be a smooth, projective, Frobenius split variety with

$$H^{2i-1}(X, W(\mathcal{O}_X)) \otimes K \neq 0.$$

Then $\lambda_i\colon\ \mathrm{CH}^i(X_{\bar{k}})_{alg,\ell\text{-}\mathrm{tor}} \to H^{2i-1}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ is not surjective for any $\ell \neq p$.

Remark 11.3 If X is smooth projective Calabi–Yau, Frobenius split variety of dimension $2n-1$, for $n \geq 2$, then $\lambda_n\colon\ \mathrm{CH}^n(X_{\bar{k}})_{\mathrm{alg},\ell\text{-}\mathrm{tor}} \to H^{2n-1}(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ is not surjective. Indeed, it is easy to see by Theorem 5.1 and the fact that X is Frobenius split, Calabi–Yau variety of dimension $2n-1$ that $H^{2n-1}(X, W(\mathcal{O}_X))$ is nonzero and of finite type.

We say that a p-torsion abelian group is of finite cotype if its Pontryagin dual is direct sum of finite number of copies of $\mathbb{Z}_p$ and finite group.

Remark 11.4 Let X be a smooth projective threefold over a perfect field k of characteristic $p > 0$. Assume that $H^3(X, W(\mathcal{O}_X))$ is of finite type over W. Then

$$\mathrm{CH}^2(X_{\bar{k}})_{p\text{-}\mathrm{tor}}$$

is an abelian group of finite cotype. To see this one argues as follows. By the hypothesis and Theorem 6.1, we see that the differential $H^2(X, W\Omega^1_X) \to H^2(X, W\Omega^2_X)$ is zero. Then we are done by [7, Corollary 3.7]. In particular we also deduce the following.

Remark 11.5 If X is a smooth, projective, Frobenius split threefold over a perfect field k of characteristic $p > 0$, then $\mathrm{CH}^2(X_{\bar{k}})_{p\text{-}\mathrm{tor}}$ is of finite cotype.

Remark 11.6 Let $f(x_0, x_1, x_3, x_4) = 0$ be a hypersurface in $\mathbb{P}^4$. Assume that the coefficient of $(x_0x_1x_2x_3x_4)^{p-1}$ is non-zero in f^{p-1}. Then this hypersurface is F-split. If $f = 0$ defines a smooth subvariety of $\mathbb{P}^4$, then we can apply Remarks 11.1 and 11.4 to the Chow groups of $f = 0$.

Remark 11.7 We can also apply this consideration to some singular varieties. Especially to the quintic variety, given by $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 5x_0x_1x_2x_3x_4$, which was first investigated by Schoen [19]. This quintic is singular with ordinary double points, but when $p \equiv 1 \bmod 5$, it is Frobenius split, and using a criterion of [13] one checks easily that its desingularization is Frobenius split. Using Theorem 6.1 we deduce that its Chow group of codimension two cycles has p-torsion of finite cotype and using the usual formula for Chow groups of blowups we deduce that when $p = 1 \bmod 5$ the group of codimension two cycles on Schoen variety has p-torsion of finite cotype.

Remark 11.8 These results together with the conjectures of Bloch (see [2]) indicate the presence of a non-trivial filtration on Chow groups of Frobenius split varieties and the non-degeneracy of the coniveau filtration on these varieties. Over the algebraic closure of a finite field one expects the Abel–Jacobi mappings to be surjective (this has been verified by Schoen [20] under the rubric of the Tate conjecture).

Remark 11.9 Over the algebraic closure of a finite field, our results, together with the conjectural yoga of slopes, as expounded by Bloch (see [2]), and Schoen's work suggest that the Griffiths group of Frobenius split threefolds is non-trivial. We note that we do not know how to produce explicit cycles on Frobenius split threefolds, except possibly in a small number of cases.

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