Figure 1 shows a unit quarter circle in the first quadrant with the lines $y = 1$ and $y = tx$, where $t > 0$. Now C is the point $\left(\frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}}\right)$ and $\angle COA = \tan^{-1} t$. Now we have the area inequality

$$2\left[\Delta OBC\right] < 2\left[\text{Sector } OBC\right] < 2\left[\Delta OBD\right],$$

and hence

$$\frac{1}{\sqrt{1+t^2}} < \frac{\pi}{2} - \tan^{-1} t < \frac{1}{t}.$$

10.1017/mag.2023.67

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107.22 Quick proofs of two inequalities related to the digamma function

We begin with some standard facts and notations which indicate the context in which we are working. References [1, Chapter 2] and [2, p. 334] give the assumed formula (3) and its consequence (4). Let n be a positive integer and consider the harmonic number

$$H_{n-1} = \sum_{j=1}^{n-1} \frac{1}{j}. \tag{1}$$

Recall the Euler-Mascheroni constant $\gamma = \lim_{n\to\infty} \gamma_n$, where

$$\gamma_n = \int_1^n \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) dx = \sum_{j=1}^{n-1} \left(\frac{1}{j} - \int_j^{j+1} \frac{1}{t} dt\right) = H_{n-1} - \log n. \tag{2}$$

We consider the gamma function $\Gamma(t)$ as a function of the positive real number t. We assume that the digamma function, i.e. the derivative of the log of the gamma function, is represented by the formula

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} = -\gamma + \sum_{j=0}^{\infty}\left(\frac{1}{j+1} - \frac{1}{j+t}\right) = -\gamma + \sum_{j=0}^{\infty} \frac{t-1}{(j+1)(j+t)}. \tag{3}$$

This series is uniformly convergent for t bounded away from zero. It is noteworthy, as well as obvious from (3), that

$$\psi(n) = -\gamma + H_{n-1}.$$

Thus, $\psi(t)$ interpolates the sequence $-\gamma + H_{n-1}$ and $\log t$ interpolates the sequence $-\gamma_n + H_{n-1}$. Equation (3) also implies

$$\psi'(t) = \sum_{j=0}^{\infty} \frac{1}{(t + j)^2}, \tag{4}$$

which shows that ψ is strictly increasing for all $t > 0$.

The inequalities alluded to in the title above are stated in parts (i) and (ii) of the Proposition below. The work of Louis Gordon [3] motivated this paper. G. J. O. Jameson [4] has also proved the inequalities stated in our Proposition and given bounds on the differences defined by the inequalities.

The authors thank the Referee for the elegant proof of Lemma 1.

Lemma 1:

(i) For all $t > 0$, $\log t > \psi(t)$.

(ii) $\lim\limits_{t \to \infty} \left(\log t - \psi(t)\right) = 0$.

Proof:

(i) Since $\log\left(\Gamma(t + 1)\right) - \log\left(\Gamma(t)\right) = \log t$, the mean-value theorem implies that there exists ξ such that $t < \xi < t + 1$ and $\psi(\xi) = \log t$. Therefore, since $\psi(t)$ is a strictly increasing function, $\psi(t) < \log t < \psi(t + 1)$, which proves (i).

(ii) From $\log\left(\Gamma(t+1)\right) - \log\left(\Gamma(t)\right) = \log t$, by differentiating, we get $\psi(t+1) - \psi(t) = 1/t$. Thus, we have $\log t - \psi(t) < \psi(t+1) - \psi(t) = 1/t$, which implies (ii).

Lemma 2:

$$\psi'(t) - \frac{1}{t} - \frac{1}{2t^2} = \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{(t + j)^2 (t + j + 1)^2}. \tag{5}$$

Proof: We apply the partial fraction decomposition

$$\frac{1}{x^2 (x + 1)^2} = \frac{2(x + 1) + 1}{(x + 1)^2} - \frac{2x - 1}{x^2}$$

with $x = t + j$ to get the parenthesized middle of (6). By telescoping we obtain

$$\frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{(t+j)^2 (t+j+1)^2} = \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{2(t+j+1)+1}{(t+j+1)^2} - \frac{2(t+j)-1}{(t+j)^2}\right)$$

$$= \sum_{j=0}^{\infty} \frac{1}{(t+j)^2} - \frac{1}{t} - \frac{1}{2t^2}. \tag{6}$$

Use (4) to complete the proof of (5).

Lemma 3: For all $t > 0$

https://doi.org/10.1017/mag.2023.68 Published online by Cambridge University Press

$$\frac{1}{6t^3} > \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{(t+j)^2(t+j+1)^2}. \tag{7}$$

Proof: Using the equalities

$$\begin{aligned}(t+j+1)^3 - (t+j)^3 &= 3(t+j)^2 + 3(t+j) + 1\\ &= 3(t+j)(t+j+1) + 1\end{aligned}$$

we get

$$\begin{aligned}\frac{1}{6t^3} &= \frac{1}{6}\sum_{j=0}^{\infty}\left(\frac{1}{(t+j)^3} - \frac{1}{(t+j+1)^3}\right) = \frac{1}{6}\sum_{j=0}^{\infty}\frac{3(t+j)^2+3(t+j)+1}{(t+j)^3(t+j+1)^3}\\ &= \frac{1}{2}\sum_{j=0}^{\infty}\frac{1+(3(t+j)(t+j+1))^{-1}}{(t+j)^2(t+j+1)^2}\end{aligned}$$

or

$$\frac{1}{6t^3} = \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{(t+j)^2(t+j+1)^2} + \frac{1}{6}\sum_{j=0}^{\infty}\frac{1}{(t+j)^3(t+j+1)^3}, \tag{8}$$

which implies (7).

Proposition:

(i) For all $t > 0$, $\psi(t) < \log t - \frac{1}{2t}$;

(ii) For all $t > 0$, $\psi(t) > \log t - \frac{1}{2t} - \frac{1}{12t^2}$.

Proof: (i) Equation (5) implies that, for all $t > 0$, $\psi'(t) - \frac{1}{t} - \frac{1}{2t^2} > 0$. Therefore, using Lemma 1(ii) we obtain

$$\begin{aligned}0 < \int_t^{\infty}\left(\psi'(x) - \frac{1}{x} - \frac{1}{2x^2}\right)dx &= \left[\psi(x) - \log x + \frac{1}{2x}\right]_t^{\infty}\\ &= -\left(\psi(t) - \log t + \frac{1}{2t}\right).\end{aligned}$$

This proves (i).

(ii) By subtracting (7) from (5) we obtain $\psi'(t) - \frac{1}{t} - \frac{1}{2t^2} - \frac{1}{6t^3} < 0$. Therefore, $-\left(\psi(t) - \log t + \frac{1}{2t} + \frac{1}{12t^2}\right) < 0$, so (ii) is proved.

Corollary:

(i) $\psi(t) - \log t + \frac{1}{2t} = \frac{1}{2}\sum_{j=0}^{\infty}\int_t^{\infty}\frac{dx}{(x+j)^2(x+j+1)^2}$.

https://doi.org/10.1017/mag.2023.68 Published online by Cambridge University Press

(ii) $$\psi(t) - \log t + \frac{1}{2t} + \frac{1}{12t^2} = -\frac{1}{6}\sum_{j=0}^{\infty}\int_t^{\infty}\frac{dx}{(x+j)^3(x+j+1)^3}.$$

Proof: Equation (5) implies (i) and the difference, (5) – (8), implies (ii).

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10.1017/mag.2023.68

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107.23 Location of the inarc circle and its point of contact with the circumcircle

The inarc circle of a triangle

An *inarc circle* of a triangle is a circle tangent to two sides of a triangle and internally to the circumcircle of the triangle, see Figure 1. In this note we consider first the interesting problem of locating the *inarc centre*, the centre of this circle, L_A, and then as a second problem we locate the point of tangency T of the inarc circle and the circumcircle. In [1] the first problem is solved geometrically by beautiful application of inversion. We will use simple algebra, one well-known theorem and one famous formula.

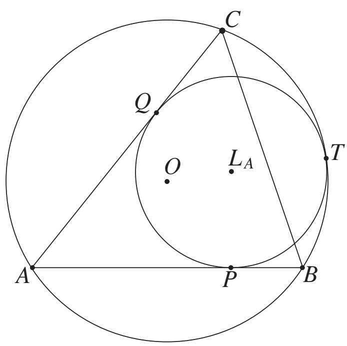

FIGURE 1: An inarc circle