A FREQUENCY DISTRIBUTION METHOD FOR VALUING AVERAGE OPTIONS

By Edwin H. Neave

School of Business,
Queen's University,
Kingston, Ontario

Abstract

This paper finds payoff frequency distributions for valuing European and American fixed strike average options on a discrete time, recombining multiplicative binomial asset price process. In comparison to other discrete valuation methods the distributions, obtained analytically from a generating function, greatly reduce the computational requirements needed for accurate valuation. Less data are needed to value geometric than arithmetic averages, but the magnitude of calculations is similar for both instruments. Calculations of order T^3 are needed to value European instruments, of order T^4 to value their American counterparts. A frequency distribution of a quantity called *path sums* is used to value geometric average options, and a joint distribution of path sums and realized prices is used to value arithmetic average options. The frequency distributions give an exact value for geometric average, an approximate value for arithmetic average instruments. The method obtains additional information from the generating function to estimate approximation errors relative to the exact binomial solution. If the errors are significant they can be reduced using still further detail from the generating function. Error reduction can be performed selectively to minimize additional calculation.

Keywords

Binomial models; options pricing; average options; Gaussian binomial coefficients; numerical methods.

1. Introduction

Average options are instruments whose payoff depends on the average price of the underlying asset, determined over a prespecified period. The averages may be either arithmetic or geometric, and there are both fixed strike and floating strike average options, with the former being by far the most common. Jarrow and Turnbull (1996: 651, hereafter JT) state that average options are used in foreign exchange and commodity trading as well as in interest rate contracts. Commodity based options are written on such assets as oil or aluminium. JT note that the use of an average price reduces an option's sensitivity to price changes in the underlying commodity, especially to price changes occurring at or near contract maturity. Reduced sensitivity to prices can prove especially important in the case of illiquid commodities.

Because of their path dependence, average options are generally regarded as difficult to value, despite the following considerable progress. European fixed strike geometric average options have known analytic solutions for both continuously and discretely determined averages. Valuing European arithmetic average options on a continuous time process is more difficult, mainly because the conventional choice of process is a geometric diffusion for which the distribution of prices' arithmetic averages is not lognormal. Nevertheless, analytic solutions for European arithmetic average options have been found for continuously determined averages by Yor (1992), and Geman and Yor (1993); Geman and Eydeland (1995) report computational experience with these methods.

There are no analytic solutions for continuous time models with discrete averaging, although Turnbull and Wakeman (1991), Levy (1991), and Curran (1992) offer approximate solutions. Neither have discrete time models been studied analytically. Hull and White (1993) approximately value arithmetic average instruments on a binomial process. Neave (1993) uses a binomial model to calculate values for European and American arithmetic average options. Ho (1992) and Tilley (1993) propose simulation with bundling techniques for reducing calculations, and Tilley uses his approach to value both European and American average options. Ritchken, Sankarasubramanian and Vijh (1993) approximately value European arithmetic average options with up to 64 reset points, American options for up to 16 reset points, and compare their approximations to values obtained by simulation.

Methods such as Turnbull and Wakeman's are sufficiently accurate for processes with an annual volatility of 0.40 or less, but some price processes (e.g., those for aluminum and crude oil) exhibit higher volatilities. Moreover, for a fixed number of time periods T convergence of approximate to exact values becomes slower as volatility increases.This paper reduces the computational tasks in valuation for any volatility. It both offers new approximation methods with greater accuracy than those in the literature, and shows how the approximations can be amended to find exact valuations. It achieves these goals by organizing the data along lines indicated by a generating function.

While less data are needed to value geometric than arithmetic average instruments, in both cases the calculations are of order T^3 for European instruments, of order T^4 for American instruments, where T is the number of time periods. (Both n and T are used to denote time in the literature; T is used in at least two recent texts.) The calculations employ sets of paths called *bundles*, where a bundle is defined as the set of all possible paths of the same length and having a common end point. Each bundle can be broken into *sub-bundles*, where a sub-bundle consists of the paths in a bundle that have the same *path sum*, the latter being defined as the sum of path price indices. The number of paths in each sub-bundle is described by the so-called Gaussian binomial coefficients, for which

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

analytic formulae are available. Distributions of path sums can be used to value European and American geometric average options exactly, the latter by recursive methods.

Another description of path characteristics, the joint frequency distribution of path sums and realized prices, is obtained from the same generating function in this paper. The joint distribution can be used to obtain good approximate values of both European and American arithmetic average options. In the European case, exact solutions can be found from the approximations with relatively little additional computing. Further experimentation is needed to determine the best way of refining the approximations to obtain exact values of the American options.

The methods can be applied to a variety of options, but for illustrative purposes the paper only values fixed strike average calls. The discussion is organized as follows. Section 2 specifies the asset price process and defines the options. Section 3 describes the problem structure, defines the generating function, and specifies the frequency distributions. Section 4 values a European and an American geometric average call. Section 5 values the corresponding arithmetic average calls; Section 6 concludes. Appendices detail some features of the methods.

2. The Price Process and the Options

This section defines the price process and formulates European call valuation problems. A recursive form of the European valuation problem is developed to show how bundling methods can be extended to value American as well as European instruments.

2.1. The Process and its Averages

Let $S_0 = 1$, and define $\{S_t\}$, the asset price process, by:

$$S_t = US_{t-1}; \tag{2.1}$$

where for $t \in \{1, 2, ..., T\}$,

$$U = \begin{cases} u; & p \\ u^{-1}; & q \end{cases}$$

with $u > 1$. The realized price cannot become negative, and remains finite for finite values of T and u. Cox and Rubinstein (1985) show that one continuous time limit of the binomial process is the lognormal; Feller (1957) provides parameter values for which the limiting distribution is the Poisson.

It is helpful to rewrite (2.1) as

$$S_t = u^{J_t};\ J_t \equiv \sum_{s=1}^{t} X_s;\ t = 1, 2, ..., T \tag{2.2}$$

where the X_s, $s = 1, 2, \ldots, T$ are independent, identically distributed random variables:

$$X_s \equiv \begin{cases} 1; & p \\ -1; & q. \end{cases}$$

The values J_t, $t = 0, 2, \ldots, T$ are called *node* values. Since $S_0 \equiv 1$, $J_0 \equiv 0$. The cumulative sums of node values

$$V_t \equiv \sum_{s=0}^{t} J_s = \sum_{s=0}^{t} (t-s)\, X_{s+1} \tag{2.3}$$

are called *path sums*. Define the process averages, geometric and arithmetic respectively, by

$$\begin{aligned} G_t &\equiv \left[\prod_{s=0}^{t} S_s\right]^{1/(t+1)} \\ &= \left[\prod_{s=0}^{t} u^{J_s}\right]^{1/(t+1)} \\ &= u^{V_t/(t+1)} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} H_t &\equiv \left[\sum_{s=0}^{t} S_s\right] / (t+1) \\ &= \left[\sum_{s=0}^{t} u^{J_s}\right] / (t+1) \end{aligned} \tag{2.5}$$

Given u, (2.4) shows that the V_t are needed to determine geometric averages, while (2.5) shows that the J_s; $s = 0, \ldots, t$ are needed to determine arithmetic averages.

2.2. Standard Indices

It is convenient to represent the possible outcomes of (2.1) as in Figure 1. For $T = 4$, the Figure arrays successive periods' outcomes along the main diagonals, starting with $t = 0$ in the lower left hand corner. Price increases are represented by upward moves, decreases by horizontal moves to the right.

$$
\begin{array}{ccccc}
u^4 & & & & \\
u^3 & u^2 & & & \\
u^2 & u^1 & u^0 & & \\
u^1 & u^0 & u^{-1} & u^{-2} & \\
u^0 & u^{-1} & u^{-2} & u^{-3} & u^{-4}
\end{array}
$$

FIGURE 1: Original Indexing.

Price paths can be described either by a sequence of realized prices or, as Figure 1 and the subsequent discussion suggest, by the timing and signs of their first differences. For example, the path with price indices

$$0 - 1 - 2 - 3 - 4$$

is described by the vector of first differences

$$(X_1 \;\; X_2 \; X_3 \; X_4)' = (-1 \;\; -1 \;\; -1 \;\; -1)';$$

while the path with indices

$$0 \; 1 \; 0 - 1 - 2$$

is described by

$$(X_1 \;\; X_2 \; X_3 \; X_4)' = (+1 \;\; -1 \;\; -1 \;\; -1)'.$$

The respective path sums –10 and –2 can be calculated either from the node values or, using (2.3), from the X_t. *Using* (2.3), the path sum for the second path above is

$$4 - 3 - 2 - 1 = -2.$$

Information regarding paths and path sums can be determined systematically from a generating function that recognizes the sign and timing of first differences. For example, when $t=4$ a suitable generating function is

$$f_4^0(y, w) = \prod_{j=1}^{4} \left(y^{-1}w^{-j} + yw^j\right) =$$

$$y^{-4}w^{-10} + y^{-2}\left(w^{-8} + w^{-6} + w^{-4} + w^{-2}\right) \tag{2.6}$$

$$+y^0\left(w^{-4} + w^{-2} + 2w^0 + w^2 + w^4\right)$$

$$+y^2\left(w^2 + w^4 + w^6 + w^8\right) + y^4w^{10}.$$

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

The terms y^{-1} in the first line of (2.6) record the effect on the time 4 price of any differences such $X_t = -1$, while the terms y^1 record the effect on the time 4 price of any differences such that $X_t = 1$, $t \in \{1, 2, 3, 4\}$. The terms w^{5} record the effect on the path sum if $X_{4+1-s} = -1$, while the terms w^5 record the effect on the path sum if $X_{4+1-s} = 1$; $s \in \{1, 2, 3, 4\}$.

Lines 2 through 4 of (2.6) suggest grouping paths according to powers of y. Let a *bundle* $B(t, j)$ be the set of all paths ending at (t, j). For any given bundle defined by (2.6), the associated polynomial in w defines the distribution of path sums: powers of w indicate the values of the sums, coefficients of individual terms indicate the frequencies with which the sums are attained. For later use, let a *sub-bundle* $B(t, j, V)$ be the set of all paths in $B(t, j)$whose indices sum to V. The number of paths in each sub-bundle is given by the coefficients of the appropriate polynomial in w.

Function (2.6) and Figure 1 help both to structure valuation problems and to simplify path descriptions. With regard to structure, Figure 1 indicates that the attainable set of realized indices for paths in $B(4,0)$ is defined by the rectangle with lower left-hand corner at $(0,0)$ upper righthand corner at $(4,0)$. The Figure can be used to verify that $B(4,0)$ consists of $4!/2!2! = 6$ paths, all with the same probability of occurrence, and that the maximal and minimal path sums in $B(4,0)$ are 4 and –4 respectively. Accordingly, the set of possible values for path sums in $B(4,0)$ is

$$-4 \; -2 \; 0 \; 2 \; 4,$$

and (2.6) shows these values respectively occur with the frequencies

$$1 \; 1 \; 2 \; 1 \; 1.$$

With regard to simplifying path descriptions, the paths in a given bundle are distinguished by different orderings of price increases and decreases, but the timing of the increases implies the timing of the decreases. For example, since all paths in $B(4, 0)$ have two increases and two decreases, the path for which $X_1 = X_2 = 1$ must also have $X_3 = X_4 = -1$, from which it follows that the path's node values are

$$0 \; 1 \; 2 \; 1 \; 0.$$

Generalizing the example, the paths in any bundle can be described fully just by specifying the values of s, $s \in \{1, 2, ..., t\}$, for which $X_s = 1$. More formally, path characteristics can be inferred from a standardized process which replaces S_t in (2. 1) with S_t^*, where

$$S_t^* \equiv J_t^*; \; J_t^* \equiv \sum_{s=1}^{t} X_s^*; \tag{2.7}$$

$$X_s^* \equiv \begin{cases} 1; & X_s = 1 \\ 0; & X_s = -1. \end{cases}$$

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

A standardized process for $t = 4$ is displayed in Figure 2.

U^4				
U^3	U^3			
U^2	U^2	U^2		
U^1	U^1	U^1	U^1	
U^0	U^0	U^0	U^0	U^0

FIGURE 2: Standard Indexing.

The generating function for the standardized process is obtained by rewriting (2.6) as:

$$f_4^0(y, w) = \prod_{j=1}^{4} \left(y^{-1}w^{-1} + yw^j\right) = \\ y^{-4}w^{-10} \prod_{j=1}^{4} \left(1 + xv^j\right), \tag{2.8}$$

where $x \equiv y^2$ and $v \equiv w^2$. When using the standardized process the multiplicative constant $x^4 y^{10}$ is ignored and the generating function written in the simpler form

$$\begin{aligned} f_4(x, v) &= \prod_{j=1}^{4} \left(1 + xv^j\right) = \\ & 1 + x\left(v^1 + v^2 + v^3 + v^4\right) \\ & +x^2\left(v^3 + v^4 + 2v^5 + v^6 + v^7\right) \\ & +x^3\left(v^6 + v^7 + v^8 + v^9\right) + x^4v^{10} = \\ & 1 + xv\left[\left(v^4 - 1\right)/(v - 1)\right] + x^2v^3.\left[\left(v^4 - 1\right)\left(v^3 - 1\right)/\left(v^2 - 1\right)(v - 1)\right] \\ & +x^3v^6\left[\left(v^4 - 1\right)/(v - 1)\right] + x^4v^{10} = \\ & \equiv g_{4,0}(v) + xvg_{4,1}(v) + x^2v^3g_{4,2}(v) + \\ & x^3v^6g_{4,3}(v) + x^4g^{10}g_{4,4}(v). \end{aligned} \tag{2.9}$$

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

More generally, the generating function for the standardized process is:

$$f_t(x,\ v) = \prod_{j*=1}^{t} \left(1 + xv^{j*}\right) =$$

$$\sum_{j*=0}^{t} g_{t,j*}(v) v^{j*(j*+1)/2} x^{j*}. \tag{2.10}$$

The functions $g_{t,j*}(v)$ in (2. 10), known as *Gaussian polynomials*, take the form

$$g_{t,j*}(v) = \prod_{k=1}^{j*} \frac{v^{t+1-k} - 1}{v^k - 1};\ 1 \le j^* \le t, \tag{2.11}$$

and $g_{t,0^*}(v) \equiv 1$; cf. Berman and Fryer (1972). The coefficients of the $g_{j^*}(v)$, which can be written as polynomials, define the so-called *Gaussian binomial coefficients*. In the present setting the Gaussian binomial coefficients $g_{t,j^*}(v)$ describe the frequencies of V^*, conditional on $J^* = j^*$. It is clear from comparing (2.6), (2.7) and (2.9) that

$$\begin{aligned} J_t &= 2 \cdot J_t^* - t; \\ V_t &= 2 \cdot V_t^* - t(t+1)/2. \end{aligned} \tag{2.12}$$

The possible values J_t^* are the integers from 0 to t, those of V_t^* the integers from zero to $t(t+1)/2$.

The sub-bundles defined by (2.10) can be used directly to value European average options, but recursion relations between the sub-bundles are needed to value American options. As Section 3 will show in greater detail, the necessary relations can be determined from (2.10)

$$f_t(x,\ v) = f_{t-1}(x,\ v)(1 + xv^t),$$

in conjunction with:

$$g_{t,0} = g_{t-1,\,0};$$

$$g_{t,j} = g_{t-1,\,j} + g_{t-1,\,j-1} v^{t-j},\ j = 1,\ 2,\ \ldots,\ t-1;$$

$$g_{t,\,t} = g_{t-1,\,t-1}$$

2.3. European Fixed Strike Average Calls

The payoff to a European fixed strike average call with exercise date T is

$$C_T \equiv (A_T - K)^+ \tag{2.13a}$$

where A_T is a random variable representing either a geometric or an arithmetic average and X_+ means $max\{X, 0\}$. The geometric average call uses $A_T \equiv G_T$, where G_T is defined by (2.4), and the arithmetic average uses $A_T \equiv H_T$, where H_T is defined by (2.5). Given a probability measure p, the time zero values of the European options are

$$C_0 \equiv R^{-T}E(A_T - K)^+ \tag{2.13b}$$

where E denotes expectation under p and $R^t \equiv (1+r)^t$ indicates the t-period accumulation of \$1 at the single-period risk free interest rate r. Recursive approaches can be used with either a martingale or with objective probability measures; cf. Dixit and Pindyck (1994). Schwartz (1994) discusses the theoretical correctness of using the different measures. In consistency with option pricing theory, we assume no arbitrage opportunities, market completeness, and that transactions costs are zero. Then a unique martingale measure $p^* = (R - u^{-1})/(u - u^{-1})$ can be obtained from the normalized process $S_t^* = S_t/R^t$. The paper uses p rather than p^*, reserving the asterisks to denote standard indexing.

To value the American analogues to (2.13a) and (2.13b), it is convenient first to formulate (2.13b) with the states defined as individual price paths.The methods will then be adapted to find recursions between states defined as path sums. We first number paths according to

$$Z \equiv \sum_{s=1}^{T} X_s^* \cdot 2^{T-s}, \tag{2.14}$$

and note the state variable Z can assume the realized values $z \in \{0, 1, ..., 2^T - 1\}$. Identifying the paths using values of z (2.13b) can be written:

$$C_0(z) \equiv R^{-T}E\{(A(z) - K)^+\}, \tag{2.15}$$

where $A(z)$ indicates an average over path z. There are 2^T possible realized values of Z, making computation infeasible when T is large. The states are later redefined so that for computational purposes it is only necessary to recognize distinct values of order T^3. Since $H(z) \geq G(z)$ for all z, (2.15) immediately confirms the result, first pointed out by Kemna and Vorst (1990), that the value of a European arithmetic average call is never less than the value of the corresponding geometric average call.

A recursive formulation is not needed to solve problem (2.15), but will help relate our methods for valuing European options to those for their American counterparts. Suppose henceforth that the z are arranged in increasing order at time T i.e.,

$$0,\ 1,\ ...,\ 2^T - 1.$$

Examination of Z shows the path numbers are lexicographically ordered by the signs of path first differences. For example, the pair of paths 2^T-1 and 2^T-2 differ in the sign of the first difference taken between times $T-1$ and T. The same is true for the pair 2^T-3 and 2^T-4, and for all remaining pairs of adjacent paths. After the expected value at time $T-1$ is taken over pairs of paths that are adjacent in terms of z, the states then requiring to be distinguished are indicated by

$$z \in \{0,\ 2,\ \ldots,\ 2(2^{T-1}-1)\}.$$

Again adjacent pairs of the remaining paths differ in the sign of what is now the first difference between times $T-1$ and $T-2$. That is, the remaining states are

$$z \in \{0,\ 4,\ \ldots,\ 4(2^{T-2}-1)\}.$$

The process continues until time 0, when the single state denoted by $z = 0$ is reached. The path numbering method is further illustrated in Table 3 below.

Using the relations between values of Z, (2.15) can be written recursively as:

$$C_T(z) \equiv (A_T(z) - K)^+;$$

$$z \in \{0,\ 1,\ \ldots,\ 2^T-1\} \equiv Z_T;$$

$$C_{T-1}(z) \equiv R^{-1}\{pC_T(z+1) + qC_T(z)\};$$

$$z \in \{j \cdot 2 : j = 0,\ 2,\ \ldots,\ 2^{T-1}-1\} \equiv Z_{T-1}. \tag{2.16a}$$

In (2.16a) $C_T(z)$ is the value of the European call at time T if the price path from time 0 time time T is described by z. In general,

$$C_{T-t}(z) \equiv R^{-1}\{pC_{T-t+1}(z+2^{t-1}) + qC_{T-t+1}(z)\}; \tag{2.16b}$$

$$z \in \{j \cdot 2^t : j = 0,\ 1,\ \ldots\ 2^{T-t}-1\} \equiv Z_{T-t}.$$

When $t = T$, (2.16b) defines the time zero call value.

2.4. American Fixed Strike Average Calls

To write the recursion for the American call, (2.16) is amended to recognize the effect of early exercise. Let $D_t(z)$ be the time t value of the call if the price process has followed path z from time 0 to time t:

$$D_T(z) \equiv C_T(z);$$

$$z \in Z_T;$$

$$D_{T-1}(z) \equiv max\left\{(A_{T-1}(z) - K)^+,\ R^{-1}[pD_T(z+1) + qD_T(z)]\right\};$$

$$z \in Z_{T-1};$$

and (2.17)

$$D_{T-t}(z) \equiv max\left\{(A_{T-t}(z) - K)^+,\ R^{-1}\left[pD_{T-t+1}\left(z + 2^{t-1}\right) + qD_{T-t+1}(z)\right]\right\};$$

$$z \in Z_{T-t}.$$

In conformity with the standard result that the value of an American call is never less than that of its European counterpart, equations (2.17) show immediately that $D_t(z) \geq C_t(z)$ for all feasible values of z and t.

Since they recognize 2^T distinct paths, computations based on (2.16) and (2.17) increase exponentially in T. To reduce computation, the rest of this paper defines state variable values as the values defining path sub-bundles. In the American case the paper further determines how sub-bundles at a given time point are related to sub-bundles for the immediately preceding time. This approach reduces the number of calculations to cubic or fourth degree polynomials in T, according to whether European or American options are being valued. (The higher degree of polynomial for American options results from having to repeat the calculations at each of the T stages in the problem.) The approach gives exact values for geometric average options, approximate values for arithmetic average options. In the latter case, approximation error can be estimated and eliminated using relatively little additional calculation.

3. Problem Data and Valuation Methods

This section states process parameters, then discusses how paths can be bundled for valuation purposes. The methods use properties of (2.10) to adapt (2.16) and (2.17).

3.1. Process Parameters; Option Specifications

To enhance comparisons among different types of instruments, the same process parameters are used to value examples of four options – European and American geometric and arithmetic fixed strike average calls. As specified in Table 2 below, the examples value instruments on (2.1) with $T = 6$ quarterly time intervals, an annual volatility $\sigma = 0.40$, and a risk free rate $r = 0.10$ per annum. The initial asset price is $S_0 = 1.00$. Let k be defined as the solution to $u^k = K$ and take $K = 1.00$, so that $k = 0$. All options are assumed to expire at time T. If an option is exercised at time t, its path averages are defined over times $0, \ldots, t$. For European options $t = T$, for American options the early exercise feature means $t \leq T$.

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

Section 6.1, reporting computational experience, values European arithmetic average calls for $t \in \{6, 12, ..., 48\}$, $\sigma \in \{0.40, 0.60, \text{and } 0.80\}$, and the remaining parameters as indicated in Table 2.

TABLE 2

DATA FOR VALUE COMPARISONS

$\Delta t = 0.250000$		
$T = 6$	$\sigma = 0.400000$ per annum	
$r = 0.100000$ per annum		
$R = 1.100000^{.25}$	$= 1.024114$ per quater	
$u = 1.221403$	$= \exp\,(0.400000/(0.250000^{.5})$	
$p = 0.510051$	$= (R - u^{-1})/(u - u^{-1})$	
$q = 0.489949$	$= 1 - p$	$K = u^0 = 1.000000$

3.2. Ordering and Bundling Paths

Valuing a fixed strike average call involves finding a probability distribution of paths whose averages exceed K. These calculations' efficiency can be enhanced by organizing the data as indicated in Figure 3. Figure 3 shows the relations between a bundle and its sub-bundles, as well as the behaviour of the bundle's arithmetic averages when the paths are organized as shown. Each cell in Figure 3 represents a path in the bundle B(8,0), and the cell height indicates a path arithmetic average, in this case when $\sigma = 0.80$. Each (horizontal) bar of cells represents one of the sub-bundles of B(8, 0), with the length of the bar indicating the number of paths in the sub-bundle. The different heights within a bar indicate distinct sub-bundle arithmetic averages, the number of which is generally very much less than the sub-bundle' s number of paths. All the information conveyed by the graph can be obtained analyticaly, and all features of the graph except the cell heights are invariant with respect to volatility. Grouping paths into sub-bundles as indicated by the graph orders both the sub-bundle geometric means and the minima and maxima of sub-bundle arithmetic averages, properties used to advantage in the subsequent valuations.

Using the approach suggested by Figure 3, Table 4 organizes the data needed to obtain sub-bundle means of arithmetic averages in$B(6, 0)$. Each line of the Table 4 records, in the first thirteen columns, data needed to obtain such a sub-bundle mean. (Table 4 is shown with more columns than would normally be used in practice.) Column g indicates the numbers of paths in each sub-bundle, column V the path sum defining each sub-bundle, and column V^* the standardized path sums. Column M/g calculated from the index frequency section of Table 4, defines the sub-bundle mean of path arithmetic averages. For example, in the row for$B(6, 0, -5)$ $M/g = 6.126512$, indicating the mean of the arithmetic averages of the two paths in the subbundle is *6.126512 / 7*. When the sub-bundles are ordered by V, the values in column M/g increase monotonically, as illustrated by the example.

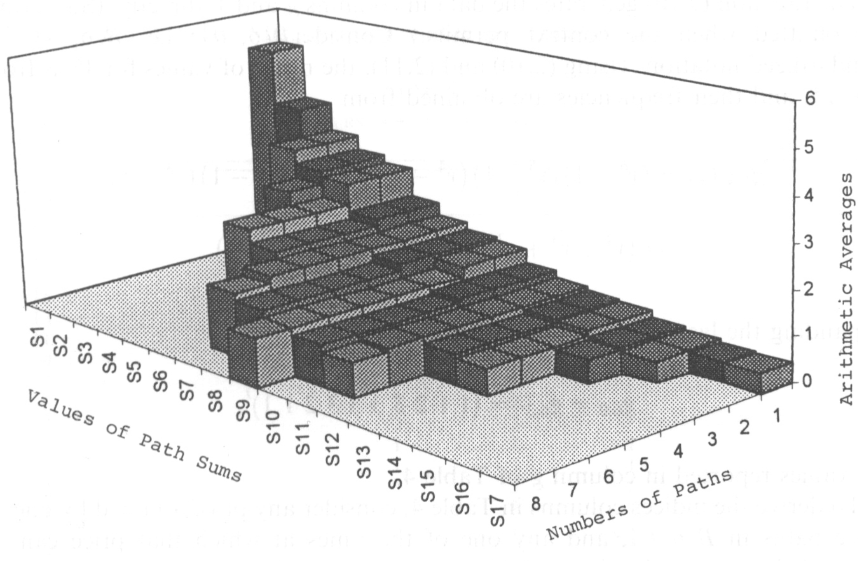

FIGURE 3. Arithmetic Averages in B(8,0); σ = 0.80.

TABLE 4

FREQUENCY DISTRIBUTIONS FOR B(6, 0)

Indices																
–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	g	V	V^*	M/g
0	0	0	1	2	2	2	0	0	0	0	0	0	1	–9	6	5.526912
0	0	0	0	2	3	2	0	0	0	0	0	0	1	–7	7	5.796831
0	0	0	0	2	6	6	0	0	0	0	0	0	2	–5	8	6.126512
0	0	0	0	2	7	10	2	0	0	0	0	0	3	–3	9	6.504853
0	0	0	0	0	6	12	3	0	0	0	0	0	3	–1	10	6.858864
0	0	0	0	0	3	12	6	0	0	0	0	0	3	1	11	7.261537
0	0	0	0	0	2	10	7	2	0	0	0	0	3	3	12	7.723644
0	0	0	0	0	0	6	6	2	0	0	0	0	2	5	13	8.156035
0	0	0	0	0	0	2	3	2	0	0	0	0	1	7	14	8.647860
0	0	0	0	0	0	2	2	2	1	0	0	0	1	9	15	9.248576

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

3.3. Frequency Distributions

Let the vector g_{t,j^*} represent the coefficients $g_{t,j^*}(v)$. It follows from (2.11) that g_{t,j^*} describes the frequency distributions of path sums for both $B(t,\ j^*,\ V^*)$, and $B(t,\ j,\ V)$. Columns g and V of Table 4 can thus be written directly from the $g_{t,j*}(v)$. The index frequency data in the first thirteen columns of Table 4 can be obtained using two-fold convolutions of (2.11). Consider each in turn.

The function (2.10) generates the data in columns g and V directly. (Subscripts are omitted when the context permits.) Consider*B(6, 0);* i.e. *B(6, 3*)* in standardized notation. Using (2.10) and (2.11), the range of values for V^* is from *6* to *15*, and their frequencies are obtained from

$$g_{6,3^*}(v) = \left(v^6 - 1\right)\left(v^5 - 1\right)\left(v^4 - 1\right)/(v - 1)\left(v^2 - 1\right)\left(v^3 - 1\right)$$

$$= \left(v^4 + v^3 + v^2 + v + 1\right)\left(v^3 + 1\right)\left(v^2 + 1\right).$$

Expanding the last line, it follows immediately that

$$g_{6,0} = g_{6,3^*} = (1\ 1\ 2\ 3\ 3\ 3\ 3\ 2\ 1\ 1)',$$

the values reported in column g of Table 4.

To derive the indices columns in Table 4, consider any price attained by one or more paths in *B(6, 0)*, and any one of the times at which that price can be attained. Then, consider the twofold convolution describing how those time-index combinations are related to the path sums at time T. A term from this convolution gives a frequency distribution of path sums for paths attaining the given time-price combination. Finally, since a given price can be attained at more than one point in time, the frequency distributions are summed across time to find the frequency distribution of path sums associated with the price index. Calculating these distributions for all attainable prices gives the joint frequency distribution for the bundle. The frequency data are generated column by column, as shown in Appendix II. Effectively, this method circumvents the analytical difficulty that the sum of lognormal variables is not lognormal.

In practice the data of Table 4 are computed using a forward recursion.The manner of constructing the data means they remain the same for all options of the type discussed here, so the valuation problem involves a setup cost that only needs to be incurred once.

4. Valuing Geometric Average Calls

This section values the European and then the American geometric average call.

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

4.1. Valuing the European Geometric Average Call

European geometric average options can be valued from just columns g and V of arrays like Table 4. Table 5, organized in a fashion similar to Table 4, shows all the data needed to value the European geometric average call. That is, Table 5 displays the frequency distributions for all sub-bundles

$$B(6,\ j,\ V); j \in \{-6,\ -4,\ \ldots,\ 6\};\ V \in \{-21,\ -19,\ \ldots,\ 21\}.$$

As in Table 4, blanks indicate unattainable combinations.

TABLE 5

NUMBERS OF PATHS BY SUB-BUNDLE

V/J	−6	−4	−2	0	2	4	6
−21	1						
−19		1					
−17		1					
−15		1	1				
−13		1	1				
−11		1	2				
−9		1	2	1			
−7			3	1			
−5			2	2			
−3			2	3			
−1			1	3	1		
1			1	3	1		
3				3	2		
5				2	2		
7				1	3		
9				1	2	1	
11					2	1	
13					1	1	
15					1	1	
17						1	
19						1	
21							1
Totals	1	6	15	20	15	6	1

The Table 5 data and the parameters of Table 2 are used to calculate the call payoffs shown in Table 6. For example, the contribution to call value of $B(6,0,5)$ is:

$$2 \cdot \left(1.221403^{5/7} - 1,\ 0\right)^{+} = 0.307130.$$

The 2 is the number of paths in $B(6,0,5)$, 1.221403 is the value of u, 5/7 is the index of the geometric average over the periods 0 through 6, and 1 is the exercise price.

The entries in each column of Table 6 are summed and multiplied by the appropriate probabilities as shown in the Table's last three lines. For example, column 2 adds to 3.489205 and the probability for each of its paths is $p^4 q^2 = 0.016246$ when $p = .510051$ and $q = 1 - p$. The third line, the product of sums and probabilities, is summed over all columns and multiplied by R^{-6} to obtain the time 0 discounted call value of 0.121869.

TABLE 6
EVALUATING THE EUROPEAN GEOMETRIC AVERAGE CALL

V\J	−6	−4	−2	0	2	4	6
−21	0.000000						
−19		0.000000					
−17		0.000000					
−15		0.000000	0.000000				
−13		0.000000	0.000000				
−11		0.000000	0.000000				
−9		0.000000	0.000000	0.000000			
−7			0.000000	0.000000			
−5			0.000000	0.000000			
−3			0.000000	0.000000			
−1			0.000000	0.000000	0.000000		
1			0.028984	0.086951	0.028984		
3				0.268485	0.178990		
5				0.307130	0.307130		
7				0.221403	0.664209		
9				0.293230	0.586460	0.293230	
11					0.738563	0.369281	
13					0.449805	0.449805	
15					0.535064	0.535064	
17						0.625336	
19						0.720918	
21							0.822120
Column Sums							
	0.000000	0.000000	0.028984	1.177199	3.489205	2.993634	0.822120
Probabilities							
	0.013833	0.014400	0.014991	0.015606	0.016246	0.016913	0.017607
Expected Values at Time 6							
	0.000000	0.000000	0.000343	0.018371	0.056687	0.050631	0.014475

European Geometric Average Call	0.121869

4.2. Valuing the American Geometric Average Call: Recursions

The American geometric average call is valued using a specialized version of (2.16) that defines recursions between sub-bundles:

$$D_{T-t}(j,\ z) \equiv g_{T-t,\,j,\,z}\ \max\{(G_{T-t}(j,\ z) - K)^+,$$

$$R^{-1}[pD_{T-t+1}(j+1,\ z+j+1)/g_{T-t+1,\,j+1,\,z+j+1}+$$

$$+qD_{T-t+1}(j-1,\ z+j-1)/g_{T-t+1,j-1,z+j-1}]\}; \qquad (4.1)$$

$$j \in \{-(T-t), -(T-t)+2,\ \ldots,\ T-t\};$$

$$z \in \{V_{T-t},j\};\ t \in \{0,\ \ldots,\ T\},\ D_{T+1}(\cdot) \equiv 0,$$

where $g_{T-t,\,j,\,z}$ is the number of paths in $B(T\text{-}t, j, z)$ and $\{V_{T-t,\,j}\}$ is the set of values defined by the coefficients of $g_{T-t,\,j*}$. The recursion relations between sub-

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

bundles can be derived from (2.10). For example, time 6 and time 5 frequency distributions are related by:

$$f_6(x, v) =$$

$$(1 + xv^6) \sum_{j^*=0}^{5} g_{5, j^*}(v) v^{j^*(j^*+1)/2} x^{j^*}. \tag{4.2}$$

A bundle defined at time 6 combines paths from adjacent end points at time 5. In terms of standard notation $B(6, j^*)$, has a distribution of path sums $g_{6, j*}$ determined by summing the generating function terms $v^6 g_{5, (j-1)*}$ and $g_{5, j*}$. A backward recursion to a bundle at time 5 must employ the relevant path sums and their frequencies taken from adjacent end points at time 6; again cf. (4.2). To perform the backward induction calculations at time 5 for an American option, the time 5 payoffs (with frequency distribution $g_{5, j*}$) are compared with the expected value of the time 6 payoffs (with frequency distributions determined by $v^6 g_{5, j*}$and $g_{5, j*}$ respectively).

To illustrate the recursions using the original indices, consider *B(5, –5, –15)*. This subbundle's single path extends to the single path in *B(6,–6,–21)* if the price decreases between times 5 and 6, to a path in *B(6,–4,–19)* if the price increases. (Remaining paths in *B(6,–4,–19)* are reached from *B(5,–3)*, and form a part of the calculation of expected payoffs for *B(5,–3).)* For *B(5,–5,–15)*, the payoff to holding the option is the expected value of the payoff from proceeding either to $B>(6, -6, -21)$ or to *B(6,–4,–19)*. The payoff to immediate exercise for *B(5,–5, –15)* is zero, determined by comparing the geometric average $u^{-15/6}$ to the exercise price of u^0. In this case no further calculation is necessary: the expected value of continuing from *B(5, –5, –15)* cannot be less than the value of immediate exercise, and therefore it is only necessary to record the expected value of continuing. Table 7 shows in greater detail how the frequencies at time 6 are generated from the relevant frequencies at time 5, and thus also shows how time 6 frequencies can be divided to carry out the backward inductions just described.

Sub-bundles can contain many paths, but examining (4. 1) for *T, T-1, ..., 0* shows that each sub-bundle is defined to contain only paths whose payoffs are the same (for geometric average instruments) regardless of time point or nature of optimal policy. (The result is not true for arithmetic average instruments; see Section 5.) Thus bundling methods can be used for valuing both European and American geometric average options. In the latter case, for each of the two time 6 parts of Table 7, a payoff table similar to Table 6 is constructed. The payoff tables for time 6 are then used to construct a table of expected discounted payoffs at time 5, and these are compared to the payoffs for immediate exercise at time 5.

For example, there is one path ending at (6, 4) with a path sum of 19 and one path ending at (6, 6) with a path sum of 21. Both these paths emanated from a single path at (5, 5) with a path sum of 15. Since the payoffs at time 6 are *0.720918* and *0.822120* respectively, the expected discounted payoff at time 5 is

$$.754346 = [(.489949)(.720918) + (.510051)(.822120)]/1.024114$$

TABLE 7

RELATIONS BETWEEN PATH SUMS, TIMES 5 AND 6

Frequencies at time 5

V\J	−5	−3	−1	1	3	5
−15	1					
−13		1				
−11		1				
−9		1	1			
−7		1	1			
−5		1	2			
−3			2	1		
−1			2	1		
1			1	2		
3			1	2		
5				2	1	
7				1	1	
9				1	1	
11					1	
13					1	
15						1

Frequencies at time 6

V\J	−6	−4	−4	−2	−2	0	0	2	2	4	4	6
−21	1											
−19		1										
−17			1									
−15			1	1								
−13			1	1								
−11			1	1	1							
−9			1	1	1	1						
−7				1	2	1						
−5					2	2						
−3					2	2	1					
−1					1	2	1	1				
1					1	1	2	1				
3						1	2	2				
5							2	2				
7							1	2	1			
9							1	1	1	1		
11								1	1	1		
13									1	1		
15									1	1		
17										1		
19											1	
21												1

To allow for early exercise, these expected values are compared to the values of immediate exercise at time 5, and for each comparison the maximum is recorded. In the present example, the payoff to *B(5,5,15),0.648722*, is calculated just as in Table 6. Since *0.648722* < 0.754346, the optimal policy for this sub-bundle is not to exercise, and the value of *0.754346* is recorded in the payoffs to the optimal policy at time 5.

The complete set of time 5 optimal decisions is given in Table 8, where C means it is optimal not to exercise, X means it is optimal to exercise immediately, and 0 means the payoff is zero whether the option is exercised or not. (The zero payoffs are recorded to display the form of the time 5 optimal policy for all time 5 sub-bundles.) Note that while the paths in *B(5, 1, –1)* have an immediate payoff of zero – their time 5 geometric average is less than the strike price – there is still a positive payoff to continuing, as shown by the C in the position *(–1, 1)*, referring to the payoffs to *B(5, 1, –1)*.

TABLE 8

OPTIMAL DECISIONS AT $t = 5$

V\J	–5	–3	–1	1	3	5
–15	0					
–13		0				
–11		0				
–9		0	0			
–7		0	0			
–5		0	0			
–3			0	0		
–1			0	C		
1			X	C		
3			X	C		
5				C	C	
7				X	C	
9				X	C	
11					C	
13					C	
15						C

To continue with the backward induction, a time 5 frequency distribution, organized as in the second part of Table 7, is used to divide the optimal payoffs at the prices –3, –1, 1, and 3 into payoffs for upward and downward moves. (As before extreme prices are reached in only one way; price -5 by a downward move, price 5 by an upward move.) The backward induction then proceeds from time 5 to time 4, now comparing the discounted expected value of the optimal payoffs at time 5 with the payoffs to immediate exercise at time 4. Continuing the backward induction procedure until time zero is reached, choosing an optimal exercise policy at each time, gives a value for the American call of .126932.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932

5. Valuing Arithmetic Average Calls

Both European and American arithmetic average calls can be valued approximately using the joint frequency distribution of Table 4. The approximation error can then be estimated, and if it is small enough no further calculation will be needed. If greater accuracy is desired some parts of the joint distribution must be elaborated. Obtaining further detail requires the procedures described in Appendix I, but can be done selectively and typically does not require extensive additional calculations.

5.1. Initial Approximate Solution for the European Arithmetic Average Call

Approximate values of arithmetic average options can be obtained by using the kinds of data reported in the body of Table 4. Each line of Table 4 is be used to find the mean of the arithmetic averages for all paths in a given sub-bundle. The approximation is based on assuming that the arithmetic average for each path in a given sub-bundle is exactly equal to the sub-bundle mean. With this approximation, both European and American arithmetic average instruments can be valued in a manner analogous to that used for geometric average instruments in Section 4. Of course, the assumption introduces approximation error, but the error can be estimated and reduced with relatively few additional calculations as discussed in the next section.

To obtain the approximate value of a European arithmetic average option, the methods of section 4 are adapted as illustrated in Table 10. The only difference between Table 6 and Table 10 is that the latter now contains payoffs determined from the means of sub-bundle arithmetic averages. The analogous payoffs in Table 6 were determined from geometric averages, known to be equal for all paths in any given sub-bundle.

Table 10 shows a positive value for B(6, 2, −1), whereas the corresponding value in Table 6 was zero. The difference reflects the fact that arithmetic averages exceed geometric averages. The path in question is 0-1-2-1012, and has an arithmetic average of 1.003001 for $u = 1.221403$.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932
European Arithmetic Average Call	0.136520[1]
[1] Approximate Value	

https://doi.org/10.2143/AST.27.2.542047 Published online by Cambridge University Press

TABLE 10

APPROXIMATE VALUE, EUROPEAN AVERAGE CELL

	−6	−4	−2	0	2	4	6
−21	0.000000						
−19		0.000000					
−17		0.000000					
−15		0.000000	0.000000				
−13		0.000000	0.000000				
−11		0.000000	0.000000				
−9		0.000000	0.000000	0.000000			
−7			0.000000	0.000000			
−5			0.000000	0.000000			
−3			0.000000	0.000000			
−1			0.000000	0.000000	0.003001		
1			0.060526	0.112087	0.050098		
3				0.310133	0.215246		
5				0.330296	0.343032		
7				0.235409	0.734518		
9				0.321225	0.626894	0.368517	
11					0.782972	0.426042	
13					0.477303	0.496303	
15						0.582119	
17						0.686936	
19						0.814960	
21							0.971328
Column Sums							
	0.000000	0.000000	0.060526	1.30915	3.815184	3.374877	0.971328
Probabilities							
	0.013833	0.014400	0.014991	0.015606	0.016246	0.016913	0.017607
Time 6 Expected Values							
	0.000000	0.000000	0.000907	0.020431	0.061983	0.057079	0.017102
Time 0 Approx Value of European Arithmetic Average Call							0.136520

5.2. Assessing and Reducing Approximation Error

Approximation errors can be introduced by the methods of 5.1, because the arithmetic averages of paths in a sub-bundle do not generally equal their mean. The present method could be expected to give good approximations even without additional refinements. First, it can only introduce error in a limited way, as the next section shows in greater detail. Second, the approximation itself should be at least as accurate as that of Curran (1992). The present approximation is actually based on both geometric averages and path end points, whereas Curran's is only based on the former. Some computational experience supporting the claim is given in section 6.1.

More importantly, the approximation can only introduce error for a limited number of sub-bundles. Whenver the maximal path average in a sub-bundle is less than the strike price, the subbundle contributes nothmg to the value of a European call and can be ignored. Whenever the minimal path average in a sub-bundle exceeds the strike price, every path in the sub-bundle contributes to the value of the European call, and using the mean of sub-bundle arithmetic averages introduces no error. Since all paths in a sub-bundle have equal probability, the sum of the individual path averages is n times their mean, where n is the number of paths in the subbundle. Thus the individual paths' contributions to option value are n times the contribution calculated using the sub-bundle mean.

The only sub-bundles for which error can be introduced in a European option are those for which the maximal path average exceeds the strike price and the minimum falls strictly below it. Such sub-bundles (which must have more than a single path) are said to be *cut* by the strike price. The number of sub-bundles which can be cut by the strike price is relatively small, and the subbundles in question can readily be identified; see Neave and Stein (1997) for a method. To eliminate all approximation error, it is necessary to examine the sub-bundles which are actually cut by the strike price, and to correct the approximation calculations for those cases.

To illustrate error estimation and reduction, consider $B(6, 0, -1)$. The aggregate data reported in Table 4 are:

				g	V	M/g
Indices	−1	0	1			
Frequencies	6	12	3	3	−1	6.858864

The value of M in the above extract from Table 4 is found using

$$M = 60u^{-1} + 12u^{0} + 3u^{1} = 20.576593$$

when $u = 1.221403$. Since all three paths in $B(6, 0, -1)$ have the same probability, the mean of the sub-bundle arithmetic averages is $M/3g = 0.979838$. Approximation error could arise if one or more of the paths in $B(6, 0, -1)$ had an arithmetic average in excess of 1, the strike price.

To eliminate approximation error, it is necessary to determine the frequency distribution of distinct arithmetic averages in any sub-bundle which can be cut by the strike price. Using the methods of Appendix I, it can be shown that the maximal path average in $B(6, 0, -1)$ is less than the strike price, which eliminates any need to examine it further. Nevertheless, to illustrate the issues more fully, it is useful to write out the individual paths according to methods outlined in Appendix I:

Indices	−1	0	1	z	V	N
Frequencies	2	4	1	37	−1	6.858864
	2	4	1	25	−1	6.858864
	2	4	1	21	−1	6.858864

In this case the sub-bundle has only one distinct arithmetic average. However, if the sub-bundle had more than one distinct average, and if it were cut by the strike price, only the averages above the price would contribute to call value and the original approximation would have to be corrected. In the present example, checking the remaining sub-bundles shows that no other subbundle can actually be cut by the strike price, and the exact value of the European arithmetic average equals the approximate value determined earlier.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932
European Arithmetic Average Call	0.136520 [1,2]
[1] Approximate Value [2] Exact Value	

5.3. Initial Approximate Solution for the American Arithmetic Average Call

The approximate value of the American arithmetic average call is obtained by using the methods of 5.2 recursively. A recursion relation identical to (4.1), except in its use of arithmetic averages, is used:

$$D_{T-t}(j,\ z) \equiv g_{T-t,j,z} max\{H_{T-t}(j,\ z) - K)^{+},$$

$$R^{-1}[pD_{T-t+1}(j+1,\ z+j+1)/g_{T-t+1,\,j+1,\,z+j+1}+$$

$$+qD_{T-t+1}(j-1,\ z+j-1)/g_{T-t+1,j-1,z-j-1};]\} \tag{5.1}$$

$$j \in \{-(T-t),\ -(T-t)+2,\ \ldots,\ T-t\};\ z \in \{V_{tj}\};\ t \in \{0,\ \ldots,\ T\},$$

where $g_{T-t,j,z}$ is the number of paths in the sub-bundle defined by j and z. Equations (5.1) give an approximate value because they assume that arithmetic averages are equal for all paths in each sub-bundle. Recursive relations between joint frequency distributions are determined using exactly the same methods as in Table 8.

Using the mean value of payoffs for each sub-bundle, backward induction calculations can be performed just as in 3.4. The calculations give an approximate value of *0.14109.3* for the American call.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932
European Arithmetic Average Call	0.136520[1,2]
American Arithmetic Average Call	0.141093[1]
[1] Approximate Value [2] Exact Value	

5.4. Reducing approximation error

The section 5.3 assumption that all paths in a sub-bundle have equal arithmetic averages can lead to calculating a sub-optimal option value just as with the European option. However, it is possible both to assess the approximation error and to reduce it in much the same way as before.

In the backward induction calculations, the assumption of equal averages is used to divide payoffs according to the number of paths in each sub-bundle. To reduce approximation errors, it is necessary to evaluate which recursive calculations are affected by this approximation. The simplest way to eliminate all approximation error is to divide sub-bundles further on the basis of individual arithmetic averages, and then proceed exactly as in valuing the geometric average options. Unless computing resources are severely limited, this is probably the simplest way to eliminate all approximation error, since experiments indicate the number of divided sub-bundles is roughly described by a fourth-degree polynomial in T.

If the procedure of the foregoing paragraph is not followed, sub-bundles can contain differing arithmetic averages, and care needs to be taken in assessing and reducing the resulting approximation error. (The tradeoff between the two approaches is best assessed in the context of a given valuation problem.) A good rule of thumb is to begin by examining payoffs near the exercise boundary at some time period near $2T/3$, and continue backwards to earlier times if significant errors are detected. Section 5.2 demonstrated the importance of examining payoffs near the exercise boundary; the reason for choosing a time period around $2T/3$ is that typically more exercise decisions are made as option expiry nears.

In the present example, suppose it is desired to find the details of the two paths in $B(5,1,3)$, to check whether the assumption of equal arithmetic averages, which implies dividing payoffs in a 1:1 ratio, gives a nearly optimal value. Using Appendix I, the two time 5 paths are:

$$\begin{array}{cccccc} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 2 & 1. \end{array}$$

The two paths' arithmetic averages are 1.110702 and 1.125660 respectively, and thus their contribution to value under a policy of immediate exercise is .110702 and .125660 respectively. However, if the two paths are extended to time 6, their expected values are

$$(.165418p + .094887q)/R = .127646$$

for the first path, and

$$(.177884p + .107623q)/R = .140082$$

for the second. Clearly, the optimal policy for $B(5, 1, 3)$ is not to exercise at time 5. Even so, the optimal sub-bundle payoff of

$$.267728 = .127646 + .140082$$

should be divided on the basis of expected values rather than by numbers of paths as in 5.3. In the present example, this is the only refinement to the approximation needed to determine an optimum; all other divisions based on numbers of path are already optimal. Thus, a single modification suffices to obtain the exact American arithmetic average option value of 0.141269. The difference between exact and approximate valuations indicates that before relying on approximations in practice, model-based evaluations of their accuracy should be established.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932
European Arithmetic Average Call	0.136520[1,2]
American Arithmetic Average Call	0.141093[1] 0.141269[2]
[1] Approximate Value [2] Exact Value	

The literature does not stress the importance of assessing approximations in relation to a model determined optimum. However as evidenced by the approximate and exact values for the American arithmetic average call, the present example indicates that even plausible approximations can create significant valuation errors. More computational experience of the sort described in 6.1 is is needed to determine the likely incidence of errors for the American option. In practice it may prove useful to find an exact solution for a set of typical parameter values and use that value to estimate approximation errors for American instruments when they are valued accordmg to the quick methods of Section 5.3.

6. Extensions and Conclusions

This section sketches computational experience to date and also remarks on how the paper's methods can be extended to valuing other path dependent instruments.

6.1. Computational Experience

While computations using the method are still in the early stages, experience to date is encouraging. The data in Table 11, taken from Neave and Stein (1997), report our results for European arithmetic average calls with relatively large volatilities.

TABLE 11

APPROXIMATE VALUES OF ARITHMETIC AVERAGE OPTION

T	$\sigma = 0.40$	$\sigma = 0.60$	$\sigma = 0.80$	CPU Secs
6	.136520	.184712	.231945	0
12	.137026	.185367	.232823	0
18	.137214	.185685	.233290	2
24	.137322	.185862	.233547	7
30	.137392	.185972	.233710	22
36	.137441	.186046	.233822	52
42	.137476	.186100	.233901	107
48	.137502	.186142	.233963	204

All unstated parameters are the same as in Table 2. Approximation errors are discussed below.

Table 11 suggests that, in the context of discrete models, the present method both increases accuracy and reduces calculation time. With respect to accuracy, Ritchken, Sankarasubramanian and Vijh (1993) use an Edgeworth approximation to value European arithmetic average options, benchmarking their results usmg simulated values. For volatilities of 0.20 and 0.30 respectively, the standard errors in simulations for 16 to 64 periods are on the order of 0.004 to 0.005. For 25 reset points, the relative approximation errors of this paper' s method are 0.0002 and 0.0009 for volatilities of 0.40 and 0.80 respectively (Stein, 1996), and exact valuations can be found with modest amounts of additional calculation.

An examination of Figure 3 suggests the present approximation is also likely to give greater accuracy than that of Hull and White (1993). Our approach approximates arithmetic averages using sub-bundle means, while Hull and White use nonlinear interpolation between arithmetic averages determined by the maximum and minimum path averages in a bundle. Our approach only introduces error in sub-bundles cut by the strike price, whereas nonlinear interpolation can introduce error at a greater number of sub-bundles. Finally, we can estimate and reduce the error created by a sub-bundle's being cut, while Hull and White offer no way of either estimating or reducing the error of their method.

With respect to computation time, Table 11 reports the number of CPU seconds needed to set up and obtain the valuations. In addition to the data reported in Table 11, we have been able to find exact values for European geometric average calls, and approximate values for European arithmetic averages calls, for values of T up to 100. The computation times for these experiments have been about one hour on a SunSparc workstation. Computation times are comparable to recent unpublished work using the Hull and White approximation, but as already mentioned the present method gives greater accuracy. Finally, computation time is independent of the process volatility.

With respect to memory requirements, experiments with the European arithmetic average call, conducted on a SunSparc work station, mdicate the procedure uses 0.7MB (megabytes) of RAM when T = 30, 9.3 MB when T = 60, and 68.8MB when T = 100. As rough comparisons, MicroSoft Word '97 uses 2.6MB, Netscape Navigator 3.0 uses 4.5MB. Personal computers with 64MB of RAM are now standard, and some work stations offer up to 110MB.

Additional experiments are needed to assess the approach's accuracy and memory requirements in valuing the American arithmetic average option. Nevertheless, the framework organizes and reduces the numbers of computations in new ways, and also permits comparing approximations with exact optima for the same problem.

6.2. Time Weighted Averages

The methods developed above can readily be modified to value instruments whose averages are computed on a subset of the time points. For example, if arithmetic averages are computed on a subset of time points, the joint frequency distributions used in this paper need only be modified to record the frequencies with which indices are realized at chosen reset points. They can also be modified relatively easily to value instruments with time weighted averages. The approach can be extended to average strike options by determining joint distributions of the averages and path ends, readily available from the information developed in this paper.

6.3. Path Sums and Time Dependent Probabilities

Since the present model uses a constant value of u, valuation under a martingale with time varying interest rates requires using time dependent probabilities. Given time dependent probabilities, exact values can be found recursively, but the calculations are exponential in T. The task can be simplified by using the generating function to define a joint frequency distribution of path sums and time dependent probabilities, using a procedure much like that of Appendix II. Then depending on the relations between probabilities at each point in time, the difference between maximal and minimal probabilities for the paths in a sub-bundle can be assessed. If the difference is unimportant for the problem at hand, an average path probability can be used; otherwise individual probabilities need to be enumerated using methods similar to those outlined above.

6.4. Conclusions

This paper valued European and American fixed strike average calls on a discrete time, recombining multiplicative binomial asset price process. Using generating functions to find frequency distributions of option payoffs, the paper showed how to eliminate much of the calculation previously thought to be involved in valuing path dependent options. The procedures value European geometric average options analytically, and use relatively few computations to value European arithmetic average options. Both types' American counterparts are valued using recursive relations between frequency distributions.

ACKNOWLEDGEMENTS

Versions of this paper have been presented at the International Meeting on Risk Theory, Mathematisches Forschungsinstitut, Oberwolfach, Germany, 1994; the 5th AFIR International Colloquium, Brussels, 1995; and at the Northern Finance Association Meetings, London, Ont., 1995. I am grateful to Phelim Boyle, Freddy Delbaen, Daniel Dufresne, Helyette Geman, Anders Martin-Lof, Jim Tilley and Stuart Turnbull for constructive comments. David Arthurs showed me how to create Figure 3 using Excel.

REFERENCES

BERMAN, GERALD, and K. D. FRYER, Introduction to Combinatorics, New York: Addison-Wesley, 1972.

CARVERHILL, A., and L. CLEWLOW, 1990, "Flexible Convolution," Risk 3, Apr. 25-29.

CHEN, CHUANG-CHONG and KOH KHEE-MENG, Principles and Techniques in Combinatorics, Singapore: World Scientific Publishing, 1992.

CURRAN, M., "Beyond Average Intelligence, " Risk, November 1992, p. 60.

DIXIT, AVINASH K. and ROBERT S. PINDYCK, Investment Decisions Under Uncertainty, Princeton: Princeton University Press, 1994.

FELLER, WILLIAM, An Introduction to Probability Theory and its Applications, 2nd Edition, New York: Wiley, 1957.

GEMAN, HELYETTE, and A. EYDELAND, "Domino Effect," Risk, 1995, 65-67.

GEMAN, HELYETTE, and MARC YOR, "Bessel Processes, Asian Options, and Perpetuities," Mathematical Finance 3, 1993, 349 - 375.

GOLDMAN, M. Barry, HOWARD B. SOSM and M. GATTO, "Path Dependent Options: Buy at the Low, Sell at the High," Journal of Finance 34, 1979, 1111-1127.

HOO, Thomas S. Y., "Managing Illiquid Bonds and the Linear Path Space," Journal of Fixed Income, June 1992, 80-94.

HUANG, CHI-FU, and ROBERT H. LITZENBERGER, Foundations for Financial Economics, New York: North-Holland, 1988.

HULL, JOHN, and ALAN WHITE, "Efficient Procedures for Valuing European and American Path-Dependent Options," The Journal of Derivatives, Fall 1993, 21-31.

JARROW, ROBERT, and STUART M. TURNBULL, Derivative Securities, Cincinnati: South-Western Publishing Company, 1996.

KEMNA, A. and VORST, TON "A Pricing Method for Options Based on Average Asset Values," Journal of Banking and Finance, 1990.

LEVY, EDMOND, "A Note on Pricing European Average Options," Nomura Bank International Working Paper, 1991.

LEVY, EDMOND, and STUART M. TURNBULL, "Average Intelli.gence," Risk Magazine, 1992.

NEAVE, EDWM H., "Valuing European Average Options on a Recombining Binomial Random Walk," Actuarial Approach For Financial Risks, vol. 1, pp. 153-180, Third AFIR International Colloquium, Rome, 1993.

NEAVE, EDWIN H., and CHRISTOPHER L. STEIN "Approximate and Exact Valuation of European Arithmetic Average Spot Price Calls," Queen's University Working Paper, 1997.

RITCHKEN, P., L. SANKARASUBRAMANIAN, and A. VIJH, "The Valuation of Path Dependent Contracts on the Average," Management Science 39 (10), 1993, 1202-1213.

ROGERS, L. C. G. and SHI, Z., "The Value of an Asian Option," Journal of Applied Probability, 1995.

SCHWARTZ, EDUARDO S., Review of Dixit and Pindyck, Investment Under Uncertainty, Journal of Finance 49, 1994, 1924-1928.

STEIN, CHRISTOPHER L., "Computing the Exact Value of European Arithmetic Average Spot Price Calls," Queen's University Working Paper, 1996.

TILLEY, JAMES A., "Valuing American Options in a Path Simulation Model," Actuarial Approach For Financial Risks, vol. 1, pp. 249-266, Third AFIR International Colloquium, Rome, 1993.

TURNBULL, STUART M. and LEE MACDONALD WAKEMAN, "A Quick Algorithm for Pricing European Average Options," Journal of Financial and Quantitative Analysis 26, 1991, 377-389.

YOR, M., "On Some Exponential Functions of Brownian Motion," Advances in Applied Probability 24, 1992, 509-531.

ZHANG, P., "Flexible Arithmetic Asian Options," The Journal of Derivatives, Spring 1995, 53-63.

Appendix I. Relating Z and V*

The generating function can be used to study both the mapping from Z to V^* and the inverse mapping from V^* to Z. Consider the term of $f_4(x, 7, v)$ equal to $x^4 z^{43} v^{13}$, which describes the path defined by $z = 43$, ending price j^*, and path sum $v^* = 13$.

Suppose it is known that $z = 43$. The value of j^* can be determined by expressing 43 as a sum of powers of 2, and counting the number of terms. That is,

$$43 = 2^5 + 2^3 + 2^1 + 2^0$$

so that $j^* = 4$. Since Z is defined as a sum of terms 2^{j-1} while v^* is determined by a sum of indices j,

$$v^* = 6 + 4 + 2 + 1 = 13$$

To study the inverse mapping from V^* to Z, let the values of j^* and v^* be given. To continue the previous example, if $j^* = 4$ and $v^* = 13$, then from (2.3) v^* must be the sum of four integers chosen from $1, \ldots, 6$. There are only two such combinations; either the foregoing or

$$v^* = 5 + 4 + 3 + 1 = 13$$

for which $Z = 29$. The two paths in the sub-bundle $B(6,2,5)$ are thus

0101012 and *0-101212*

The maximal arithmetic average in any sub-bundle is defined by one of the extremal values of Z associated with the sub-bundle. Moreover, the maximal arithmetic average increases as the term V defining the sub-bundles increases. Finally, a minimal path average can be characterized in terms of Z. However the geometric average is also a lower bound on the arithmetic averages, and is in any case recorded as a part of the valuation method.

Appendix II. Finding the Joint Frequency Distribution: Example

This Appendix develops an analytic method of finding joint frequency distributions of indices and path sums. While in practice it is usually convenient to calculate the joint frequency distributions recursively, the analytic approach of this Appendix makes it possible to organize the computations efficiently. The joint frequency distribution is obtained analytically using two-fold convolutions of (2.17) taking the form $f^3(x, v) * f^{T-5}(x, v)$. A term of the convolved functions can be interpreted as follows. Consider any feasible index in $B(T, j)$, say (s, k). The number of paths through (s, k) is readily shown to be $b(s, k) \cdot b(T - s, j - k)$, where $b(T, j) \equiv T!/(T - j)!j!$. From (2.16) the distribution of path sums at (s, k) is $g_{s,k}(v) v^{k(k+1)/2} x^k$. Since any path in in $B(T, j)$ arriving at index (s, k) must still take T-s steps, the distribution of path sums at (t, j) is

$$v^{k(T-s)} \cdot g_{s,k}(v) v^{k(k+1)/2} x^k \cdot g_{T-s,\, j-k}(v) v^{(j-k)(j-k+1)/2} x^{j-k} = \qquad \text{(A.1)}$$

$$v^{k(T-s)+[k(k+1)+(j_k)(j-k+1)]/2} x^j g_{s,k}(v) \cdot g_{T-s,j-k}(v)$$

The term $v^{k(T-s)}$ compensates for the fact that the remaining T-s steps begin at (s, k), while $g_{T-s,\, j-k}(v)$ begins its counting from $(0, 0)$.

To illustrate the calculations, Table A.I repeats the joint frequency distribution of path sums and indices realized reported in Table 4 for $B(6, 3^*)$. Blanks indicate combinations which cannot be realized by paths in $B(6, 3^*)$.

TABLE A.I
Joint Frequency Distribution for B(6, 3*)

V\j	-3	-2	-1	0	1	2	3	Row Totals
9	1	2	2	2				7
7		2	3	2				7
5		2	6	6				14
3		2	7	10	2			21
1			6	12	3			21
1			3	12	6			21
3			2	10	7	2		21
5				6	6	2		14
7				2	3	2		7
9				2	2	2	1	7
	1	8	29	64	29	8	1	

Note that the row totals equal the product of the 7 indices in each path and the path frequencies. The second section of Table A. 1 supplements the column totals at the bottom of the first section by showing the frequencies with which individual indices are realized at different times. As before, blanks represent unattainable combinations.

Terms of the two-fold convolutions are used to calculate the joint frequency distributions are employed for each time-index combination, as shown in the detailed calculations of Table A.II. Each column of Table A.II represents a time-index combination; for example, the index -2 can be realized at time 2 or at time 4. These two columns then indicate the frequency distributions of path sums at time 6 for paths attaining the index -2 at either time 2 or time 4. Similarly, the index -1 can be realized at times 1, 3, and 5. The convolution describes only a single frequency distribution at times 1 and 5, but three at time 3. This is because paths arriving at index -1, time 3 can have three values at that point, and each path from that point to the end can also take on any one of three incremental values. The positioning of the frequencies within the columns is determined by the range of path sums, as described both in Section 3.2 and at the beginning of this Appendix.

TABLE A.II

Obtaining the Joint Frequency Distribution

Indices	−3	−2	−2	−1	−1	−1	−1	−1	0	0	0	0	0	0
Times	3	2	4	1	3	3	3	5	0	2	2	4	4	6
V														
−9	1	1	1	1				1	1					1
−7		1	1	1	1			1	1					1
−5		1	1	2	1	1		2	2	1		1		2
−3		1	1	2	1	1	1	2	3	1	1	1	1	3
−1				2		1	1	2	3	2	1	2	1	3
1				1			1	1	3	1	2	1	2	3
3				1				1	3	1	1	1	1	3
5									2		1		1	2
7									1					1
9									1					1
	1	4	4	10	3	3	3	10	20	6	6	6	6	20

Appendix III: Finding distinct arithmetic averages in a sub-bundle

To see how the distinct arithmetic averages in a sub-bundle can be found, consider the subbundles (8, 3, 15), (8, 3, 14), (8, 3, 13), and (8, 3, 12). Each sub-bundle contains six paths, as shown by the path numbers m the following rows:

8 6 1	8 5 2	8 4 3	7 6 2	7 5 3	6 5 4
8 5 1	8 4 2	7 6 1	7 5 2	7 4 3	6 5 3
8 4 1	8 3 2	7 5 1	7 4 2	6 5 2	6 4 3
8 3 1	7 4 1	7 3 2	6 5 1	6 4 2	5 4 3

The arithmetic averages for the foregoing paths are shown next for $\sigma = 0.40$.

0.9056653	0.8909149	0.8909149	0.8909149	0.8909149	0.8828047
0.8609239	0.8609239	0.8542838	0.8461735	0.8461735	0.8461735
0.8242928	0.8242928	0.8161825	0.8095424	0.8095424	0.8095424
0.7943017	0.7795514	0.7795514	0.7795514	0.7795514	0.7729113

The example shows the need, when exact valuation is desired, for carefully investigating any particular sub-bundles cut by the strike price. In the present each sub-bundle has exactly three distinct averages, but the frequency distributions of the three distinct averages vary. Thus, if one or more of these sub-bundles were cut by the strike price, the valuation effect would depend on the particular sub-bundle or sub-bundles affected. So far, it seems necessary to determine the frequency distribution of the distinct averages in any such sub-bundle.

Frequency distributions of distinct averages can be found either by enumerating the subbundle's path numbers or by using a dynamic programming search to find the distinct path averages, then determining the frequency of each distinct average using linear programming. In large subbundles, the second method is more efficient than complete enumeration, because the number of paths can be large while the number of distinct averages is very much less than the number of paths.