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AN APPROXIMATION THEOREM FOR COARSE V-TOPOLOGIES ON RINGS

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ABSTRACT. An approximation theorem for V-topologies on not necessarily commutative rings is proved. This holds for a certain class of rings (called *rings with enough units*) and a certain class of V-topologies (called *coarse V-topologies*). This has application, for example, to V-topologies induced by orderings.

The purpose of this paper is to prove an approximation theorem for V-topologies on (not necessarily commutative) rings along the lines of the approximation theorem for valuations in [8], [9], [10], [11]. The result is valid for a certain class of rings called *rings with enough units*, and a certain class of V-topologies called *coarse V-topologies*. This work was motivated, in part, by a question raised in [22] concerning V-topologies induced by orderings.

In the field case there are several proofs [2], [23], [26], [28]. Of these, the proof in [28] is by far the simplest and, moreover, this proof is also valid for skew fields. The proof given here is patterned along the lines of the proof in [28].

0. **Preliminaries.** Let A be a ring. A *ring topology* on A is a (not necessarily Hausdorff) topology τ such that the operations $(x,y) \longmapsto x-y$, $(x,y) \longmapsto xy$ are continuous. We identify τ with the set of τ-neighbourhoods of zero, *i.e.*, the statement $N \in \tau$ means that N is a τ-neighbourhood of zero. A subset S in A is said to be (*left*) *τ-bounded* if, for every $M \in \tau$, there exists $N \in \tau$ such that $NS \subseteq M$. Sums, differences, products, and finite unions of τ-bounded sets are τ-bounded. For a ring homomorphism $\alpha: A \longrightarrow A'$ and a ring topology τ on A', we have the induced ring topology $\tilde{\tau}$ on A (the weakest topology such that α is continuous). We say $S \subseteq A$ is *τ-bounded* if $\alpha(S)$ is τ-bounded. Thus S is τ-bounded $\Rightarrow$ S is $\tilde{\tau}$-bounded. We make extensive use of the following almost trivial result; see [28, Lemma 2.1].

LEMMA 0.1. *Let $\alpha_i: A \longrightarrow A_i$ be a ring homomorphism, τ_i a ring topology on A_i, $i = 1, \ldots, n$. Suppose τ is any topology on A such that the maps $y \longmapsto x + y$, $x \in A$, are continuous, and no $S \in \tau$ is τ_i-bounded, $i = 1, \ldots, n$. Then, for any $S \in \tau$ and any τ_i-bounded sets $B_i \subseteq A$, $i = 1, \ldots, n$, $S \not\subseteq B_1 \cup \cdots \cup B_n$.*

PROOF. This is true by hypothesis if $n = 1$. Suppose $n \geq 2$ and that $S \subseteq B_1 \cup \cdots \cup B_n$ for some $S \in \tau$ and some τ_i-bounded sets B_i and that n is minimal with this property. Replacing S by its interior, we can assume S is open. Since $B_n - B_n$ is τ_n-bounded,

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$S \not\subseteq B_n - B_n$ so $\exists x \in S$, $x \notin B_n - B_n$. Pick $S' \in \tau$, $S' \subseteq S$, $x + S' \subseteq S$. Then, for any $y \in B_n \cap S'$, $x+y \in S$ so $x+y \in B_i$ for some i. If $x+y \in B_n$, then $x = (x+y)-y \in B_n - B_n$, contradiction. Thus $x+y \in B_i$, $i < n$, so $y \in \bigcup_{i=1}^{n-1}(B_i - x)$. Thus $S' \subseteq \bigcup_{i=1}^{n-1}\big(B_i \cup (B_i - x)\big)$. Since $B_i \cup (B_i - x)$ is τ_i-bounded, this contradicts the choice of n. ∎

1. **V-topologies on skew fields.** Let F be a skew field. A *V-topology* on F is a non-discrete Hausdorff ring topology τ on F such that $(F \setminus U)^{-1} := \{x^{-1} \mid x \in F \setminus U\}$ is τ-bounded for each $U \in \tau$ [15]. One also has the following characterization:

LEMMA 1.1. *For a non-discrete Hausdorff ring topology τ on F, the following are equivalent:*

(1) τ is a V-topology.

(2) There exists a τ-bounded set $I \in \tau$ such that, for each $x \in F$, either $x \in I$ or $x^{-1} \in I$.

PROOF. (1) $\Rightarrow$ (2). Let $I = (F \setminus U)^{-1} \cup \{0\}$ where $U \in \tau$ is chosen so that $1 \notin UU$. Then $U \subseteq I$ so everything is clear. (2) $\Rightarrow$ (1). I is bounded so for each $S \in \tau$ there exists $x \in F^*$ such that $xI \subseteq S$. Since $y \longmapsto xy$ is a homeomorphism of F onto itself, xI is a neighbourhood of zero. Thus the sets xI, $x \in F^*$ form a basis of neighbourhoods of 0. Since $(F \setminus xI)^{-1} \subseteq Ix^{-1}$ for any $x \in F^*$, the rest is clear. ∎

A *field topology* on F is a ring topology such that the mapping $x \longmapsto x^{-1}$ is continuous at each $x \neq 0$ in F. The following is well-known:

LEMMA 1.2. *Every V-topology on F is a field topology.*

PROOF. Let $a \in F^*$. Fix $b \in F^*$ such that $(a + bI) \cap bI = \emptyset$. Then, for $x \in bI$, $a+x \notin bI$ so $(a+x)^{-1} \in Ib^{-1}$ so $a^{-1} - (a+x)^{-1} = a^{-1}x(a+x)^{-1} \in a^{-1}xIb^{-1}$. Now take x close to zero and use the fact that I is τ-bounded. ∎

EXAMPLES. (1) Any Archimedean absolute value on F induces a V-topology on F (called an *Archimedean* V-topology).

(2) For any proper (invariant) valuation ring in F [25], the associated valuation topology is a V-topology.

(3) By results in [6], [16], any V-topology on F which bounds the commutator group of F is either Archimedean as in (1) or is induced by a valuation ring as in (2).

THEOREM 1.3. *If $\tau_1, \ldots, \tau_n$ are distinct V-topologies on F and $a_1, \ldots, a_n \in F$ are given, then there exists $a \in F$ such that a is arbitrarily close to a_i with respect to τ_i, $i = 1, \ldots, n$.*

We give a quick proof of this, essentially, Weber's proof in [28], which will serve to motivate the generalization to rings given in Theorem 2.1.

PROOF. By Lemma 1.1, we can choose τ_i-bounded sets $I_i \in \tau_i$ such that $x \notin I_i \Rightarrow x^{-1} \in I_i$. It suffices to show the existence of elements $z_1, \ldots, z_n$ with z_i arbitrarily close to 1 with respect to τ_i and arbitrarily close to 0 with respect to τ_j, $j \neq i$. (For then we can take $a = \sum_{i=1}^{n} a_i z_i$.) This is clear if $n = 1$. Suppose $n = 2$. It suffices to construct one of z_1, z_2 since then we can choose the other using $z_1 + z_2 = 1$. We know $\tau_1 \not\subseteq \tau_2$ or $\tau_2 \not\subseteq \tau_1$, say $\tau_2 \not\subseteq \tau_1$. For $y_1, y_2 \in F^*$, if $y_1 I_1 \subseteq I_2 y_2^{-1}$ then, for any $y \in F^*$, $yy_1 I_1 y_2 \subseteq yI_2$ contradicting $\tau_2 \not\subseteq \tau_1$. Thus $y_1 I_1 \not\subseteq I_2 y_2^{-1}$ so we get $x_1 \in y_1 I_1, x_1 \notin I_2 y_2^{-1}$ (so $x_1^{-1} \in y_2 I_2$). Take $z_1 = (x_1 + 1)^{-1}$. Now suppose $n \geq 3$. By symmetry, it suffices to construct z_1. Let $y_1, \ldots, y_n \in F^*$. By the proof in the case $n = 2$ we know $\tau_i \not\subseteq \tau_j$ for $i \neq j$ and that $y_1 I_1$ is τ_i-unbounded if $i \geq 2$ (for each choice of $y_1 \in F^*$) so, by Lemma 0.1, $y_1 I_1 \not\subseteq \bigcup_{i=2}^{n} I_i y_i^{-1}$. Thus we have $x_1 \in y_1 I_1$, $x_1^{-1} \in y_i I_i$, $i \geq 2$, so we can take $z_1 = (x_1 + 1)^{-1}$. ∎

To be able to apply Theorem 1.3 it is necessary to know when two V-topologies are the same. The following facts are well-known (and not difficult to prove):

(1) Two proper valuation rings A, B in F induce the same V-topology iff A, B are dependent (*i.e.*, $AB \neq F$).

(2) Two Archimedian absolute values induce the same V-topology iff they are equivalent.

(3) Archimedian V-topologies are never equal to V-topologies induced by valuations.

EXAMPLE. Order topologies are V-topologies. Each ordering P on F [4] gives rise to a place $\lambda_P: F \longrightarrow \mathbb{R} \cup \{\infty\}$ with valuation ring $B_P = \{x \in F \mid n^2 - x^2 \in P \text{ for some } n \in \mathbb{N}\}$. If $B_P \neq F$ the order topology coincides with the V-topology induced by B_P. If $B_P = F$ the order topology is the Archimedian V-topology induced by the embedding $\lambda_P: F \longrightarrow \mathbb{R}$. Two distinct orderings P_1, P_2 on F give rise to the same V-topology iff $B_{P_1} B_{P_2} \neq F$. All this extends to orderings of higher level [1], [21].

2. ***V*-topologies on rings.** Let A be a (not necessarily commutative) ring with 1. An *epic A-field* is a pair consisting of a skew field F and a ring homomorphism $\alpha: A \longrightarrow F$ such that $\alpha(A)$ generates F as a skew field [3]. Two epic A-fields (F, α), (F', α') are considered *equivalent* if there exists an A-homomorphism $\gamma: F \longrightarrow F'$. If such a γ exists, it is unique and is an isomorphism. In the commutative case, epic A-fields are identified with prime ideals of A, but in general, in the absence of Ore conditions, the structure of epic A-fields is pretty complicated.

A *V-topology* on A is a triple (F, α, τ) where (F, α) is an epic A-field and τ is a V-topology on F. We say two V-topologies (F, α, τ), (F', α', τ') are *equivalent* if (F, α), (F', α') are equivalent and the unique A-isomorphism $\gamma: F \longrightarrow F'$ preserves the topology.

We begin by proving a general criterion for approximation which, in particular, is a generalization of Theorem 1.3:

THEOREM 2.1. *Let (F_i, α_i, τ_i), $i = 1, \ldots, n$ be inequivalent V-topologies on A. Then the following are equivalent:*

(1) $\forall b \in A$, $\exists b' \in A$ such that $\alpha_i(b')$ is arbitrarily close to $\alpha_i(b)^{-1}$ for all i such that $\alpha_i(b) \neq 0$.

(2) $\forall x_i \in F_i$, $i = 1, \ldots, n$, there exists $a \in A$ such that $\alpha_i(a)$ is arbitrarily close to x_i, $i = 1, \ldots, n$.

PROOF. The non-trivial assertion is (1) $\Rightarrow$ (2). Fix a τ_i-bounded set $I_i \in \tau_i$ such that, for $x \in F_i$, either $x \in I_i$ or $x^{-1} \in I_i$.

Suppose $n = 1$. We must show $\alpha_1(A)$ is dense in F_1. Denote by E_1 the closure of $\alpha_1(A)$ in F_1. This is clearly a subring of F_1 and $\alpha_1(A) \subseteq E_1$ so it suffices to show E_1 is a skew field. Let $x \in E_1$, $x \neq 0$. Pick $a \in A$ with $\alpha_1(a)$ close to x. By (1) we have $a' \in A$ with $\alpha_1(a')$ close to x^{-1}. This proves $x^{-1} \in E_1$.

Now suppose $n \geq 2$. It suffices to construct $a_1, \ldots, a_n \in A$ with $\alpha_i(a_i)$ close to 1 and $\alpha_j(a_i)$ close to 0 if $j \neq i$. Suppose $n = 2$. In this case, it suffices to construct one of a_1, a_2 (since then the other can be defined using $a_1 + a_2 = 1$). Suppose first that τ_1, τ_2 induce distinct topologies $\tilde{\tau}_1, \tilde{\tau}_2$ on A, say $\tilde{\tau}_2 \not\subseteq \tilde{\tau}_1$. Let $x_i \in F_i^*$, $i = 1, 2$. Suppose $\alpha_1^{-1}(x_1 I_1) \subseteq \alpha_2^{-1}(I_2 x_2^{-1})$ for some $x_i \in F_i^*$, $i = 1, 2$. Since $\alpha_i(A)$ is dense in F_i, and F_i is τ_i-unbounded, A is τ_i-unbounded, so we can apply Lemma 0.1 to $\tau = \{A\}$ to get $a \in A$ so that $\alpha_i(a)^{-1}$ is close to zero, $i = 1, 2$. ($A \not\subseteq B_1 \cup B_2$ for any τ_i-bounded sets $B_i \subseteq A$. Taking $B_i = \alpha_i^{-1}(I_i y_i^{-1})$, $y_i \in F_i^*$, this yields $a \notin \alpha_1^{-1}(I_1 y_1^{-1}) \cup \alpha_2^{-1}(I_2 y_2^{-1})$, so $\alpha_i(a)^{-1} \in y_i I_i$, $i = 1, 2$.) Then $\alpha_1^{-1}\big(\alpha_1(a)^{-1} x_1 I_1\big) \subseteq \alpha_2^{-1}\big(\alpha_2(a)^{-1} I_2 x_2^{-1}\big)$, and this contradicts $\tilde{\tau}_2 \not\subseteq \tilde{\tau}_1$. Thus $\alpha_1^{-1}(x_1 I_1) \not\subseteq \alpha_2^{-1}(I_2 x_2^{-1})$ so we have $b \in A$ with $\alpha_1(b) \in x_1 I_1$, $\alpha_2(b)^{-1} \in x_2 I_2$. Now use (1) to pick $a_1 \in A$ such that $\alpha_i(a_1)$ is close to $\alpha_i(1 + b)^{-1}$, $i = 1, 2$.

This leaves the case where $\tilde{\tau}_1 = \tilde{\tau}_2$. We claim (F_1, α_1, τ_1) and (F_2, α_2, τ_2) are equivalent in this case (so this case cannot occur). Since we are not assuming commutativity, we have to be a bit careful here. By hypothesis, $\ker(\alpha_1) = \ker(\alpha_2)$ and we have a topological A-isomorphism $\gamma: \alpha_1(A) \longrightarrow \alpha_2(A)$ and we want to show this extends to a topological A-isomorphism $\gamma: F_1 \longrightarrow F_2$. Suppose inductively we have a pair of rings $\alpha_i(A) \subseteq R_i \subseteq F_i$, $i = 1, 2$ and have extended γ to a topological A-isomorphism $\gamma: R_1 \longrightarrow R_2$. Let $S_i \subseteq F_i$, $i = 1, 2$, denote the set of all finite sums of elements of the form $x_1^{e_1} \cdots x_s^{e_s}$ where $x_1, \ldots, x_s \in R_i$, $e_i \in \{-1, 1\}$, $x_i \neq 0$ if $e_i = -1$. This is a subring of F_i. Clearly it suffices to show we have an extension $\gamma: S_1 \longrightarrow S_2$. The point is the following: For each $x \in R_1$, $x \neq 0$, we have $a \in A$ such that $\alpha_1(a)$ is close to x, so $\alpha_2(a) = \gamma\big(\alpha_1(a)\big)$ is close to $\gamma(x)$. Thus, by (1), we have $a' \in A$ such that $\alpha_i(a')$ is close to x^{-1} and $\alpha_2(a') = \gamma\big(\alpha_1(a')\big)$ is close to $\gamma(x)^{-1}$. Applying this many times, if we have any word $x = \cdots + x_1^{e_1} \cdots x_s^{e_s} + \cdots$ in S_1 and look at the corresponding word $x' = \cdots + \gamma(x_1)^{e_1} \cdots \gamma(x_s)^{e_s} + \cdots$ in S_2, then we can find $a \in A$ with $\alpha_1(a)$ close to x and $\alpha_2(a) = \gamma\big(\alpha_1(a)\big)$ close to x'. Thus we have a well-defined extension $\gamma: S_1 \longrightarrow S_2$.

Finally, suppose $n \geq 3$. By symmetry, it suffices to construct a_1. We use Lemma 0.1: We know from the case $n = 2$ that $\alpha_1^{-1}(x_1 I_1)$ is τ_i-unbounded for $i \geq 2$ for each choice of $x_1 \in F_1^*$. Thus, by Lemma 0.1, $\alpha_1^{-1}(x_1 I_1) \not\subseteq \bigcup_{i=2}^n \alpha_i^{-1}(I_i x_i^{-1})$ for all $x_i \in F_i^*$, $i = 1, \ldots, n$. Thus we have $a \in A$ with $\alpha_1(a) \in x_1 I_1$, $\alpha_i(a)^{-1} \in x_i I_i$, $i \geq 2$. Now use (1) again to pick $a_1 \in A$ with $\alpha_i(a_1)$ close to $\alpha_i(1 + a)^{-1}$, $i = 1, \ldots, n$. ∎

We say a V-topology (F, α, τ) on A is *Archimedian* if τ is Archimedian. We say (F, α, τ) is *coarse* if A is τ-unbounded. Clearly, coarseness is a necessary condition for $\alpha(A)$ to be dense in F.

NOTE. (1) Any V-topology on a skew field is coarse.

(2) Any Archimedian V-topology on A is coarse (since $\mathbb{Z} \subseteq A$ is τ-unbounded).

(3) In the non-Archimedian case, if B is a proper valuation ring in F inducing τ then (F, α, τ) is coarse iff $\alpha(A)B = F$.

Coarse V-topologies are fairly numerous: Suppose (F, α, τ) is a non-Archimedian V-topology on A and B is a proper valuation ring in F inducing τ with $\alpha(A) \not\subseteq B$. (*e.g.*, if F is commutative and $\mathbb{Q} \subseteq A$ then such B always exists.) By considering the composite homomorphism $\alpha' : A \longrightarrow \bar{F}$, where $\bar{F}$ is the residue skew field of the (not necessarily invariant) valuation ring $\alpha(A)B$, and the push-down $\bar{B}$ of B to $\bar{F}$, we end up with a specialization (F', α', B') of (F, α, B) where F' is the skew subfield of $\bar{F}$ generated by $\alpha'(A)$ and $B' = \bar{B} \cap F'$. Moreover, $\alpha'(A)B' = F'$, so this process constructs a coarse specialization (F', α', τ') of (F, α, τ).

EXAMPLE. An *ordering* on A is a triple (F, α, P) where (F, α) is an epic A-field and P is an ordering on F. Thus, to every ordering on A, we have an associated V-topology (F, α, τ_P) on A. Let $B_P \subseteq F$ be the valuation ring of the place $\lambda_P : F \longrightarrow \mathbb{R} \cup \{\infty\}$ associated to P. Then (F, α, τ_P) is coarse iff $\alpha(A)B_P = F$. Moreover, if $\alpha(A)B_P \neq F$ then we have a *unique* specialization (F', α', P') of (F, α, P) with $\alpha'(A)B_{P'} = F'$ (obtained by going to the residue skew field of the valuation ring $\alpha(A)B_P$). Note: There is no requirement now that $\alpha(A) \not\subseteq B_P$. If $\alpha(A) \subseteq B_P$, it just means that the resulting specialization is Archimedian. One would expect all this to generalize to higher level orderings as well.

To get concrete results we must also assume the existence of enough units in A. Rather tentatively, we say A has *enough units* if, for each $b \in A$, there exists a non-empty set τ_b of subsets of Ab satisfying

(1) $\forall S \in \tau_b$, $0 \in S$.
(2) $\forall S, S' \in \tau_b$, $\exists S'' \in \tau_b$ such that $S'' \subseteq S \cap S'$.
(3) $\forall S \in \tau_b$, $\forall x \in S$, $\exists S' \in \tau_b$ such that $x + S' \subseteq S$.
(4) $\forall S \in \tau_b$, $S - S \supseteq Ab^m$ for some $m \geq 1$.
(5) $\forall S \in \tau_b$, $1 + S \subseteq A^*$.

Conditions (1), (2), (3) just mean that τ_b is a basis of open neighbourhoods of 0 for a topology on A such that the maps $y \longmapsto x + y$, $x \in A$, are continuous.

EXAMPLES. (1) Let A^m denote the set of finite products $a_1^m \cdots a_k^m$, $a_1, \ldots, a_k \in A$, $k \geq 1$, and let $\sum A^m$ denote the set of all finite sums of elements of A^m. If $1 + \sum A^m \subseteq A^*$ for some (even) $m \geq 1$ [18], [22], then A has enough units. *Proof:* By a polynomial identity in [12, Theorem 8.8.2], $\sum A^m - \sum A^m = A$. Thus we can take $\tau_b = \{S\}$ where $S = \sum A^m b^m$. ∎

(2) As in the commutative case [19], [27] let us say A has *many units* or is *local-global* if for each integer $k \geq 1$ and each polynomial f in non-commuting variables $X_1, \ldots, X_k$ with coefficients in A, if for each residue skew field $A/\mathfrak{m}$ of A there exists $x_1, \ldots, x_k \in A$ (depending on $\mathfrak{m}$) such that $f(x_1, \ldots, x_k) \notin \mathfrak{m}$, then there exists $x_1, \ldots, x_k \in A$ such that $f(x_1, \ldots, x_k) \in A^*$.

(3) Suppose A has many units and all residue skew fields of A are infinite. Then A has enough units and, in this case, we can take $m = 1$. *Proof:* Take τ_b to be the set of all sets

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$S_V := \{xb \in Ab \mid xb + vb + 1 \in A^* \ \forall v \in V\}$ where V runs through all finite subsets of A such that $0 \in V$ and $vb + 1 \in A^* \ \forall v \in V$. We must verify conditions (1) through (5). (1) and (5) are clear. For (2), just observe that $S_V \cap S_{V'} = S_{V \cup V'}$. For (3), note that, if $yb \in S_V$, then $yb + S_{V'} \subseteq S_V$ where $V' = (V+y) \cup \{0\}$. For (4), if $a \in A$, we want $ab \in S_V - S_V$, so we want $x \in A$ satisfying $(*)$ $xb + vb + 1, xb - ab + vb + 1 \in A^*, \forall v \in V$. Take $f(X)$ to be the product of the factors $Xb + vb + 1, Xb - ab + vb + 1, v \in V$ in some order, multiplied by the product of these same factors in the reverse order. Using $pqp \in A^* \Leftrightarrow p, q \in A^*$, we see that $x \in A$ satisfies $(*)$ iff $f(x) \in A^*$. Thus, by our hypothesis, we have $x \in A$ satisfying $(*)$. ∎

(4) Suppose $A/\mathfrak{J}$ is strongly regular [7], where $\mathfrak{J}$ is the Jacobson radical. Then A has many units, and hence has enough units if the residue skew fields of A are infinite.

(5) In (3) and (4), it may be possible to remove the requirement that the residue skew fields are infinite.

THEOREM 2.2. *Suppose A has enough units and that (F_i, α_i, τ_i), $i = 1, \ldots, n$, are inequivalent coarse V-topologies on A. Then, given elements $x_i \in F_i$, there exists $a \in A$ such that $\alpha_i(a)$ is arbitrarily close to x_i, $i = 1, \ldots, n$.*

PROOF. It suffices to verify condition (1) of Theorem 2.1. Let $b \in A$ be given. We may as well assume $\alpha_i(b) \neq 0$ for all $i = 1, \ldots, n$. Fix a τ_i-bounded set $I_i \in \tau_i$ such that, for $x \in F_i$, either $x \in I_i$ or $x^{-1} \in I_i$. Since $\alpha_i(b) \neq 0$ and A is τ_i-unbounded, Ab^m is also τ_i-unbounded. Since $S - S \supseteq Ab^m$ for each $S \in \tau_b$, this means S is τ_i-unbounded for each $S \in \tau_b$. Fix $S \in \tau_b$. By Lemma 0.1, $S \not\subseteq \bigcup_{i=1}^n B_i$ for any τ_i-bounded sets B_i, $i = 1, \ldots, n$. Taking $B_i = \alpha_i^{-1}(I_i x_i^{-1})$, $x_i \in F_i^*$, this yields $a \in A$, $1 + ab \in A^*$, $ab \notin \bigcup_{i=1}^n \alpha_i^{-1}(I_i x_i^{-1})$. Sorting this out, we see that $\alpha_i(ab)^{-1} \in x_i I_i$. To complete the proof, take $b' = (1+ab)^{-1}a$. ∎

REMARK. In particular, this settles the question raised in [22].

For certain kinds of "weak" approximation on commutative rings, we can drop the requirement that A has enough units. In the non-commutative case, localization is more complicated and this method doesn't seem to work.

COROLLARY 2.3. *Suppose A is any commutative ring and (F_i, α_i, τ_i), $i = 1, \ldots, n$ are inequivalent coarse V-topologies on A. Suppose $T_i^* \subseteq F_i^*$ is a subgroup which is a τ_i-neighbourhood of* 1 *such that the factor group F_i^*/T_i^* is torsion, $i = 1, \ldots, n$. Then the natural map $A \setminus \bigcup_{i=1}^n \ker(\alpha_i) \longrightarrow \prod_{i=1}^n F_i^*/T_i^*$ is surjective.*

PROOF. Go to the semi-local ring $B = S^{-1}A$ where $S = A \setminus \bigcup_{i=1}^n \ker(\alpha_i)$. This has enough units, so we can apply Theorem 2.2. Let $x_i \in F_i^*$. We want $a \in A \setminus \bigcup_{i=1}^n \ker(\alpha_i)$ such that $x_i T_i^* = \alpha_i(a) T_i^*$. Let $U_i \in \tau_i$ be such that $1 + U_i \subseteq T_i^*$. By Theorem 2.2 there exists $a/s \in B$ such that $\alpha_i(a/s) \in x_i + x_i U_i$, $i = 1, \ldots, n$. Replacing a by $as^{\ell-1}$ and s by s^ℓ where ℓ is the least common multiple of the orders of the elements $\alpha_i(s)T_i^*$, we can assume $\alpha_i(s) \in T_i^*$. Thus $\alpha_i(a) \in \alpha_i(s) x_i (1 + U_i) \subseteq x_i T_i^*$. ∎

In Corollary 2.3, the hypothesis that the (F_i, α_i, τ_i) are coarse can be relaxed quite a bit. Obviously we only need the (F_i, α_i, τ_i) to be coarse *after* semi-localizing. For example, this is true automatically if there are no proper inclusion relations between the kernels of the α_i.

NOTE. See [13], [14] for approximation theorems for valuations and places on ternary fields (sometimes called *planar ternary rings*).

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