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QUASI-DIFFERENTIABLE NORMS

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Let E be a real Banach space with norm ρ. Let $S=\{x \in E: \rho(x)=1\}$. A norm on E is admissible if it generates the same topology as ρ.

The norm ρ is *Gateaux differentiable* if for each $x \in S$ and $u \in E$,

$$G(x, u) = \lim_{h\to 0} \frac{\rho(x+hu)-\rho(x)}{h}$$

exists.

It is well-known that, if ρ is Gateaux differentiable, for each $x \in S$, $G(x, \cdot)$ is a norm one linear functional on E and the mapping $x \mapsto G(x, \cdot)$ is norm-to-w^* continuous. Also, each separable real Banach space has an admissible Gateaux differentiable norm [2, 3].

Let $U \subseteq E$, $V \subseteq F$ be open subsets of the Banach spaces E and F. $f: U \to V$ is said to be *differentiable at* $x \in U$ if there is $Df(x) \in L(E, F)$, the set of bounded linear transformations from E to F, such that

$$f(x+u) = f(x)+Df(x)(u)+\rho(x)\cdot r$$

where $\lim_{u\to 0} r=0$. f is said to be *quasi-differentiable at* $x \in U$ if there is $f'x \in L(E, F)$ such that for each continuous path $\alpha:(-1, 1)\to U$, $\alpha(0)=x$, α differentiable at 0, then $f(\alpha(t))$ is differentiable at $t=0$ and $D_t f(\alpha(0))=(f'x)(D\alpha(0))$ where D_t is differentiation with respect to t. f is of *class* Q^1 on U if f is continuous and quasi-differentiable on U, $f': U\to L(E, F)$ is bounded in the $L(E, F)$ norm and the mapping $(x, y)\mapsto (f'x)(y)$ is continuous on $U\times U$. We will say $f \in Q^1(U, V)$.

V. Goodman considers quasi-differentiable functions in [4]. By characterizing quasi-differentiable norms as Gateaux differentiable norms, using the renorming result mentioned above and using the methods of Bonic and Frampton [1] we obtain some of Goodman's results together with an improved approximation theorem.

1. Theorem. *Let E be a Banach space and let σ be a continuous norm on E. Then $\sigma \in Q^1(E\backslash\{0\}, R)$ if and only if σ is Gateaux differentiable.*

Proof. It is easily shown that if α is quasi-differentiable, σ is Gateaux differentiable and $G(x, u)=(\sigma'x)(u)$.

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Now suppose σ is Gateaux differentiable. Let $\alpha:(-1,1)\to E$, $\alpha(0)=x$, α differentiable at $t=0$. So $\alpha(h)=\alpha(0)+D\alpha(0)h+|h|\cdot r$ where $\lim_{h\to 0} r=0$. Hence we get the inequalities

$$\frac{\sigma(x+h\,D\alpha(0))-\sigma(x)}{h}-\frac{|h|}{h}\sigma(r)\le\frac{\sigma(\alpha(h))-\sigma(\alpha(0))}{h}\le\frac{\sigma(x+h\,D\alpha)(0)-\sigma(x)}{h}+\frac{|h|}{h}\sigma(r).$$

Thus σ is quasi-differentiable at x and $\sigma'x=G(x,\cdot)$.

$\sigma':E\backslash\{0\}\to E^*$ is bounded since $\sigma'x=G(x,\cdot)$ and $G(x,\cdot)$ has norm one. Also $(x,y)\mapsto(\sigma'x)(y)$ is continuous since $x\mapsto\sigma'x$ is norm-to-w^* continuous and $\sigma'x$ is a continuous functional.

2. THEOREM. *Let E be a real Banach space that admits a norm σ that is Gateaux differentiable. Then there is $f\in Q^1(E,R)$ such that f has bounded non-empty support.*

Proof. Choose $h\in C^1(R,R)$ such that h' is bounded, $h(0)=1$, $h(t)>0$ if $t>1$. Then f is obtained by extending $h\cdot\sigma\in Q^1(E\backslash\{0\},R)$ to E.

3. COROLLARY (Goodman). *If E is separable then E admits Q^1-partitions of unity.*

Comment on Proof. Since E is separable, E admits a Gateaux differentiable norm. By Theorem 2 there is a non-trivial real valued Q^1 function on E with bounded support. Evidently Q^1 is not a smoothness category as defined by Bonic and Frampton [1] but the composition of a Q^1 function and C^1 function with bounded derivative is Q^1, so their construction [1, Theorem 1] with minor modifications gives the desired partition of unity.

4. THEOREM. *If E is separable, then each real-valued continuous function on E can be uniformly approximated by a Q^1-function.*

Proof. Let $f\in C(E,R)$. Cover R with open intervals of radius $\varepsilon/2$. Let U be the open cover of E obtained by pulling the open intervals back by f. By Corollary 3 there is a Q^1-partition of unity $\{\psi_\alpha\}$ subordinate to U. $g=\sum_\alpha f(x_\alpha)\psi_\alpha$ is of class Q^1, where x_α is chosen so that $\psi_\alpha(x_\alpha)>0$. Then

$$|f(x)-g(x)|<\varepsilon \quad \text{for each } x\in E.$$

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