Pieri rules for skew dual immaculate functions

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Abstract. In this paper, we give Pieri rules for skew dual immaculate functions and their recently
discovered row-strict counterparts. We establish our rules using a right-action analogue of the skew
Littlewood–Richardson rule for Hopf algebras of Lam–Lauve–Sottile. We also obtain Pieri rules for
row-strict (dual) immaculate functions.

1 Introduction

Schur-like functions are a new and flourishing area since the discovery of qua-
sisymmetric Schur functions in 2011 [11], which led to numerous other similar
functions being discovered, for example, [1, 4, 6, 10, 14–17]. In essence, Schur-like
functions are functions that refine the ubiquitous Schur functions and reflect many
of their properties, such as their combinatorics [2, 9], their representation theory
[5, 7, 21, 22], and in the case of quasisymmetric Schur functions have already been
applied to resolve conjectures [13]. Of the various Schur-like functions to arise after
the quasisymmetric Schur functions, two were naturally related to them: the dual
immaculate functions [6] and the row-strict quasisymmetric Schur functions [17].
Recently, a fourth basis that interpolates between these latter two bases, the *row-
strict dual immaculate functions*, was discovered [19], thus completing the picture.
The representation theory of these functions was revealed in [20], in addition to the
fundamental combinatorics in [19]. In this paper, we extend the combinatorics to
uncover skew Pieri rules in the spirit of [3, 12, 23] for both row-strict and classical
dual immaculate functions.

More precisely, our paper is structured as follows. In Section 2, we establish a
right-action analogue of [12, Theorem 2.1] in Theorem 2.6. We then recall required
background for the Hopf algebras of quasisymmetric functions, QSym, and non-
commutative symmetric functions, NSym, in Section 3. Finally, in Section 4, we give
(left) Pieri rules for row-strict immaculate functions and row-strict dual immaculate
functions in Corollaries 4.3 and 4.5, respectively. Our final theorem is Theorem 4.7,

in which we establish Pieri rules for skew dual immaculate functions, and row-strict skew dual immaculate functions.

2 The right-action skew Littlewood–Richardson rule for Hopf algebras

We begin by recalling and deducing general Hopf algebra results that will be useful later. Following Tewari and van Willigenburg [23], let H and H^* be a pair of dual Hopf algebras over a field k with duality pairing $\langle \, , \, \rangle : H \otimes H^* \to k$ for which the structure of H^* is dual to that of H and vice versa. Let $h \in H, a \in H^*$. By Sweedler notation, we have coproduct denoted by $\Delta h = \sum h_1 \otimes h_2$, and similarly $h_1 h_2 = h_1 \cdot h_2$ denotes product. We define the action of one algebra on the other one by the following:

$$(2.1) \qquad\qquad h \rightharpoonup a = \sum \langle h, a_2 \rangle a_1,$$

$$(2.2) \qquad\qquad a \rightharpoonup h = \sum \langle h_2, a \rangle h_1.$$

Let $S : H \to H$ denote the antipode map. Then for $\Delta h = \sum h_1 \otimes h_2$,

$$(2.3) \qquad\qquad \sum (Sh_1)h_2 = \varepsilon(h)1_H = \sum h_1(Sh_2),$$

where ε and 1 denote counit and unit, respectively. Following Montgomery [18], we can define the convolution product $*$ for f and g in H by

$$(f * g)(a) = \sum \langle f, a_1 \rangle \langle g, a_2 \rangle = \langle fg, a \rangle.$$

Then it follows that

$$\langle g, f \rightharpoonup a \rangle = \langle gf, a \rangle.$$

Similarly, $\langle a \rightharpoonup f, b \rangle = \langle f, ba \rangle$. Since H^* is a left H-module algebra under $\rightharpoonup$, we have that

$$h \rightharpoonup (a \cdot b) = \sum (h_1 \rightharpoonup a) \cdot (h_2 \rightharpoonup b).$$

Lemma 2.1 [12]   *For $g, h \in H$ and $a \in H^*$,*

$$(a \rightharpoonup g) \cdot h = \sum (S(h_2) \rightharpoonup a) \rightharpoonup (g \cdot h_1),$$

where $S : H \to H$ is the antipode.

As in Montgomery [18], define a right action by the following:

$$(2.4) \qquad\qquad h \leftharpoonup a = \sum \langle h, a_1 \rangle a_2,$$

$$(2.5) \qquad\qquad a \leftharpoonup h = \sum \langle h_1, a \rangle h_2.$$

As before, it follows that $\langle g, f \leftharpoonup a \rangle = \langle fg, a \rangle$ and $\langle a \leftharpoonup f, b \rangle = \langle f, ab \rangle$.

Lemma 2.2   *Let $f \in H$ and $a, b \in H^*$. Then*

$$f \leftharpoonup (a \cdot b) = \sum (f_1 \leftharpoonup a) \cdot (f_2 \leftharpoonup b).$$

Proof   Let $f, g \in H$ and $a, b \in H^*$. Then

$$\langle g, f \leftharpoonup (a \cdot b)\rangle = \langle fg, ab\rangle$$
$$= \langle a \leftharpoonup (fg), b\rangle$$
$$= \sum \langle f_1 g_1, a\rangle \langle f_2 g_2, b\rangle$$
$$= \sum \langle g_1, f_1 \leftharpoonup a\rangle \langle g_2, f_2 \leftharpoonup b\rangle$$
$$= \sum \langle g, (f_1 \leftharpoonup a) \cdot (f_2 \leftharpoonup b)\rangle.$$

Thus, $f \leftharpoonup (a \cdot b) = \sum (f_1 \leftharpoonup a) \cdot (f_2 \leftharpoonup b)$.   ■

Lemma 2.3   *Let $a \in H^*$. Then*

$$\varepsilon(h) \cdot 1_H \leftharpoonup a = a$$

for any $h \in H$.

Proof   Let $a \in H^*$ and $h \in H$. Then

$$\varepsilon(h) \cdot 1_H \leftharpoonup a = \sum \langle \varepsilon(h) \cdot 1_H, a_1\rangle a_2.$$

This is only nonzero when $a_1 = 1_{H^*}$.   ■

Lemma 2.4   *Let $h \in H$ and $a, b \in H^*$. Then*

$$a \cdot (h \leftharpoonup b) = \sum h_1 \leftharpoonup ((S(h_2) \leftharpoonup a) \cdot b).$$

Proof   Expand the sum using Lemma 2.2 and coassociativity, $(\Delta \otimes 1) \circ \Delta(h) = (1 \otimes \Delta) \circ \Delta(h) = \sum h_1 \otimes h_2 \otimes h_3$, to get

$$\sum h_1 \leftharpoonup ((S(h_2) \leftharpoonup a) \cdot b) = \sum (h_1 \leftharpoonup (S(h_2) \leftharpoonup a)) \cdot (h_3 \leftharpoonup b)$$
$$= \sum (h_1 \cdot S(h_2) \leftharpoonup a) \cdot (h_3 \leftharpoonup b) \text{ since } H^* \text{ is an } H\text{-module}$$
$$= ((\varepsilon(h) \cdot 1_H) \leftharpoonup a) \cdot (h \leftharpoonup b) \text{ by (2.3)}$$
$$= a \cdot (h \leftharpoonup b) \text{ by Lemma 2.3.}$$ ■

Lemma 2.5   *Let $g, h \in H$ and $a \in H^*$. Then*

$$h \cdot (a \leftharpoonup g) = \sum (S(h_2) \leftharpoonup a) \leftharpoonup (h_1 \cdot g).$$

Proof   Let $g, h \in H$ and $a, b \in H^*$. Then

$$\langle h \cdot (a \leftharpoonup g), b\rangle = \langle a \leftharpoonup g, h \leftharpoonup b\rangle$$
$$= \langle g, a \cdot (h \leftharpoonup b)\rangle$$
$$= \langle g, \sum (h_1 \leftharpoonup ((S(h_2) \leftharpoonup a) \cdot b)\rangle \text{ by Lemma 2.4}$$
$$= \sum \langle g, h_1 \leftharpoonup ((S(h_2) \leftharpoonup a) \cdot b)\rangle$$
$$= \sum \langle h_1 \cdot g, (S(h_2) \leftharpoonup a) \cdot b\rangle$$
$$= \sum \langle (S(h_2) \leftharpoonup a) \leftharpoonup (h_1 \cdot g), b\rangle.$$ ■

We can use the right action to obtain an algebraic Littlewood–Richardson formula analogous to [12, Theorem 2.1] for those bases whose skew elements appear as the right tensor factor in the coproduct.

Let $\{L_\alpha\} \subset H$ and $\{R_\beta\} \subset H^*$ be dual bases with indexing set $\mathcal{P}$. Then

$$(2.6) \qquad L_\alpha \cdot L_\beta = \sum_\gamma b^\gamma_{\alpha,\beta} L_\gamma \qquad \Delta(L_\gamma) = \sum_{\alpha,\beta} c^\gamma_{\alpha,\beta} L_\alpha \otimes L_\beta,$$

$$(2.7) \qquad R_\alpha \cdot R_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} R_\gamma \qquad \Delta(R_\gamma) = \sum_{\alpha,\beta} b^\gamma_{\alpha,\beta} R_\alpha \otimes R_\beta,$$

where $b^\gamma_{\alpha,\beta}$ and $c^\gamma_{\alpha,\beta}$ are structure constants. We can also write

$$(2.8) \qquad \Delta(L_\gamma) = \sum_\delta L_\delta \otimes L_{\gamma/\delta} \qquad \Delta(R_\gamma) = \sum_\delta R_\delta \otimes R_{\gamma/\delta}.$$

Note that $L_\alpha \leftharpoonup R_\beta = R_{\beta/\alpha}$ and $R_\beta \leftharpoonup L_\alpha = L_{\alpha/\beta}$. Further,

$$(2.9) \qquad \Delta(L_{\alpha/\beta}) = \sum_{\pi,\rho} c^\alpha_{\pi,\rho,\beta} L_\pi \otimes L_\rho \qquad \Delta(R_{\alpha/\beta}) = \sum_{\pi,\rho} b^\alpha_{\pi,\rho,\beta} R_\pi \otimes R_\rho.$$

The antipode acts on L_ρ by $S(L_\rho) = (-1)^{\theta(\rho)} L_{\rho^*}$ where $\theta : \mathcal{P} \to \mathbb{N}$ and $* : \mathcal{P} \to \mathcal{P}$.

Theorem 2.6   *For* $\alpha, \beta, \gamma, \delta \in \mathcal{P}$,

$$L_{\alpha/\beta} \cdot L_{\gamma/\delta} = \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} L_{\nu/\mu}.$$

Proof   We use Lemma 2.5 and the preceding facts about the product, coproduct, and antipode maps on H and H^* to obtain

$$
\begin{aligned}
L_{\alpha/\beta} \cdot L_{\gamma/\delta} &= L_{\alpha/\beta} \cdot (R_\delta \leftharpoonup L_\gamma) \\
&= \sum_{\pi,\rho} c^\alpha_{\pi,\rho,\beta} (S(L_\rho) \leftharpoonup R_\delta) \leftharpoonup (L_\pi \cdot L_\gamma) \\
&= \sum_{\pi,\rho} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} (L_{\rho^*} \leftharpoonup R_\delta) \leftharpoonup (L_\pi \cdot L_\gamma) \\
&= \sum_{\pi,\rho} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} \left(R_{\delta/\rho^*} \leftharpoonup \left(\sum_\nu b^\nu_{\pi,\gamma} L_\nu \right) \right) \\
&= \sum_{\pi,\rho,\nu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} (R_{\delta/\rho^*} \leftharpoonup L_\nu) \\
&= \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} (R_\mu \leftharpoonup L_\nu) \\
&= \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} L_{\nu/\mu}. \qquad \blacksquare
\end{aligned}
$$

3   **The dual Hopf algebras** QSym **and** NSym

We now focus our attention on the dual Hopf algebra pair of noncommutative symmetric functions and quasisymmetric functions, and introduce our main objects of study the (row-strict) dual immaculate functions.

A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, denoted by $\alpha \vDash n$ is a list of positive integers such that $\sum_{i=1}^{k} \alpha_i = n$. We call n the *size* of α and sometimes denote it by $|\alpha|$, and call k the *length* of α and sometimes denote it by $\ell(\alpha)$. If $\alpha_{j_1} = \cdots = \alpha_{j_m} = i$, we sometimes abbreviate this to i^m, and denote the *empty composition* of 0 by $\varnothing$. There exists a natural correspondence between compositions $\alpha \vDash n$ and subsets $S \subseteq \{1, \ldots, n-1\} = [n-1]$. More precisely, $\alpha = (\alpha_1, \ldots, \alpha_k)$ corresponds to $\mathrm{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$, and conversely $S = \{s_1, \ldots, s_{k-1}\}$ corresponds to $\mathrm{comp}(S) = (s_1, s_2 - s_1, \ldots, n - s_{k-1})$. We also denote by S^c the set complement of S in $[n-1]$.

Given a composition α, its *diagram*, also denoted by α, is the array of left-justified boxes with α_i boxes in row i from the *bottom*. Given two compositions α, β we say that $\beta \subseteq \alpha$ if $\beta_j \leq \alpha_j$ for all $1 \leq j \leq \ell(\beta) \leq \ell(\alpha)$, and given α, β such that $\beta \subseteq \alpha$, the *skew diagram* α/β is the array of boxes in α but not β when β is placed in the bottom-left corner of α. If, furthermore, $\beta \subseteq \alpha$ and $\alpha_j - \beta_j \in \{0,1\}$ for all $1 \leq j \leq \ell(\beta) \leq \ell(\alpha)$, then we call α/β a *vertical strip*.

Example 3.1   If $\alpha = (3, 4, 1)$, then $|\alpha| = 8, \ell(\alpha) = 3$, and $\mathrm{set}(\alpha) = \{3, 7\}$. Its diagram is
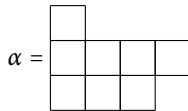
and if $\beta = (2, 4)$, then
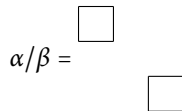
is a vertical strip.

Definition 3.2   Given a composition α, a *standard immaculate tableau* T of *shape* α is a bijective filling of its diagram with $1, \ldots, |\alpha|$ such that:

(1)   The entries in the leftmost column increase from bottom to top.
(2)   The entries in each row increase from left to right.

We obtain a *standard skew immaculate tableau* of shape α/β by extending the definition to skew diagrams α/β in the natural way.

Given a standard (skew) immaculate tableau, T, its *descent set* is

$$\mathrm{Des}(T) = \{i : i + 1 \text{ appears strictly above } i \text{ in } T\}.$$

Example 3.3   A standard skew immaculate tableau of shape $(3, 4, 1)/(1)$ is

$$T = \begin{array}{|c|c|c|c|}\hline 7 & & & \\\hline 2 & 3 & 4 & 6 \\\hline & 1 & 5 & \\\hline \end{array}$$

with $\mathrm{Des}(T) = \{1, 5, 6\}$.

We are now ready to define our Hopf algebras and functions of central interest.

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$ and commuting variables $\{x_1, x_2, \ldots\}$ we define the *monomial quasisymmetric function* M_α to be

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

the *fundamental quasisymmetric function* F_α to be

$$F_\alpha = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin \mathrm{set}(\alpha)}} x_{i_1} \cdots x_{i_n}$$

the *dual immaculate function* $\mathfrak{S}_\alpha^*$ to be

$$\mathfrak{S}_\alpha^* = \sum_T F_{\mathrm{comp}(\mathrm{Des}(T))}$$

and the *row-strict dual immaculate function* $\mathcal{R}\mathfrak{S}_\alpha^*$ to be

$$\mathcal{R}\mathfrak{S}_\alpha^* = \sum_T F_{\mathrm{comp}(\mathrm{Des}(T)^c)},$$

where the latter two sums are over all standard immaculate tableaux T of shape α. These extend naturally to give *skew* dual immaculate and row-strict dual immaculate functions $\mathfrak{S}_{\alpha/\beta}^*$ [6] and $\mathcal{R}\mathfrak{S}_{\alpha/\beta}^*$[19], where α/β is a skew diagram.

Example 3.4   We have that $M_{(2)} = x_1^2 + x_2^2 + x_3^2 + \cdots$ and $F_{(2)} = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots = \mathfrak{S}_{(2)}^* = \mathcal{R}\mathfrak{S}_{(1^2)}^*$ from the following standard immaculate tableau T with $\mathrm{Des}(T) = \varnothing$.

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\\hline \end{array}$$

The set of all monomial or fundamental quasisymmetric functions forms a basis for the *Hopf algebra of quasisymmetric functions* QSym, as does the set of all (row-strict) dual immaculate functions. There exists an involutory automorphism ψ defined on fundamental quasisymmetric functions by

$$\psi(F_\alpha) = F_{\mathrm{comp}(\mathrm{set}(\alpha^c))}$$

such that [19]

$$\psi(\mathfrak{S}_\alpha^*) = \mathcal{R}\mathfrak{S}_\alpha^*$$

for a composition α. This extends naturally to skew diagrams α/β to give

$$\psi(\mathfrak{S}^*_{\alpha/\beta}) = \mathcal{R}\mathfrak{S}^*_{\alpha/\beta}.$$

Dual to the Hopf algebra of quasisymmetric functions is the *Hopf algebra of noncommutative symmetric functions* NSym. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$ and *noncommuting variables* $\{y_1, y_2, \ldots\}$ we define the *nth elementary noncommutative symmetric function* $\boldsymbol{e}_n$ to be

$$\boldsymbol{e}_n = \sum_{i_1 < \cdots < i_n} y_{i_1} \cdots y_{i_n}$$

and the *elementary noncommutative symmetric function* $\boldsymbol{e}_\alpha$ to be

$$\boldsymbol{e}_\alpha = \boldsymbol{e}_{\alpha_1} \cdots \boldsymbol{e}_{\alpha_k}.$$

Meanwhile, we define the *nth complete homogeneous noncommutative symmetric function* $\boldsymbol{h}_n$ to be

$$\boldsymbol{h}_n = \sum_{i_1 \leq \cdots \leq i_n} y_{i_1} \cdots y_{i_n}$$

and the *complete homogeneous noncommutative symmetric function* $\boldsymbol{h}_\alpha$ to be

$$\boldsymbol{h}_\alpha = \boldsymbol{h}_{\alpha_1} \cdots \boldsymbol{h}_{\alpha_k}.$$

The set of all elementary or complete homogeneous noncommutative symmetric functions forms a basis for NSym. The duality between QSym and NSym is given by

$$\langle M_\alpha, \boldsymbol{h}_\alpha \rangle = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. This induces the bases dual to the (row-strict) dual immaculate functions via

$$\langle \mathfrak{S}^*_\alpha, \mathfrak{S}_\alpha \rangle = \delta_{\alpha\beta} \qquad \langle \mathcal{R}\mathfrak{S}^*_\alpha, \mathcal{R}\mathfrak{S}_\alpha \rangle = \delta_{\alpha\beta}$$

and implicitly defines the bases of *immaculate* and *row-strict immaculate functions*. While concrete combinatorial definitions of these functions have been established [6, 19], we will not need them here. However, what we will need is the involutory automorphism in NSym corresponding to ψ in QSym, defined by $\psi(\boldsymbol{e}_\alpha) = \boldsymbol{h}_\alpha$ that gives $\psi(\mathfrak{S}_\alpha) = \mathcal{R}\mathfrak{S}_\alpha$ [19].

4 The Pieri rules for skew dual immaculate functions

A left Pieri rule for immaculate functions was conjectured in [6, Conjecture 3.7] and proved in [8]. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we say that $\mathrm{tail}(\alpha) = (\alpha_2, \ldots, \alpha_k)$. If $\beta \in \mathbb{Z}^k$, then $\mathrm{neg}(\alpha - \beta) = |\{i : \alpha_i - \beta_i < 0\}|$. Let $\mathrm{sgn}(\beta) = (-1)^{\mathrm{neg}(\beta)}$ with $\mathrm{neg}(\beta) = |\{i : \beta_i < 0\}|$.

Following [8], we define $Z_{s,\alpha}$ to be a set of all $\beta \in \mathbb{Z}^k$ such that:

(1) $\beta_1 + \cdots + \beta_k = s$ and $\beta_1 + \cdots + \beta_i \leq s$ for all $i < k$.
(2) $\alpha_i - \beta_i \geq 0$ for all $1 \leq i \leq k$ and $|i : \alpha_i - \beta_i = 0| \leq 1$.
(3)   For all $1 \leq i \leq k$,

- if $\alpha_i > s - (\beta_1 + \cdots + \beta_{i-1})$, then $0 \le \beta_i \le s - (\beta_1 + \cdots + \beta_{i-1})$,
- if $\alpha_i < s - (\beta_1 + \cdots + \beta_{i-1})$, then $\beta_i < 0$, and
- if $\alpha_i = s - (\beta_1 + \cdots + \beta_{i-1})$, then either $\beta_i < 0$ or $\beta_i = \alpha_i$ and $\beta_{i+1} = \cdots = \beta_k = 0$.

Now we are ready to define the coefficients of the immaculate basis appearing in the left Pieri rule.

Definition 4.1 [8] For a positive integer s and compositions α, γ with $|\alpha| - |\gamma| = s$, let $1 \le j \le k$ be the smallest integer such that $\alpha_i = \gamma_{i-1}$ for all $j < i \le k$ where $j = k$ when $\alpha_k \ne \gamma_{k-1}$. Let $j \le r \le k$ be the largest integer such that $\alpha_j < \alpha_{j+1} < \cdots < \alpha_r$. Let $\alpha^{(i)} = (\alpha_1, \ldots, \alpha_i)$. Then define

$$c_{s,\alpha}^{\gamma} = \begin{cases} \mathrm{sgn}(\alpha - \gamma), & \text{if } \ell(\gamma) = \ell(\alpha) \text{ and } \alpha - \gamma \in Z_{s,\alpha}, \\ \mathrm{sgn}(\alpha^{(j-1)} - \gamma^{(j-1)}), & \text{if } \ell(\gamma) = \ell(\alpha) - 1, \\ & r - j \text{ is even, and} \\ & (\alpha^{(j-1)} - \gamma^{(j-1)}, \alpha_j, 0, \ldots, 0) \in Z_{s,\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.2 [6, 8] Let $m > 0$ and α be a composition. Then

$$h_m \mathfrak{S}_\alpha = \sum_{\substack{\beta \vDash |\alpha|+m \\ \beta_1 \ge m \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c_{\beta_1-m,\alpha}^{\mathrm{tail}(\beta)} \mathfrak{S}_\beta.$$

Applying ψ to both sides of the left Pieri rule in Theorem 4.2 immediately yields a left Pieri rule for row-strict immaculate functions.

Corollary 4.3 Let $m > 0$ and α be a composition. Then

$$e_m \mathcal{R}\mathfrak{S}_\alpha = \sum_{\substack{\beta \vDash |\alpha|+m \\ \beta_1 \ge m \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c_{\beta_1-m,\alpha}^{\mathrm{tail}(\beta)} \mathcal{R}\mathfrak{S}_\beta.$$

Lemma 3.1 of [8] shows that for $s \ge 0$, $r > 0$ and compositions α, β with $|\alpha| = |\beta| + s$,

$$\langle \mathfrak{S}_\alpha, F_{(s)} \mathfrak{S}_\beta^* \rangle = \langle h_r \mathfrak{S}_\alpha, \mathfrak{S}_{(s+r,\beta)}^* \rangle.$$

This leads to the following Pieri rule for dual immaculate functions.

Theorem 4.4 [8] Let $s > 0$ and α be a composition. Then

$$F_{(s)} \mathfrak{S}_\alpha^* = \sum_{\substack{\beta \vDash |\alpha|+s \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c_{s,\beta}^{\alpha} \mathfrak{S}_\beta^*.$$

Again, applying ψ to both sides gives a Pieri rule for row-strict dual immaculate functions.

Corollary 4.5 *Let $s > 0$ and α be a composition. Then*

$$F_{(1^s)}\mathcal{R}\mathfrak{S}^*_\alpha = \sum_{\substack{\beta \vDash |\alpha| + s \\ 0 \leq \ell(\beta) - \ell(\alpha) \leq 1}} c^\alpha_{s,\beta} \mathcal{R}\mathfrak{S}^*_\beta.$$

We use these results together with Hopf algebra computations to construct a Pieri rule for skew dual immaculate functions. Using the map ψ, this also gives a Pieri rule for row-strict skew dual immaculate functions. But first, we have a small, yet crucial, lemma.

Lemma 4.6 *Let α and γ be compositions. Then $\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha = \mathfrak{S}^*_{\alpha/\gamma}$.*

Proof Recall that if $H = \mathrm{QSym}$ and $H^* = \mathrm{NSym}$ are our pair of dual Hopf algebras, then we know $\Delta\mathfrak{S}^*_\alpha = \sum_\beta \mathfrak{S}^*_\beta \otimes \mathfrak{S}^*_{\alpha/\beta}$ and we have that

$$\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha = \sum_\beta \langle \mathfrak{S}_\gamma, \mathfrak{S}^*_\beta \rangle \mathfrak{S}^*_{\alpha/\beta} = \mathfrak{S}^*_{\alpha/\gamma}$$

since $\langle \mathfrak{S}_\gamma, \mathfrak{S}^*_\beta \rangle = \delta_{\gamma\beta}$, where $\delta_{\gamma\beta} = 1$ if $\gamma = \beta$ and 0 otherwise. ∎

We can now give our Pieri rule for (row-strict) skew dual immaculate functions.

Theorem 4.7 *Let $\gamma \subseteq \alpha$. Then*

$$\mathfrak{S}^*_{(s)}\mathfrak{S}^*_{\alpha/\gamma} = \sum_{\beta/\tau}(-1)^{|\gamma|-|\tau|} \cdot c^\alpha_{|\beta|-|\alpha|,\beta} \, \mathfrak{S}^*_{\beta/\tau},$$

and hence by applying ψ to both sides

$$\mathcal{R}\mathfrak{S}^*_{(s)}\mathcal{R}\mathfrak{S}^*_{\alpha/\gamma} = \sum_{\beta/\tau}(-1)^{|\gamma|-|\tau|} \cdot c^\alpha_{|\beta|-|\alpha|,\beta} \, \mathcal{R}\mathfrak{S}^*_{\beta/\tau},$$

where $|\beta/\tau| = |\alpha/\gamma| + s$, γ/τ is a vertical strip of length at most s, $\ell(\beta) - \ell(\alpha) \in \{0, 1\}$ and $c^\alpha_{|\beta|-|\alpha|,\beta}$ is the coefficient of Definition 4.1. These decompositions are multiplicity-free up to sign.

Proof Note that $\mathfrak{S}^*_{(1^s)} = F_{(1^s)}$ and $\mathfrak{S}^*_{(s)} = F_{(s)}$. Recall that

$$(4.1) \qquad\qquad \Delta F_\alpha = \sum_{\substack{(\beta,\gamma) \text{ with} \\ \beta \cdot \gamma = \alpha \text{ or} \\ \beta \odot \gamma = \alpha}} F_\beta \otimes F_\gamma,$$

where for $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_l)$, $\beta \cdot \gamma = (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_l)$ is the *concatenation* of β and γ, and $\beta \odot \gamma = (\beta_1, \ldots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \ldots, \gamma_l)$ is the *near-concatenation* of β and γ.

Then we have that

$$\Delta(F_{(s)}) = \sum_{i=0}^{s} F_{(i)} \otimes F_{(s-i)}.$$

Thus,

$$
\begin{aligned}
\mathfrak{S}^*_{(s)}\mathfrak{S}^*_{\alpha/\gamma} &= \mathfrak{S}^*_{(s)}(\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha) \qquad \text{by Lemma 4.6} \\
&= F_{(s)}(\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha) \\
&= \sum_{i=0}^{s}(S(F_{(s-i)}) \leftharpoonup \mathfrak{S}_\gamma) \leftharpoonup (F_{(i)}\mathfrak{S}^*_\alpha) \quad \text{by Lemma 2.5.}
\end{aligned}
$$

We first compute $S(F_{(s-i)}) \leftharpoonup \mathfrak{S}_\gamma$. Since it is well known that $S(F_\alpha) = (-1)^{|\alpha|}F_{\text{comp}(\text{set}(\alpha)^c)}$ we have that $S(F_{(s-i)}) = (-1)^{s-i}F_{(1^{s-i})}$. Furthermore, we can write the coproduct as

$$
\Delta(\mathfrak{S}_\gamma) = \sum_{\delta,\tau} b^\gamma_{\delta,\tau}\mathfrak{S}_\delta \otimes \mathfrak{S}_\tau.
$$

Thus,

$$
\begin{aligned}
S(F_{(s-i)}) \leftharpoonup \mathfrak{S}_\gamma &= (-1)^{s-i}F_{(1^{s-i})} \leftharpoonup \mathfrak{S}_\gamma \\
&= \sum_{\delta,\tau}(-1)^{s-i}b^\gamma_{\delta,\tau}\langle F_{(1^{s-i})}, \mathfrak{S}_\delta\rangle\mathfrak{S}_\tau \\
&= \sum_{\delta,\tau}(-1)^{s-i}b^\gamma_{\delta,\tau}\langle \mathfrak{S}^*_{(1^{s-i})}, \mathfrak{S}_\delta\rangle\mathfrak{S}_\tau \\
&= \sum_{\tau}(-1)^{s-i}b^\gamma_{(1^{s-i}),\tau}\mathfrak{S}_\tau.
\end{aligned}
$$

By the definition of product and coproduct on NSym, we have that

$$
b^\gamma_{\delta,\tau} = \langle\Delta\mathfrak{S}_\gamma, \mathfrak{S}^*_\delta \otimes \mathfrak{S}^*_\tau\rangle = \langle\mathfrak{S}_\gamma, \mathfrak{S}^*_\delta \cdot \mathfrak{S}^*_\tau\rangle.
$$

To compute this for $\delta = (1^{s-i})$, we use Proposition 3.34 from [6] which states that $F^\perp_{(1^r)}\mathfrak{S}_\alpha = \sum_\beta \mathfrak{S}_\beta$, where $\beta \in \mathbb{Z}^{\ell(\alpha)}$, $\alpha_k - \beta_k \in \{0,1\}$ for all k and $|\beta| = |\alpha| - r$. The operator $F^\perp$ is used throughout [6], and has the property that $\langle F^\perp\mathfrak{S}_\alpha, \mathfrak{S}^*_\beta\rangle = \langle\mathfrak{S}_\alpha, F\mathfrak{S}^*_\beta\rangle$.

Thus,

$$
\begin{aligned}
b^\gamma_{(1^{s-i}),\tau} &= \langle\mathfrak{S}_\gamma, \mathfrak{S}^*_{(1^{s-i})}\mathfrak{S}^*_\tau\rangle \\
&= \langle\mathfrak{S}_\gamma, F_{(1^{s-i})}\mathfrak{S}^*_\tau\rangle \\
&= \langle F^\perp_{(1^{s-i})}\mathfrak{S}_\gamma, \mathfrak{S}^*_\tau\rangle \\
&= \left\langle\sum_\beta \mathfrak{S}_\beta, \mathfrak{S}^*_\tau\right\rangle \\
&= \delta_{\beta\tau},
\end{aligned}
$$

where the sum is over all β such that $\beta \in \mathbb{Z}^{\ell(\gamma)}$, $\gamma_k - \beta_k \in \{0,1\}$ for all k, and $|\beta| = |\gamma| - (s-i)$.

Then using the above calculations, Theorem 4.4 and Lemma 4.6, we have that

$$
\mathfrak{S}^*_{(s)}\mathfrak{S}^*_{\alpha/\gamma} = \mathfrak{S}^*_{(s)}(\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha)
$$

$$
= \sum_{i=0}^{s}\left((S(F_{(s-i)}) \leftharpoonup \mathfrak{S}_\gamma) \leftharpoonup (F_{(i)}\mathfrak{S}^*_\alpha)\right)
$$

$$
= \sum_{i=0}^{s}\left((-1)^{(s-i)}\sum_{\substack{\tau \in \mathbb{Z}^{\ell(\gamma)} \\ \gamma_k - \tau_k \in \{0,1\} \\ |\tau| = |\gamma| - (s-i)}}\mathfrak{S}_\tau\right)\leftharpoonup\left(\sum_{\substack{\beta \vDash |\alpha| + i \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c^\alpha_{i,\beta}\mathfrak{S}^*_\beta\right)
$$

$$
= \sum_{i=0}^{s}\sum_{\substack{\tau, \beta \\ \tau \in \mathbb{Z}^{\ell(\gamma)} \\ \gamma_k - \tau_k \in \{0,1\} \\ |\tau| = |\gamma| - (s-i) \\ \beta \vDash |\alpha| + i \\ \ell(\beta) - \ell(\alpha) \in \{0,1\}}}(-1)^{(s-i)}\cdot c^\alpha_{i,\beta}\,\mathfrak{S}^*_{\beta/\tau}
$$

$$
= \sum_{\beta/\tau}(-1)^{|\gamma|-|\tau|}\cdot c^\alpha_{|\beta|-|\alpha|,\beta}\,\mathfrak{S}^*_{\beta/\tau},
$$

where $|\beta/\tau| = |\alpha/\gamma| + s$, γ/τ is a vertical strip of length at most s, and $\ell(\beta) - \ell(\alpha) \in \{0,1\}$. ∎

Example 4.8 Let us compute $\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)}$.

First, we need to compute all compositions $\beta \vDash 4 + i$ for $i \in \{0,1,2\}$ and $\ell(\beta) = 3$ or 4. We list all possible choices for β as the set

$$
\begin{aligned}
A = \{&(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,2), (1,1,2,1), (1,2,1,1),\\
&(2,1,1,1), (1,1,3), (1,2,2), (1,3,1), (2,1,2), (2,2,1), (3,1,1), (1,1,1,3),\\
&(1,1,2,2), (1,1,3,1), (1,2,1,2), (1,2,2,1), (1,3,1,1), (2,1,1,2),\\
&(2,1,2,1), (2,2,1,1), (3,1,1,1), (1,1,4), (1,2,3), (1,3,2), (1,4,1),\\
&(2,1,3), (2,2,2), (2,3,1), (3,1,2), (3,2,1), (4,1,1)\}.
\end{aligned}
$$

Next, we need to find τ by removing a vertical strip of length at most $s = 2$ from $\gamma = (1,1)$. We list all options for τ as the set $B = \{\varnothing, (1), (1,1)\}$.

By Theorem 4.7, now we expand $\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)}$ by finding all valid pairs (β, τ) such that $|\beta/\tau| = 4$. Thus,

$$
\begin{aligned}
\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)} = {}& c^{(1,2,1)}_{0,(1,1,1,1)}\mathfrak{S}^*_{(1,1,1,1)} + c^{(1,2,1)}_{0,(1,1,2)}\mathfrak{S}^*_{(1,1,2)}\\
&+ c^{(1,2,1)}_{0,(1,2,1)}\mathfrak{S}^*_{(1,2,1)} + c^{(1,2,1)}_{0,(2,1,1)}\mathfrak{S}^*_{(2,1,1)}\\
&- c^{(1,2,1)}_{1,(1,1,1,2)}\mathfrak{S}^*_{(1,1,1,2)/(1)} - c^{(1,2,1)}_{1,(1,1,2,1)}\mathfrak{S}^*_{(1,1,2,1)/(1)}\\
&- c^{(1,2,1)}_{1,(1,2,1,1)}\mathfrak{S}^*_{(1,2,1,1)/(1)} - c^{(1,2,1)}_{1,(2,1,1,1)}\mathfrak{S}^*_{(2,1,1,1)/(1)}\\
&- c^{(1,2,1)}_{1,(1,1,3)}\mathfrak{S}^*_{(1,1,3)/(1)} - c^{(1,2,1)}_{1,(1,2,2)}\mathfrak{S}^*_{(1,2,2)/(1)}
\end{aligned}
$$

$$- c_{1,(1,3,1)}^{(1,2,1)} \mathfrak{S}_{(1,3,1)/(1)}^* - c_{1,(2,1,2)}^{(1,2,1)} \mathfrak{S}_{(2,1,2)/(1)}^*$$

$$- c_{1,(2,2,1)}^{(1,2,1)} \mathfrak{S}_{(2,2,1)/(1)}^* - c_{1,(3,1,1)}^{(1,2,1)} \mathfrak{S}_{(3,1,1)/(1)}^*$$

$$+ c_{2,(1,1,1,3)}^{(1,2,1)} \mathfrak{S}_{(1,1,1,3)/(1,1)}^* + c_{2,(1,1,2,2)}^{(1,2,1)} \mathfrak{S}_{(1,1,2,2)/(1,1)}^*$$

$$+ c_{2,(1,1,3,1)}^{(1,2,1)} \mathfrak{S}_{(1,1,3,1)/(1,1)}^* + c_{2,(1,2,1,2)}^{(1,2,1)} \mathfrak{S}_{(1,2,1,2)/(1,1)}^*$$

$$+ c_{2,(1,2,2,1)}^{(1,2,1)} \mathfrak{S}_{(1,2,2,1)/(1,1)}^* + c_{2,(1,3,1,1)}^{(1,2,1)} \mathfrak{S}_{(1,3,1,1)/(1,1)}^*$$

$$+ c_{2,(2,1,1,2)}^{(1,2,1)} \mathfrak{S}_{(2,1,1,2)/(1,1)}^* + c_{2,(2,1,2,1)}^{(1,2,1)} \mathfrak{S}_{(2,1,2,1)/(1,1)}^*$$

$$+ c_{2,(2,2,1,1)}^{(1,2,1)} \mathfrak{S}_{(2,2,1,1)/(1,1)}^* + c_{2,(3,1,1,1)}^{(1,2,1)} \mathfrak{S}_{(3,1,1,1)/(1,1)}^*$$

$$+ c_{2,(1,1,4)}^{(1,2,1)} \mathfrak{S}_{(1,1,4)/(1,1)}^* + c_{2,(1,2,3)}^{(1,2,1)} \mathfrak{S}_{(1,2,3)/(1,1)}^*$$

$$+ c_{2,(1,3,2)}^{(1,2,1)} \mathfrak{S}_{(1,3,2)/(1,1)}^* + c_{2,(1,4,1)}^{(1,2,1)} \mathfrak{S}_{(1,4,1)/(1,1)}^*$$

$$+ c_{2,(2,1,3)}^{(1,2,1)} \mathfrak{S}_{(2,1,3)/(1,1)}^* + c_{2,(2,2,2)}^{(1,2,1)} \mathfrak{S}_{(2,2,2)/(1,1)}^*$$

$$+ c_{2,(2,3,1)}^{(1,2,1)} \mathfrak{S}_{(2,3,1)/(1,1)}^* + c_{2,(3,1,2)}^{(1,2,1)} \mathfrak{S}_{(3,1,2)/(1,1)}^*$$

$$+ c_{2,(3,2,1)}^{(1,2,1)} \mathfrak{S}_{(3,2,1)/(1,1)}^* + c_{2,(4,1,1)}^{(1,2,1)} \mathfrak{S}_{(4,1,1)/(1,1)}^* .$$

We can compute all the coefficients $c_{|\beta|-|\alpha|,\beta}^{\alpha}$ using Definition 4.1, and most of them turn out to be zero. Hence, we have the following expansion after simplification:

$$\mathfrak{S}_{(2)}^* \cdot \mathfrak{S}_{(1,2,1)/(1,1)}^* = \mathfrak{S}_{(1,2,1)}^* - \mathfrak{S}_{(1,1,2,1)/(1)}^* - \mathfrak{S}_{(2,2,1)/(1)}^* + \mathfrak{S}_{(2,1,2,1)/(1,1)}^*$$
$$+ \mathfrak{S}_{(3,2,1)/(1,1)}^* .$$

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