A MOUNTAIN-CLIMBING PROBLEM

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1. Introduction. Suppose that two men stand at the same elevation on opposite sides of a mountain range and begin to climb in such a way that their elevations remain equal at all times. Will they ever meet along the way? It is this question, restated in mathematical terms, that we shall consider. We replace the mountain range by the graph of a continuous, real-valued function $f(x)$ defined for $x \in [0, 1]$, where $f(0) = f(1) = 0$, and we ask whether there exist continuous mappings $\phi(t)$, $\psi(t)$ from $[0, 1]$ into $[0, 1]$ such that

$$\phi(0) = 0, \qquad \phi(1) = 1, \qquad \psi(0) = 1, \qquad \psi(1) = 0, \tag{1}$$

$$f(\phi(t)) = f(\psi(t)), \qquad t \in [0, 1]. \tag{2}$$

Thus $\phi(t)$, $\psi(t)$ represent the x-coordinates of the two men at time t. From (1) and the continuity of ϕ and ψ, it follows that their graphs must cross at some point $t_0 \in [0, 1]$. Then $\phi(t_0) = \psi(t_0)$ is the x-coordinate of the point where the two men meet. We shall show that if f does not change sign and consists of a finite number of monotone non-increasing or non-decreasing pieces, then ϕ and ψ can always be found. When f is allowed to change sign, the problem may have no solution, as the example $f(x) = \sin 2\pi x$ shows, but we are able to give a necessary and sufficient condition for the existence of ϕ and ψ in this case.

2. The strictly monotone case. It will be convenient to restate (1) and (2) in terms of the graph of $f^{-1}f$ which is the set of

$$(x, y) \in [0, 1] \times [0, 1]$$

such that $f(x) = f(y)$. Now (2) implies that the point $(\phi(t), \psi(t))$ lies on $f^{-1}f$ for each $t \in [0, 1]$, and the mapping $t \rightarrow (\phi(t), \psi(t))$ defines a path joining $(0, 1)$ with $(1, 0)$. Thus our problem can be restated: are the points $(0, 1)$ and $(1, 0)$ pathwise connected in the graph of $f^{-1}f$? We first consider the case where f is made up of a finite number of monotone, strictly increasing or decreasing pieces. Such pieces we shall call strictly monotone.

THEOREM 1. *Suppose f is a continuous function defined on $[0, 1]$ that does not change sign, $f(0) = f(1) = 0$, and f consists of a finite number of strictly monotone pieces. Then the points $(0, 1)$ and $(1, 0)$ are pathwise connected in the graph of $f^{-1}f$.*

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Proof. Let $A = \{a_0, a_1, \ldots, a_k\}$ be the set of extreme points of f, where $a_0 = 0, a_k = 1$, and f is monotone in each interval $[a_{i-1}, a_i]$. Then $B = f^{-1}(f(A))$ is also a finite set, $B \supset A$, and B divides $[0, 1]$ into a set of disjoint, open intervals $C_1, \ldots, C_n$. For each $i, j = 1, \ldots, n$, we have either $f(C_i) = f(C_j)$ or $f(C_i) \cap f(C_j) = \emptyset$. To see this, we note that $f(B) \cap f(C_i) = \emptyset$, so that $f(B)$ divides the range of f into a set of disjoint open intervals of the form $f(C_i)$. If we set $f_i = f|C_i$, the restriction of f to C_i, then the graph of $f^{-1}f$ contains all sets $f_i^{-1}f_j$, together with a certain subset D of $B \times B$. Since f_i and f_j are strictly monotone, so is $f_i^{-1}f_j$, and its end points belong to D. Thus $f_i^{-1}f_j$ is either empty or homeomorphic to an open interval, and we may regard the graph of $f^{-1}f$ as a one-dimensional geometric complex with vertices in D and the $f_i^{-1}f_j$ as its edges. The number of edges which meet at a vertex $(b_i, b_j) \in D$ is

(i) zero, in case f has a maximum at b_i and a minimum at b_j, or vice versa,

(ii) two, in case f has at most one extremum at b_i and b_j where $b_i, b_j \neq 0, 1$,

(iii) four, in case f has a maximum at b_i and b_j or a minimum at b_i and b_j where $b_i, b_j \neq 0, 1$.

Since the edges in the graph of $f^{-1}f$ all lie in disjoint rectangles, the number of edges that meet at a vertex on the boundary of $[0, 1] \times [0, 1]$ is clearly zero, one, or two. The main diagonal E of $[0, 1] \times [0, 1]$ always belongs to $f^{-1}f$, so that one edge leaves $(0, 0)$ and $(1, 1)$. Since f does not change sign in $[0, 1]$, it has the same type of extremum, maximum or minimum, at 0, 1, and any other point where f is 0. Thus one edge leaves $(0, 1)$ and $(1, 0)$, while two edges meet at any other vertex on the boundary of $[0, 1] \times [0, 1]$. We now construct a new complex G from $f^{-1}f$ by introducing an overpass for one of the two monotone paths that cross at a vertex of type (iii), and thus obtain a one-dimensional complex that does not cross itself. The connected components of G are points, arcs, or simple closed curves. The component that contains $(0, 1)$ must be an arc with one end point at $(0, 1)$ and the other at $(0, 0)$, $(1, 0)$, or $(1, 1)$. Since E belongs to G, $(0, 1)$ and $(1, 0)$ must be the end points of an arc in G, and this arc also lies in the graph of $f^{-1}f$, although it may now cross itself.

We next consider a slightly more general question than our original one. If two men stand at the foot of two different mountain ranges and climb so that their elevations always remain equal, can they ever cross their respective mountain ranges?

THEOREM 2. *Suppose f and g are continuous functions defined on $[a, b]$ and $[c, d]$, respectively, which do not change sign and have the same sign,*

$$f(a) = f(b) = g(c) = g(d) = 0,$$

and f and g consist of a finite number of strictly monotone pieces. Then the points (a, d) and (b, c) are pathwise connected in the graph of $g^{-1}f$ if and only if

$$\sup\{|f(x)| : x \in [a, b]\} = \sup\{|g(y)| : y \in [c, d]\}. \tag{3}$$

Proof. Let the left- and right-hand members of (3) be denoted by $m(f)$ and $m(g)$, respectively, and suppose that $m(f) > m(g)$. If (a, d) and (b, c) were pathwise connected in $g^{-1}f$, then we could find continuous mappings ϕ and ψ from $[0, 1]$ into $[a, b]$ and $[c, d]$, respectively, such that $f(\phi(t)) = g(\psi(t))$ for each $t \in [0, 1]$. This equation holds, in particular, for the point t_0 such that $|f(\phi(t_0))| = m(f)$, which is clearly impossible. Similarly, the assumption $m(f) < m(g)$ leads to a contradiction. Hence, the pathwise connectedness of (a, d) and (b, c) in $g^{-1}f$ implies (3).

Suppose, conversely, that (3) holds, and let $x_0 \in [a, b]$ and $y_0 \in [c, d]$ be chosen so that

$$|f(x_0)| = m(f) = m(g) = |g(y_0)|.$$

Let K and L denote constants such that $x_0 < y_0 + K$ and $x_0 + L > y_0$. We now define two new functions F and G:

$$F(x) = \begin{cases} f(x) & \text{for } x \in [a, x_0], \\ f(x_0)(1 + (x - x_0)(y_0 + K - x)) & \text{for } x \in [x_0, y_0 + K], \\ g(x - K) & \text{for } x \in [y_0 + K, d + K], \end{cases} \tag{4}$$

$$G(y) = \begin{cases} g(y) & \text{for } y \in [c, y_0], \\ g(y_0)(1 + (y - y_0)(x_0 + L - y)) & \text{for } y \in [y_0, x_0 + L], \\ f(y - L) & \text{for } y \in [x_0 + L, b + L]. \end{cases} \tag{5}$$

Since F and G do not change sign, we can apply Theorem 1 to each of them and obtain continuous mappings ϕ_1, ψ_1 from $[0, 1]$ into $[a, d + K]$ and ϕ_2, ψ_2 from $[0, 1]$ into $[c, b + L]$ with the properties

$$\begin{cases} \phi_1(0) = a, \quad \phi_1(1) = d + K, \quad \psi_1(0) = d + K, \quad \psi_1(1) = a, \\ \phi_2(0) = c, \quad \phi_2(1) = b + L, \quad \psi_2(0) = b + L, \quad \psi_2(1) = c, \\ F(\phi_1(t)) = F(\psi_1(t)), \quad G(\phi_2(t)) = G(\psi_2(t)), \quad t \in [0, 1]. \end{cases} \tag{6}$$

If $x_0 < x < y_0 + K$, then the only other point $y \in [a, d + K]$ such that $F(x) = F(y)$ must also satisfy $x_0 < y < y_0 + K$. Hence the only points (x, y) in the graph of $F^{-1}F$ which satisfy

$$x_0 < x < y_0 + K \quad \text{or} \quad x_0 < y < y_0 + K$$

are those on the two diagonals in $[x_0, y_0 + K] \times [x_0, y_0 + K]$. Thus the path $t \to (\phi_1(t), \psi_1(t))$ must cross from the region $[a, x_0] \times [y_0 + K, d + K]$ to the region $[y_0 + K, d + K] \times [a, x_0]$ along the line joining the points $(x_0, y_0 + K)$ and $(y_0 + K, x_0)$. In other words, there is a point $t_1 \in [0, 1]$ with the properties

$$\begin{cases} \phi_1(t_1) = x_0, \quad \psi_1(t_1) = y_0 + K, \\ \phi_1([0, t_1]) \subset [a, x_0], \quad \psi_1([0, t_1]) \subset [y_0 + K, d + K]. \end{cases} \tag{7}$$

A similar argument applied to G yields a point $t_2 \in [0, 1]$ satisfying

$$\begin{cases} \phi_2(t_2) = x_0 + L, \quad \psi_2(t_2) = y_0, \\ \phi_2([t_2, 1]) \subset [x_0 + L, b + L], \quad \psi_2([t_2, 1]) \subset [c, y_0]. \end{cases} \tag{8}$$

We now construct a path $t \to (\phi(t), \psi(t))$ which lies in the graph of $g^{-1}f$ and joins the points (a, d) and (b, c). Choose the constant M so that $t_1 < t_2 + M$, and let

$$\phi(t) = \begin{cases} \phi_1(t) & \text{for } t \in [0, t_1], \\ x_0 & \text{for } t \in [t_1, t_2 + M], \\ \phi_2(t - M) - L & \text{for } t \in [t_2 + M, 1 + M], \end{cases}$$

$$\psi(t) = \begin{cases} \psi_1(t) - K & \text{for } t \in [0, t_1], \\ y_0 & \text{for } t \in [t_1, t_2 + M], \\ \psi_2(t - M) & \text{for } t \in [t_2 + M, 1 + M]. \end{cases}$$

Evidently ϕ and ψ are continuous, and

$$\phi(0) = a, \qquad \phi(1 + M) = b, \qquad \psi(0) = d, \qquad \psi(1 + M) = c.$$

If $t \in [0, t_1]$, then (4), (6), and (7) imply

$$\begin{aligned} f(\phi(t)) &= f(\phi_1(t)) = F(\phi_1(t) \\ &= F(\psi_1(t)) = g(\psi_1(t) - K) = g(\psi(t)). \end{aligned}$$

If $t \in [t_2 + M, 1 + M]$, then (5), (6), and (8) imply

$$\begin{aligned} f(\phi(t)) &= f(\phi_2(t - M) - L) = G(\phi_2(t - M)) \\ &= G(\psi_2(t - M)) = g(\psi_2(t - M)) = g(\psi(t)). \end{aligned}$$

If $t \in [t_1, t_2 + M]$, then clearly $f(\phi(t)) = g(\psi(t))$. Therefore, $f(\phi(t)) = g(\psi(t))$ for all $t \in [0, 1 + M]$, and the path $t \to (\phi(t), \psi(t))$ lies in the graph of $g^{-1}f$.

3. The weakly monotone case. We are now in a position to analyse the case where f is made up of a finite number of monotone non-decreasing or non-increasing pieces. We shall call such pieces weakly monotone. We first restate Theorem 1 for this case.

THEOREM 3. *The conclusion of Theorem* 1 *remains valid if, in the hypothesis, "strictly monotone" is replaced by "weakly monotone."*

Proof. We first subdivide $[0, 1]$ by means of points $0 = a_0, a_1, \ldots, a_k = 1$ so that f is weakly monotone in each interval $A_i = [a_{i-1}, a_i]$, and set $f_i = f|A_i$ for $1 \leqslant i \leqslant k$. We then partition A_i into equivalence classes of the form $f_i^{-1}(r)$, as r ranges over the real numbers, and form the resulting quotient space Q_i endowed with the quotient topology, where ω_i is the canonical mapping from A_i onto Q_i. We define the function F_i from Q_i into the real numbers by the equation $F_i \omega_i = f_i$ and note that F_i is one-to-one. Since the domains of f_i and ω_i are compact, f_i and ω_i map closed sets into closed sets, whence F_i and F_i^{-1} do the same, and F_i is a homeomorphism. From $\omega_i^{-1} = f_i^{-1}F_i$ we infer that ω_i^{-1} maps connected sets into connected sets, so that ω_i is monotone. Thus Q_i is homeomorphic to $f_i(A_i)$ which is a closed interval, and we choose the ordering for Q_i so as to make ω_i non-decreasing. Evidently F_i is strictly monotone. We now join the Q_i together by identifying the right-hand end point of Q_i with the left-hand end point of Q_{i+1}. The resulting space Q is again an interval. We combine the ω_i and the F_i in the obvious way to obtain a non-decreasing mapping ω from $[0, 1]$ onto Q and a continuous mapping F from Q into the real numbers that satisfy $F\omega = f$. Since F is made up of a finite number of strictly monotone pieces, the graph of $F^{-1}F$ is a one-dimensional complex, as was shown in the proof of Theorem 1. Thus $F^{-1}F$ contains a path consisting of vertices $(p_1, q_1), \ldots, p_n, q_n)$ in $Q \times Q$ and the edges joining consecutive pairs of them where (p_1, q_1) is the upper left corner of $Q \times Q$, and (p_n, q_n) is the lower right corner. We choose points

$$(b_i, c_i) \in [0, 1] \times [0, 1]$$

so that $\omega(b_i) = p_i$ and $\omega(c_i) = q_i$ for $1 \leqslant i \leqslant n$, where $(b_1, c_1) = (0, 1)$ and $(b_n, c_n) = (1, 0)$. Let H_i be that part of the path in $F^{-1}F$ which lies between (p_{i-1}, q_{i-1}) and (p_i, q_i) and includes its end points. Then H_i is a homeomorphism whose domain and range are intervals in Q. From $H_i \subset F^{-1}F$ we infer that

$$\omega^{-1}H_i\,\omega \subset \omega^{-1}F^{-1}F\omega = f^{-1}f,$$

and also that $(b_{i-1}, c_{i-1}), (b_i, c_i) \in f^{-1}f$. Now continuity implies that ω, H_i, and ω^{-1} are closed subsets of $[0, 1] \times Q$, $Q \times Q$, and $Q \times [0, 1]$, respectively. To see that $\omega^{-1}H_i\,\omega$ is closed, let $\{(x_m, y_m)\}$ be a sequence in $\omega^{-1}H_i\,\omega$ which converges to $(x, y) \in [0, 1] \times [0, 1]$. Then we can find $u_m, v_m \in Q$ satisfying $(x_m, u_m) \in \omega$, $(u_m, v_m) \in H_i$, and $(v_m, y_m) \in \omega^{-1}$. Since Q is compact, there are points $u, v \in Q$ and subsequences $\{u_{s(m)}\}$, $\{v_{s(m)}\}$ of $\{u_m\}$, $\{v_m\}$ that converge to u, v, respectively. Thus $\{(x_{s(m)}, u_{s(m)})\}$, $\{(u_{s(m)}, v_{s(m)})\}$, and $\{(v_{s(m)}, y_{s(m)})\}$ converge to $(x, u) \in \omega$, $(u, v) \in H_i$, and $(v, y) \in \omega^{-1}$, respectively, so that $(x, y) \in \omega^{-1}H_i\,\omega$, and the latter is a closed and compact subset of $[0, 1] \times [0, 1]$. To see that $\omega^{-1}H_i\,\omega$ is locally connected, let U and V be open intervals of real numbers. The domain of the relation

$$G = (U \times V) \cap \omega^{-1}H_i\,\omega$$

is evidently

$$W = U \cap (\omega^{-1}H_i\,\omega)^{-1}(V) = U \cap \omega^{-1}H_i^{-1}\omega(V).$$

Since ω is monotone, $\omega^{-1}H_i^{-1}\omega(V)$ is an interval, and W is connected. Moreover, if $x \in W$, then $\omega^{-1}H_i\,\omega(x)$ is an interval, and $G(x) = V \cap \omega^{-1}H_i\,\omega(x)$ is connected. Thus any separation of G could be projected vertically downward to yield a separation of W, which is impossible. Hence, G is connected. If we choose U and V so as to contain $[0, 1]$, it follows that $\omega^{-1}H_i\,\omega$ is also connected, so that $\omega^{-1}H_i\,\omega$ is a compact, connected, locally connected metric space. By the Arcwise Connectedness Theorem (**1**, p. 36), (b_{i-1}, c_{i-1}) and (b_i, c_i) can be joined by a simple arc C_i lying in $\omega^{-1}H_i\,\omega \subset f^{-1}f$. If we join the arcs C_i $(1 \leqslant i \leqslant n)$ together in the obvious way, we obtain a path that joins $(0, 1)$ with $(1, 0)$ and lies entirely in $f^{-1}f$.

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THEOREM 4. *The conclusion of Theorem 2 remains valid if, in the hypothesis, "strictly monotone" is replaced by "weakly monotone."*

Proof. The proof of Theorem 2 can be taken over unchanged, except for the reference to Theorem 1 which should be replaced by Theorem 3.

Suppose that f is continuous on $[0, 1]$, $f(0) = f(1) = 0$, f consists of a finite number of weakly monotone pieces, but f is allowed to change sign. The sets $A = \{x: f(x) > 0\}$ and $B = \{x: f(x) < 0\}$ have a finite number of connected components $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$, respectively, where we allow $m = 0$ and $n = 0$. We select points

$$0 = c_0, c_1, \ldots, c_p = 1 \in [0, 1] - (A \cup B)$$

with the property that when the collection $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$ is arranged according to increasing order, precisely one c_i occurs between consecutive terms A_j, B_k or B_k, A_j. If f does not change sign, then $p = 1$. The c_i are unique unless there is an open interval between A_j and B_k, but in this case c_i can be any point between A_j and B_k. Roughly speaking, the intervals into which the c_i divide $[0, 1]$ are the largest in which f has constant sign. For $1 \leqslant i \leqslant p$, we set $C_i = [c_{i-1}, c_i]$ and define

$$M_i = \begin{cases} \sup \{f(x): x \in C_i\} & \text{if } f(C_i) \geqslant 0, \\ \inf \{f(x): x \in C_i\} & \text{if } f(C_i) \leqslant 0. \end{cases}$$

It is clear that if any one of the M_i is zero, then $p = 1$ and f is identically 0 in $[0, 1]$. In all other cases the M_i alternate in sign. We define $\{r(i)\}$ to be the sequence of integers in which $r(1) = 1$, $r(2) = 2$, and $r(i)$ is the least integer j such that $M_j > M_{r(i-2)}$ if $M_{r(i-2)} > 0$, or $M_j < M_{r(i-2)}$ if $M_{r(i-2)} < 0$ for $i > 2$. The sequence is defined on a subset of $\{i: 1 \leqslant i \leqslant p\}$, and since the M_{2i-1} have constant sign, as well as the M_{2i}, it follows that $r(2i - 1)$ is odd and $r(2i)$ is even. Evidently both $r(2i - 1)$ and $r(2i)$ increase monotonically with i. If we arrange $R = \{r(i): 1 \leqslant i \leqslant p\}$ in the natural order which it inherits as a subset of the integers, then R can be decomposed into its "connected" components $\{R_j: 1 \leqslant j \leqslant q\}$, where R_j is a maximal string of consecutive elements of R with the same parity (even or odd). We can order the R_j in the obvious way so that $R_1 < R_2 < \ldots < R_q, R_1 = \{1\}$, $2 \in R_2$, and we denote the largest member of R_j by $u(j)$. We shall also need the analogues of the above definitions in which we start from the right-hand end of the interval instead of the left. Let $s(1) = p$, $s(2) = p - 1$, and $s(i)$ be the greatest integer j such that $M_j > M_{s(i-2)}$ if $M_{s(i-2)} > 0$, or $M_j < M_{s(i-2)}$ if $M_{s(i-2)} < 0$ for $i > 2$. Then $s(2i - 1)$ has the same parity as p, $s(2i)$ has the same parity as $p - 1$, each decreases monotonically with i, and $S = \{s(i): 1 \leqslant i \leqslant p\}$ can be decomposed into its "connected" components

$$\{S_j: 1 \leqslant j \leqslant q'\},$$

where $S_1 > S_2 > \ldots > S_{q'}$, $S_1 = \{p\}$, $p - 1 \in S_2$, and we denote the least member of S_j by $v(j)$. Evidently $M_{u(2j-1)}$, $M_{u(2j)}$, $M_{v(2j-1)}$, and $M_{v(2j)}$ are monotone functions of j.

THEOREM 5. *Suppose f is a continuous function on* $[0, 1]$, $f(0) = f(1) = 0$, *and f consists of a finite number of weakly monotone pieces. Then the points* $(0, 1)$ *and* $(1, 0)$ *are pathwise connected in $f^{-1}f$ if and only if $q = q'$ and*

$$M_{u(j)} = M_{v(j)}, \qquad 1 \leqslant j \leqslant q. \tag{9}$$

Proof. Suppose that $(0, 1)$ and $(1, 0)$ are pathwise connected in $f^{-1}f$, and $t \to (\phi(t), \psi(t))$ is the connecting path. Now $u(1) = 1$, $v(1) = p$, and we shall show that $M_1 = M_p$. If M_1 and M_p have opposite signs, then $f(\phi(t)) = f(\psi(t))$ cannot possibly hold for t near 0. Suppose $M_1 > M_p > 0$ or $M_1 < M_p < 0$, and let $t^* = \inf\{t: f(\phi(t)) = M_1\}$. Then $t^* > 0$ and $M_1 = f(\phi(t^*)) = f(\psi(t^*))$, so that $\psi(t^*) < c_{p-1}$ and $f(\psi(t))$ must change sign for $t \in [0, t^*]$. But

$$f(\phi(t)) = f(\psi(t)),$$

while $f(\phi(t)) \in [0, M_1]$ does not change sign for $t \in [0, t^*]$, which is impossible. Similarly, the assumption $0 < M_1 < M_p$ or $0 > M_1 > M_p$ leads to a contradiction. Hence, $M_1 = M_p$, and p is odd. From $2 \in R_2$ and $p - 1 \in S_2$ we infer that $u(2)$ and $v(2)$ are defined.

Suppose now that $u(i)$ and $v(i)$ are defined for all i less than a certain j, and that $M_{u(i)} = M_{v(i)}$. Since $u(1)$ and $v(1)$ have the same parity, so also $u(i)$ and $v(i)$ have the same parity, and $M_{u(i)}$ and $M_{v(i)}$ have like signs. If $j > 2$, then $u(j)$ is defined if and only if $M_k > M_{u(j-2)} > 0$ or $M_k < M_{u(j-2)} < 0$ for some integer k. For if such a k exists, then the definition of $r(i)$ implies that $k > u(j-1)$. Similarly, $v(j)$ is defined if and only if $M_k > M_{v(j-2)} > 0$ or $M_k < M_{v(j-2)} < 0$ for some k. Thus $u(j)$ is defined if and only if $v(j)$ is defined. If neither is defined, then $q = q' = j - 1$. Assuming that $u(j)$ and $v(j)$ are defined, and $M_{u(j)} > M_{v(j)} > 0$ or $M_{u(j)} < M_{v(j)} < 0$, let

$$t' = \inf\{t: f(\phi(t)) = M_{u(j)}\}.$$

Then $t' > 0$ and

$$M_{u(j)} = f(\phi(t')) = f(\psi(t')),$$

so that $v(j+2)$ is defined, and $\psi(t') < c_{v(j+2)}$. Moreover, $v(j+1)$ must be defined, and we can find $t'' \in [0, t']$ such that $f(\psi(t'')) = M_{v(j+1)}$. But $\phi([0, t']) \subset [0, c_{u(j)}]$ and

$$f[\phi(t'')) = f(\psi(t'')) = M_{v(j+1)},$$

whereas either

$$M_{v(j+1)} < M_{v(j-1)} = M_{u(j-1)} < 0 \quad \text{or} \quad M_{v(j+1)} > M_{v(j-1)} = M_{u(i-1)} > 0,$$

which is impossible. Similarly, the assumption $0 < M_{u(j)} < M_{v(j)}$ or $0 > M_{u(j)} > M_{v(j)}$ leads to a contradiction. Thus $M_{u(j)} = M_{v(j)}$, and the induction step is completed. Therefore, $q = q'$ and (9) is verified.

Suppose, conversely, that (9) holds. When $1 \leqslant j \leqslant q$, we set

$$a_j = \inf \{x \colon f(x) = M_{u(j)}\},$$

$$b_j = \inf \left\{x \colon \begin{matrix} f(x) > M_{u(j)} \text{ if } M_{u(j)} > 0, \\ f(x) < M_{u(j)} \text{ if } M_{u(j)} < 0 \end{matrix}\right\},$$

$$d_j = \sup \left\{x \colon \begin{matrix} f(x) > M_{v(j)} \text{ if } M_{v(j)} > 0, \\ f(x) < M_{v(j)} \text{ if } M_{v(j)} < 0 \end{matrix}\right\},$$

$$e_j = \sup \{x \colon f(x) = M_{v(j)}\},$$

$$a_0 = 0, \quad b_0 = c_1, \quad b_{q-1} = b_q = 1,$$

$$d_0 = c_{p-1}, \quad d_{q-1} = d_q = 0, \quad e_0 = 1.$$

In order to reduce the number of special cases in our discussion, we shall set $M_{u(0)} = M_{v(0)} = 0$. For $0 \leqslant j \leqslant q - 1$, we see that $f - M_{u(j)}$ has constant sign in $[a_j, b_j]$, $f - M_{v(j)}$ has the same sign in $[d_j, e_j]$, and each function vanishes at the ends of its interval, except for $f - M_{u(q-1)}$ at b_{q-1} and $f - M_{v(q-1)}$ at d_{q-1}. Clearly a_j, b_j, d_j, e_j are monotone functions of j, $a_j < a_{j+1} < b_j$, and $d_j < e_{j+1} < e_j$. If we apply the notation of Theorem 2 to the functions $f - M_{u(j)}$ with domain $[a_j, b_j]$ and $f - M_{v(j)}$ with domain $[d_j, e_j]$, then (9) implies

$$\begin{aligned} m(f - M_{u(j)}) &= |M_{u(j+1)} - M_{u(j)}| \\ &= |M_{v(j+1)} - M_{v(j)}| = m(f - M_{v(j)}), \qquad 0 \leqslant j \leqslant q - 1. \end{aligned}$$

When $0 \leqslant j \leqslant q - 2$, we can apply Theorem 4 to these two functions and obtain continuous mappings ϕ_j, ψ_j from $[0, 1]$ into $[a_j, b_j]$, $[d_j, e_j]$, respectively, with the properties

$$\begin{gathered} \phi_j(0) = a_j, \qquad \phi_j(1) = b_j, \qquad \psi_j(0) = e_j, \qquad \psi_j(1) = d_j, \\ f(\phi_j(t)) - M_{u(j)} = f(\psi_j(t)) - M_{v(j)}, \qquad t \in [0, 1]. \end{gathered}$$

In view of (9), we have $f(\phi_j(t)) = f(\psi_j(t))$. If we set

$$t_j = \inf \{t \colon \phi_j(t) = a_{j+1}\},$$

then it follows that $t_j = \inf \{t \colon \psi_j(t) = e_{j+1}\}$, for (9) implies that

$$f(a_{j+1}) = f(e_{j+1}).$$

We have already seen that

$$\begin{gathered} m(f - M_{u(q-1)}) = m(f - M_{v(q-1)}), \\ (f - M_{u(q-1)})(a_{q-1}) = (f - M_{v(q-1)})(e_{q-1}) = 0, \\ (f - M_{u(q-1)})(b_{q-1}) = -M_{u(q-1)} = -M_{v(q-1)} = (f - M_{v(q-1)})(d_{q-1}) \neq 0. \end{gathered}$$

However, we can enlarge the domains of these two functions to

$$[a_{q-1}, b_{q-1} + \epsilon], \qquad [-\epsilon + d_{q-1}, e_{q-1}],$$

where $\epsilon > 0$ is arbitrary, and extend each of their graphs by a straight-line segment so that it will be zero at the new end point. Then Theorem 4 applied to the extended functions yields continuous mappings ϕ_{q-1}, ψ_{q-1} from $[0, 1]$ into $[a_{q-1}, 1 + \epsilon]$, $[-\epsilon, e_{q-1}]$, respectively, with the properties

$$\phi_{q-1}(0) = a_{q-1}, \quad \phi_{q-1}(1) = 1 + \epsilon, \quad \psi_{q-1}(0) = e_{q-1}, \quad \psi_{q-1}(1) = -\epsilon,$$
$$f(\phi_{q-1}(t)) = f(\psi_{q-1}(t)), \qquad t \in [0, 1].$$

If we set $t_{q-1} = \inf\{t: \phi_{q-1}(t) = 1\}$, then it follows that $t_{q-1} = \inf\{t: \psi_{q-1}(t) = 0\}$, for $f(0) = f(1)$. Finally, we define $\phi(t) = \phi_0(t)$ for $t \in [0, t_0]$, and, in general,

$$\phi(t) = \phi_j(t - t_0 - \ldots - t_{j-1})$$

for

$$t \in [t_0 + \ldots + t_{j-1}, t_0 + \ldots + t_{j-1} + t_j], 0 \leqslant j \leqslant q - 1.$$

Similarly, we define

$$\psi(t) = \psi_j(t - t_0 - \ldots - t_{j-1})$$

for

$$t \in [t_0 + \ldots + t_{j-1}, t_0 + \ldots + t_{j-1} + t_j], 0 \leqslant j \leqslant q - 1.$$

Evidently ϕ and ψ are continuous mappings from $[0, \bar{t}]$ into $[0, 1]$, where $\bar{t} = t_0 + \ldots + t_{q-1}$, with the properties

$$\phi(0) = \phi_0(0) = 0, \qquad \phi(\bar{t}) = \phi_{q-1}(t_{q-1}) = 1,$$
$$\psi(0) = \psi_0(0) = 1, \qquad \psi(\bar{t}) = \psi_{q-1}(t_{q-1}) = 0,$$
$$f(\phi(t) = f(\psi(t)), \qquad t \in [0, \bar{t}].$$

Hence, the points $(0, 1)$ and $(1, 0)$ are pathwise connected in $f^{-1}f$.

COROLLARY. *With the same hypotheses as in Theorem* 5, *if* $M_j = M_{p-j+1}$ *for* $1 \leqslant j \leqslant p$, *then* $(0, 1)$ *and* $(1, 0)$ *are pathwise connected in* $f^{-1}f$.

Proof. From the symmetrical definitions of $u(j)$ and $v(j)$, we infer immediately that $v(j) = p - u(j) + 1$, and the result follows from (9).

4. Examples and questions. So far, we have limited ourselves to the case where f consists of only a finite number of monotone pieces. When f is made up of an infinite number of monotone pieces, the points $(0, 1)$ and $(1, 0)$ need not be pathwise connected in $f^{-1}f$, as the following example shows. Let

$$f(x) = \begin{cases} -4(x + 3/2\pi) & \text{for } x \in [-2/\pi, -3/2\pi], \\ 0 & \text{for } x \in [-3/2\pi, -1/\pi], \\ -4(x + 1/\pi) & \text{for } x \in [-1/\pi, -1/2\pi], \\ 4x & \text{for } x \in [-1/2\pi, 0], \\ x \sin(1/x) & \text{for } x \in [0, 2/\pi]. \end{cases}$$

Then f is continuous on $[-2/\pi, 2/\pi]$ and consists of a countable number of monotone pieces, $f(-2/\pi) = f(2/\pi) = 2/\pi$, and $f(x) \leqslant 2/\pi$ for $x \in [-2/\pi, 2/\pi]$. If two men were to climb this mountain range, the man on the left

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would have to cross the plateau twice every time the man on the right crossed from one peak to the next, and a discontinuity would occur when the man on the right reached $x = 0$. Although the points $(-2/\pi, 2/\pi)$ and $(2/\pi, -2/\pi)$ are not pathwise connected in $f^{-1}f$, an inspection of the graph of $f^{-1}f$ shows that they lie in the same connected component of $f^{-1}f$. If f is continuous on $[0, 1]$, has constant sign, and $f(0) = f(1) = 0$, do the points $(0, 1)$ and $(1, 0)$ always lie in the same connected component of $f^{-1}f$?

The question of whether $(0, 1)$ and $(1, 0)$ are pathwise connected in $f^{-1}f$ depends also on the size of the domain of f. We have already seen that $(0, 1)$ and $(1, 0)$ are not pathwise connected when $f(x) = \sin 2\pi x$ for $x \in [0, 1]$. But if the domain of f is $[0, 5/4]$, then $(0, 1)$ and $(1, 0)$ are pathwise connected in $f^{-1}f$. We can thus pose a more general problem: Given a continuous function f defined on $[0, 1]$ and two points $x_0, y_0 \in [0, 1]$ such that $f(x_0) = f(y_0)$, under what conditions can we find continuous mappings ϕ and ψ from $[0, 1]$ into $[0, 1]$ such that

$$\phi(0) = x_0, \qquad \phi(1) = y_0, \qquad \psi(0) = y_0, \qquad \psi(1) = x_0,$$
$$f(\phi(t)) = f(\psi(t)), \qquad t \in [0, 1]?$$

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