BULLETIN **13** (3), 1970

A REMARK ABOUT NONCOMMUTATIVE INTEGRAL EXTENSIONS[1]

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Let B be a ring with unity, A a unital subring of the centre C of B. Suppose further that B is A-integral. (That is, every element of B satisfies a monic polynomial with coefficients in A.) Under these assumptions, Hoechsmann [2] showed that "contraction to A" is a mapping from:

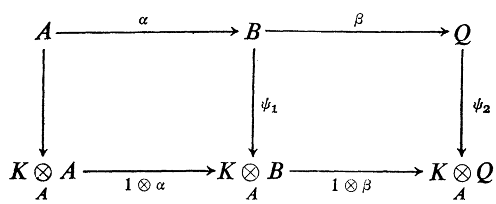
(1) The prime ideals of B *onto* the prime ideals of A,
(2) The maximal ideals of B *onto* the maximal ideals of A.

In this note we show that, under additional assumptions, a noncommutative version of the rest of the Cohen–Seidenberg "going up theorem" can be established.

LEMMA. *Let B be a prime ring with unity satisfying*:

(a) *B is integral over a unital subring A of the centre C of B*
(b) *B has a classical right quotient ring Q which is a simple ring.*
Then any nonzero prime ideal P of B satisfies $P \cap A \neq 0$.

Proof. The ring A is a subring of B (and of Q) and both B and Q are torsion-free A-modules. For, if $a \neq 0$ is in A and if $ax=0$ for some x in B, then $aBx=0$, so $x=0$. We then have the commutative diagram of A-modules



where K is the quotient field of the domain A. Since B and Q are torsion-free A-modules, ψ_1 and ψ_2 are both one-to-one. Therefore each mapping in the diagram is, in fact, a ring monomorphism. It is easily verified that $K \otimes Q$ is a right quotient ring for $K \otimes B$, that $K \otimes Q$ is simple, and that $B_1 = K \otimes B$ is integral over the subfield $K_1 = K \otimes A$ of the centre of B_1.

If $b_1 \in B_1$ has as its minimal polynomial

$$b_1^n + b_1^{n-1}a_{n-1} + \cdots + b_1 a_1 + a_0 = 0$$

Received by the editors December 12, 1969.
(1) Supported by National Research Council.

https://doi.org/10.4153/CMB-1970-067-0 Published online by Cambridge University Press

where the a_i's are in K_1 then, if $a_0 \neq 0$, b_1 must be a unit in B, while, if $a_0=0$, b_1 must be a zero-divisor in B. Therefore any element of B_1 is either a unit or a zero-divisor, so $B_1=K \otimes B$ is its own classical right quotient ring and $1 \otimes \beta$ is an isomorphism. Therefore B_1 is a simple ring. The ring $K \otimes P$ can be regarded as an A-submodule of B_1, and, as such, is an ideal. If $K \otimes P=0$, then $\psi_1(P)=0$, so $P=0$, which is false. Therefore $K \otimes P=B_1=K \otimes B$, and it follows that there are $a \neq 0$ in A and p in P for which $1 \otimes 1=(1/a) \otimes p$. Therefore $0=1 \otimes (a-p)=\psi_1(a-p)$, so $a=p \in P \cap A$. This proves the lemma.

In order to extend the results of (2), we will impose one of the following conditions on B:

(N) The ring B is right noetherian

(P) The ring B satisfies a polynomial identity $f(x_1, \ldots, x_n)=0$ for which f has coefficients in C, the centre of B, and for which, at each prime ideal P of B, f induces a nontrivial polynomial identity on B/P.

We note that if B satisfies a standard identity (see [1, p. 154]) then (P) is satisfied. Furthermore, if B is integral over a subring A of C, and if there is a bound on the degrees of the minimal polynomials of elements of B, then B satisfies (P). (To see this, one proceeds as in [1, p. 155]).

The purpose of introducing these conditions is that each of them is sufficient to guarantee that, for each prime ideal P of B, B/P has a right quotient ring which is simple. This is guaranteed by Goldie's theorem (when B has (N)) and Posner's theorem (when B has (P)) respectively. (See [1, chapter 7], for proofs of these results.)

THEOREM. *Let B be a ring with unity which is integral over a unital subring A of C, the centre of B. Suppose further that B satisfies either* (N) *or* (P). *Then*;

(a) *If P is a prime ideal of B, P is a maximal ideal of B if and only if $P \cap A$ is a maximal ideal of A*,

(b) *If P and Q are prime ideals of B, $P \subseteq Q$, and $P \cap A=Q \cap A$, then $P=Q$.*

Proof. In [2], Hoechsmann proved that P is maximal in B implies that $P \cap A$ is maximal in A.

Suppose only that P is prime in B. Then $P \cap A$ is prime in A, and we can identify $A'=A/(A \cap P)$ with the subring $(A+P)/P$ of $B/P=B'$. Also, B' is integral over A' so we can, without loss of generality, assume that $P=0$ and prove that B is simple (in (a)) and that $Q=0$ (in (b)).

In (a), we can therefore take A to be a field. The lemma can be applied to conclude that B has no proper prime ideals, and thus B has no nonzero maximal ideals. The ring B itself must then be simple.

In (b) we see, applying the lemma, that $P=0$ implies that $Q=0$, as desired.

https://doi.org/10.4153/CMB-1970-067-0 Published online by Cambridge University Press

References

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https://doi.org/10.4153/CMB-1970-067-0 Published online by Cambridge University Press