SOME DUAL SERIES AND TRIPLE INTEGRAL EQUATIONS

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1. Introduction

In this paper we first of all solve the dual series equations

$$\sum_{n=0}^{\infty} A_n \mathfrak{F}_n(a, \lambda;\ \rho) = f(\rho), \quad 0 \leqq \rho < d, \tag{1}$$

$$\sum_{n=0}^{\infty} A_n p_n(\lambda - \sigma, \lambda) \mathfrak{F}_n(a, \lambda;\ \rho) = g(\rho), \quad d < \rho \leqq 1, \tag{2}$$

where $f(\rho)$ and $g(\rho)$ are prescribed functions,

$$p_n(\lambda - \sigma, \lambda) = \frac{\Gamma(\lambda - \sigma + n)\Gamma(1 + a - \lambda + n)}{\Gamma(\lambda + n)\Gamma(1 + a + \sigma - \lambda + n)}, \tag{3}$$

and

$$\mathfrak{F}_n(a, \lambda;\ \rho) = {}_2F_1(-n, a+n;\ \lambda;\ \rho) \tag{4}$$

is the Jacobi polynomial (**2**).

Noble (**3**) has obtained an exact solution to the equations when the term $p_n(\lambda - \sigma, \lambda)$ occurs in equation (1) and not in equation (2) and his equations can be shown to include as a special case those solved by Srivastav (**5**). An account of both Noble's and Srivastav's solutions can be found in the recent book by Sneddon (**4**).

When $a+1 = 2\lambda$ equations (1) and (2) can be reduced to the type considered by Noble and therefore can be solved exactly. In general, however, it does not seem possible to obtain a closed solution and we show that when (i) $a+1 > \lambda > \sigma$, $0 < \sigma < 1$, or (ii) $a+1+\sigma > \lambda > 0$, $-1 < \sigma < 0$, the solution of the equations can be expressed in terms of the solution of a Fredholm integral equation of the second kind.

We then consider the triple integral equations

$$\int_0^\infty A(x) J_\mu(rx)\, dx = 0, \quad 0 \leqq r < a,\ b < r < \infty, \tag{5}$$

$$\int_0^\infty x^{-2\sigma} A(x) J_\mu(rx)\, dx = H(r), \quad a < r < b, \tag{6}$$

where $J_\mu(rx)$ is the Bessel function of the first kind and $H(r)$ is a known function.

The first attempt at solving these equations seems to have been made by Tranter (**6**) who assumed a series representation for the unknown function $A(x)$

and reduced the problem to that of solving a pair of dual series of the type (1) and (2). He was however able to complete the solution only when $\mu = \pm\frac{1}{2}$, $\sigma = \pm\frac{1}{2}$. Since Tranter's paper a number of solutions of equations (5) and (6) have been obtained and a description of some of the methods used can be found in (4). The most elegant and comprehensive set of solutions so far obtained has been given by Cooke (1) who used the method of Erdélyi-Kober and Hankel operators to reduce the solution of the equations to the solution of one or two Fredholm integral equations of the second kind.

As an application of the solutions of equations (1) and (2) we show that the series solution assumed by Tranter for the triple integral equations leads to solutions which are identical with some of those given by Cooke. That the method described here for the solution of the integral equations is not the most convenient or straightforward is obvious; but it is interesting to see that a development of Tranter's original method of solution yields results which are in agreement with those obtained in later work.

The analysis used in this paper is purely formal and some results which will be required are now stated below for convenient reference.

Two relations between Jacobi polynomials which are given in (3) are

$$\mathfrak{F}_n(a, \lambda;\ 1-\rho) = (-1)^n \frac{\Gamma(\lambda)\Gamma(1+a-\lambda+n)}{\Gamma(\lambda+n)\Gamma(1+a-\lambda)} \mathfrak{F}_n(a, 1+a-\lambda;\ \rho), \tag{7}$$

and

$$\frac{\Gamma(\xi+n)\Gamma(1+n)}{\Gamma(1-\sigma+n)\Gamma(\xi-\sigma+n)} \mathfrak{F}_n(\xi-\sigma, \xi;\ r) = \frac{\Gamma(\xi)(1-r)^\sigma}{\Gamma(-\sigma)\Gamma(\xi-\sigma)} \int_r^1 \frac{\mathfrak{F}_n(\xi-\sigma, \xi-\sigma;\ y)}{(y-r)^{1+\sigma}}\, dy \tag{8}$$

where $\xi-\sigma>\xi$.

From (7) we see that the Bessel function can be represented in terms of an infinite series of Jacobi polynomials by

$$(\tfrac{1}{2}tz)^{\alpha-\beta} J_\beta(tz) = t^\alpha \sum_{n=0}^{\infty} \frac{(\alpha+2n)\Gamma(\alpha+n)}{\Gamma(n+1)\Gamma(\beta+1)} J_{\alpha+2n}(z)\mathfrak{F}_n(\alpha, \beta+1;\ t^2). \tag{9}$$

The orthogonality relation for Jacobi polynomials is

$$\int_0^1 r^{\lambda-1}(1-r)^{a-\lambda}\mathfrak{F}_m(a, \lambda;\ r)\mathfrak{F}_n(a, \lambda;\ r)dr = \delta_{mn}/\Delta_n^2, \quad a+1>\lambda>0, \tag{10}$$

where δ_{mn} is the Kronecker delta and

$$\Delta_n^2(a, \lambda) = \frac{(a+2n)\Gamma(a+n)\Gamma(\lambda+n)}{\{\Gamma(\lambda)\}^2\Gamma(n+1)\Gamma(1+a-\lambda+n)}. \tag{11}$$

Noble (3) has shown that when $a+1+\sigma>\lambda>\sigma>0$, then

$$K(r,\rho)=\{\Gamma(\sigma)\}^2(r\rho)^{\lambda-1}\sum_{n=0}^{\infty}\Delta_n^2(a,\lambda)p_n(\lambda-\sigma,\lambda)\mathfrak{F}_n(a,\lambda;\,r)\mathfrak{F}_n(a,\lambda;\,\rho) \tag{12}$$

$$=\int_0^t m(x)(r-x)^{\sigma-1}(\rho-x)^{\sigma-1}dx=K_t(r,\rho), \tag{13}$$

where $m(x)=x^{\lambda-\sigma-1}(1-x)^{\lambda-\sigma-a}$ and $t=\min(r,\rho)$.

If we replace r, ρ, x, λ and σ by $(1-r)$, $(1-\rho)$, $(1-x)$, $(1+a-\lambda)$ and $(-\sigma)$ respectively in equations (12) and (13) and write $K(1-r,1-\rho)=S(r,\rho)$ we find, after using equation (7), that they become

$$S(r,\rho)=\{\Gamma(-\sigma)\}^2[(1-r)(1-\rho)]^{a-\lambda}$$
$$\sum_{n=0}^{\infty}\Delta_n^2(a,\lambda)p_n(\lambda,\lambda-\sigma)\mathfrak{F}_n(a,\lambda;\,r)\mathfrak{F}_n(a,\lambda;\,\rho) \tag{14}$$

$$=\int_u^1 n(x)(x-r)^{-\sigma-1}(x-\rho)^{-\sigma-1}dx=S_u(r,\rho), \tag{15}$$

where $a+1>a+1+\sigma>\lambda>\sigma$, $n(x)=\{m(x)\}^{-1}$, $p_n(\lambda-\sigma,\lambda)p_n(\lambda,\lambda-\sigma)=1$, and $u=\max(r,\rho)$.

2. Solution of the dual series equations

2.1. *When* $a+1>\lambda>\sigma$, $0<\sigma<1$. If we define a function $\phi(\rho)$ by the equation

$$\sum_{n=0}^{\infty}A_n\mathfrak{F}_n(a,\lambda;\,\rho)=\phi(\rho),\quad d<\rho\leqq 1, \tag{16}$$

we see that equations (1) and (16) yield the result

$$A_n=\Delta_n^2(a,\lambda)\left\{\int_0^d f(r)+\int_d^1\phi(r)\right\}r^{\lambda-1}(1-r)^{a-\lambda}\mathfrak{F}_n(a,\lambda;\,r)dr,$$
$$a+1>\lambda>0, \tag{17}$$

where we have used the orthogonality relation (10).

Substituting for A_n into equation (2) and interchanging the order of integration and summation we find that

$$\left\{\int_0^d f(r)+\int_d^1\phi(r)\right\}(1-r)^{a-\lambda}K(r,\rho)dr=\{\Gamma(\sigma)\}^2\rho^{\lambda-1}g(\rho),\quad d<\rho\leqq 1, \tag{18}$$

where $K(r,\rho)$ is defined by equation (12).

Using the notation of equation (13) the above equation can be written in the form

$$\left\{\int_0^d f(r)+\int_d^\rho\phi(r)\right\}(1-r)^{a-\lambda}K_r(r,\rho)dr+\int_\rho^1(1-r)^{a-\lambda}\phi(r)K_\rho(r,\rho)dr$$
$$=\{\Gamma(\sigma)\}^2\rho^{\lambda-1}g(\rho), \tag{19}$$

where $a+1>\lambda>\sigma>0$.

Inverting the order of integration in equation (19) we see that it becomes

$$\int_d^\rho \frac{m(x)}{(\rho-x)^{1-\sigma}}\Phi(x)dx = \{\Gamma(\sigma)\}^2\rho^{\lambda-1}g(\rho) - \int_0^d \frac{m(\xi)}{(\rho-\xi)^{1-\sigma}}[F(\xi)+\Phi_1(\xi)]d\xi, \quad (20)$$

where

$$(a)\ \Phi(x) = \int_x^1 \frac{(1-r)^{\alpha-\lambda}}{(r-x)^{1-\sigma}}\phi(r)dr, \quad (b)\ \Phi_1(x) = \int_d^1 \frac{(1-r)^{\alpha-\lambda}}{(r-x)^{1-\sigma}}\phi(r)dr, \quad (21)$$

and

$$F(\xi) = \int_\xi^d \frac{(1-r)^{\alpha-\lambda}}{(r-\xi)^{1-\sigma}}f(r)dr, \quad (22)$$

is a known function.

When $0<\sigma<1$ we can invert the Abel-type integral equations (20) and (21a) to find that

$$(1-r)^{\alpha-\lambda}\phi(r) = -\frac{\sin(\sigma\pi)}{\pi}\frac{d}{dr}\int_r^1 \frac{\Phi(x)}{(x-r)^\sigma}dx, \quad (23)$$

and

$$m(x)\Phi(x) = G(x) - \frac{\sin(\sigma\pi)}{\pi(x-d)^\sigma}\int_0^d \frac{m(\xi)(d-\xi)^\sigma}{x-\xi}[F(\xi)+\Phi_1(\xi)]d\xi, \quad (24)$$

where

$$G(x) = \frac{\Gamma(\sigma)}{\Gamma(1-\sigma)}\frac{d}{dx}\int_d^x \frac{\rho^{\lambda-1}g(\rho)}{(x-\rho)^\sigma}d\rho, \quad (25)$$

is a known function and we have used the result

$$\frac{d}{dx}\int_d^x \frac{d\rho}{(x-\rho)^\sigma(\rho-\xi)^{1-\sigma}} = \frac{(d-\xi)^\sigma}{(x-\xi)(x-d)^\sigma}, \quad 0<\sigma<1. \quad (26)$$

Substituting for $\phi(r)$ from equation (23) into equation (21*b*), performing an integration by parts and using the result

$$(1-\sigma)\int_d^x \frac{dr}{(x-r)^\sigma(r-\xi)^{2-\sigma}} = \frac{(x-d)^{1-\sigma}}{(x-\xi)(d-\xi)^{1-\sigma}}, \quad 0<\sigma<1, \quad (27)$$

it is easily shown that $\Phi_1(x)$ is given in terms of $\Phi(x)$ by the equation

$$\Phi_1(x) = (d-x)^\sigma\frac{\sin(\sigma\pi)}{\pi}\int_d^1 \frac{\Phi(y)}{(y-x)(y-d)^\sigma}dy. \quad (28)$$

If we now eliminate $\Phi_1(x)$ between equations (24) and (28) we see that $\Phi(x)$ satisfies the integral equation

$$m(x)\Phi(x) + \int_d^1 \Phi(y)U(x, y)dy = G(x) + F_1(x), \quad d<x<1, \quad (29)$$

where $U(x, y)$ is the kernel

$$U(x, y) = \frac{\sin^2(\sigma\pi)}{\pi^2[(x-d)(y-d)]^\sigma}\int_0^d \frac{m(\xi)(d-\xi)^{2\sigma}}{(x-\xi)(y-\xi)}d\xi, \quad 0<\sigma<1, \quad (30)$$

and

$$F_1(x) = -\frac{\sin(\sigma\pi)}{\pi(x-d)^\sigma}\int_0^d \frac{m(\xi)(d-\xi)^\sigma}{x-\xi} F(\xi)d\xi, \tag{31}$$

is a known function.

Equation (29) is a Fredholm integral equation of the second kind which determines $\Phi(x)$, $\phi(r)$ can then be found from equation (23) and the coefficients A_n, which are solutions of the equations (1) and (2) when $a+1>\lambda>\sigma$, $0<\sigma<1$, can be obtained from equation (17).

2.2. *When* $a+1+\sigma>\lambda>0$, $-1<\sigma<0$. In this case we set

$$\sum_{n=0}^{\infty} A_n p_n(\lambda-\sigma, \lambda)\mathfrak{F}_n(a, \lambda;\ \rho) = \psi(\rho), \quad 0 \leqq \rho < d, \tag{32}$$

and using the orthogonality relation (10) we find from equations (2) and (32) that

$$A_n = \Delta_n^2(a, \lambda)p_n(\lambda, \lambda-\sigma)\left\{\int_0^d \psi(r) + \int_d^1 g(r)\right\} r^{\lambda-1}(1-r)^{a-\lambda}\mathfrak{F}_n(a,\lambda;\ r)dr \tag{33}$$

where $a+1>\lambda>0$.

Substituting for A_n into equation (1) and interchanging the order of integration and summation we see that

$$\int_0^\rho r^{\lambda-1}\psi(r)S_\rho(r, \rho)dr + \left\{\int_\rho^d \psi(r) + \int_d^1 g(r)\right\} r^{\lambda-1}S_r(r, \rho)dr$$
$$= \{\Gamma(-\sigma)\}^2(1-\rho)^{a-\lambda}f(\rho), \tag{34}$$

where $0 \leqq \rho<d$, $a+1>a+1+\sigma>\lambda>0$ and $S_u(r, \rho)$ is defined by equation (15).

If we now invert the order of integration in the above equation and use a method similar to that used in section 2.1 we can show that when $-1<\sigma<0$ the function $\psi(r)$ is given by the equation

$$r^{\lambda-1}\psi(r) = -\frac{\sin(\sigma\pi)}{\pi}\frac{d}{dr}\int_0^r \frac{\Psi(x)}{(r-x)^{-\sigma}}dx, \quad 0<r<d, \tag{35}$$

where $\Psi(x)$ is the solution of the Fredholm integral equation of the second kind

$$n(x)\Psi(x) + \int_0^d \Psi(y)V(x, y)dy = F_2(x)+G_2(x), \quad 0<x<d, \tag{36}$$

with the kernel

$$V(x, y) = \frac{\sin^2(\sigma\pi)}{\pi^2[(d-x)(d-y)]^{-\sigma}}\int_d^1 \frac{n(\xi)(\xi-d)^{-2\sigma}}{(x-\xi)(y-\xi)}d\xi, \quad -1<\sigma<0, \tag{37}$$

and the free terms

$$F_2(x) = -\frac{\Gamma(-\sigma)}{\Gamma(1+\sigma)}\frac{d}{dx}\int_x^d \frac{(1-\rho)^{a-\lambda}}{(\rho-x)^{-\sigma}}f(\rho)d\rho, \tag{38}$$

$$G_2(x) = -(d-x)^\sigma\frac{\sin(\sigma\pi)}{\pi}\int_d^1 \frac{n(\xi)(\xi-d)^{-\sigma}}{x-\xi}d\xi\int_d^\xi \frac{\rho^{\lambda-1}g(\rho)}{(\xi-\rho)^{1+\sigma}}d\rho. \tag{39}$$

The coefficients A_n, which satisfy equations (1) and (2) when $a+1+\sigma>\lambda>0$, $-1<\sigma<0$, can then be found from equations (33), (35) and (36).

3. Solution of the triple integral equations

Tranter (6) has shown that if we assume that the solution of equations (5) and (6) is of the form

$$A(x) = x^{\sigma} \sum_{n=0}^{\infty} C_n \frac{\Gamma(1-\sigma+n)}{\Gamma(1+\mu+n)} J_{\mu+2n+1-\sigma}(bx), \tag{40}$$

then the coefficients C_n are solutions of the dual series equations

$$\sum_{n=0}^{\infty} C_n \mathfrak{F}_n(1+\mu-\sigma, 1+\mu;\ \rho) = 0, \quad 0 \leqq \rho < s, \tag{41}$$

$$\sum_{n=0}^{\infty} C_n \frac{\Gamma(1-\sigma+n)\Gamma(1+\mu-\sigma+n)}{\Gamma(1+n)\Gamma(1+\mu+n)} \mathfrak{F}_n(1+\mu-\sigma, 1+\mu;\ \rho) = \frac{2^{\sigma}\Gamma(1+\mu)}{b^{\sigma-1}\rho^{\mu/2}} H(b\sqrt{\rho}), \quad s<\rho \leqq 1, \tag{42}$$

where $b^2 s = a^2$ and (i) if $0<\sigma<1$, then $\mu>\sigma-1$, or (ii) if $-1<\sigma<0$, then $\mu>-1$.

These equations are the same as equations (1) and (2) with $a = 1+\mu-\sigma$, $\lambda = 1+\mu$, and the conditions (i) and (ii) are precisely those for which the equations can be solved.

3.1. *When* $\mu>\sigma-1$, $0<\sigma<1$. Applying the results of section 2.1 we see that the solution of equations (41) and (42) is given in terms of a function $\phi^*(r)$ by

$$C_n = \Delta_n^2(1+\mu-\sigma, 1+\mu) \int_s^1 r^{\mu}(1-r)^{-\sigma}\phi^*(r)\mathfrak{F}_n(1+\mu-\sigma, 1+\mu;\ r)dr, \tag{43}$$

where $\Delta_n^2(a, \lambda)$ is defined by equation (11).

The function $\phi^*(r)$ is found from the equation

$$(1-r)^{-\sigma}\phi^*(r) = -\frac{\sin(\sigma\pi)}{\pi}\frac{d}{dr}\int_r^1 \frac{\Phi^*(x)}{(x-r)^{\sigma}}dx, \tag{44}$$

where $\Phi^*(x)$ is the solution of the Fredholm integral equation

$$x^{\mu-\sigma}\Phi^*(x) + \int_s^1 \Phi^*(y)U^*(x, y)dy = G^*(x), \quad s<x<1, \tag{45}$$

with the kernel

$$U^*(x, y) = \frac{\sin^2(\sigma\pi)}{\pi^2[(x-s)(y-s)]^{\sigma}} \int_0^s \frac{\xi^{\mu-\sigma}(s-\xi)^{2\sigma}}{(x-\xi)(y-\xi)} d\xi, \tag{46}$$

and the free term

$$G^*(x) = \frac{2^{\sigma}\Gamma(\sigma)\Gamma(1+\mu)}{b^{\sigma-1}\Gamma(1-\sigma)} \frac{d}{dx} \int_s^x \frac{\rho^{\mu/2}H(b\sqrt{\rho})}{(x-\rho)^{\sigma}} d\rho. \tag{47}$$

It is possible to obtain the solution to the triple integral equations without computing the values of the coefficients C_n. To show this we substitute for C_n

from equation (43) into equation (40) and interchange the order of integration and summation to find that

$$A(x)=\frac{x^{\sigma}}{\Gamma(1+\mu)}\int_s^1 r^{\mu}(1-r)^{-\sigma}\phi^*(r)L(x,r)dr,\quad b^2s=a^2, \tag{48}$$

where

$$\begin{aligned}L(x,r)&=\sum_{n=0}^{\infty}\frac{(1+\mu-\sigma+2n)\Gamma(1+\mu-\sigma+n)}{\Gamma(n+1)\Gamma(1+\mu)}J_{\mu+2n+1-\sigma}(bx)\mathfrak{F}_n(1+\mu-\sigma,1+\mu;\ r)\\&=\left(\frac{bx}{2}\right)^{1-\sigma}r^{-\mu/2}J_{\mu}(bx\sqrt{r}),\end{aligned} \tag{49}$$

after using the result (9).

It follows from equations (48) and (49) that when $\mu>\sigma-1$, $0<\sigma<1$, then $A(x)$ is given by

$$A(x)=\frac{x}{\Gamma(1+\mu)}\left(\frac{b}{2}\right)^{1-\sigma}\int_s^1 r^{\mu/2}(1-r)^{-\sigma}\phi^*(r)J_{\mu}(bx\sqrt{r})dr,\quad b^2s=a^2. \tag{50}$$

The solution of the triple integral equations is given by the equations (44) (45) and (50) and this agrees with the solution obtained by Cooke (**1**, p. 62).

3.2. *When* $\mu>-1$, $-1<\sigma<0$. Using the results of section 2.2 it follows that the solution of the dual series equations (41) and (42) is given by

$$\begin{aligned}C_n=\Delta_n^2(1+\mu-\sigma,1+\mu)\frac{\Gamma(1+\mu+n)\Gamma(1+n)}{\Gamma(1+\mu-\sigma+n)\Gamma(1-\sigma+n)}&\left\{\int_0^s\psi^*(r)\right.\\&\left.+\int_s^1 g(r)\right\}r^{\mu}(1-r)^{-\sigma}\mathfrak{F}_n(1+\mu-\sigma,1+\mu;\ r)dr,\end{aligned} \tag{51}$$

where $g(r)=2^{\sigma}\Gamma(1+\mu)b^{1-\sigma}r^{-\mu/2}H(b\sqrt{r})$.

If we now make use of the result (8) we find that the above equation can be written in the form

$$\begin{aligned}C_n=\frac{\Delta_n^2(1+\mu-\sigma,1+\mu)\Gamma(1+\mu)}{\Gamma(-\sigma)\Gamma(1+\mu-\sigma)}&\left\{\int_0^s\psi^*(r)\right.\\&\left.+\int_s^1 g(r)\right\}r^{\mu}dr\int_r^1\frac{\mathfrak{F}_n(1+\mu-\sigma,1+\mu-\sigma;\ y)}{(y-r)^{1+\sigma}}dy.\end{aligned} \tag{52}$$

Substituting for C_n into equation (40), interchanging the order of integration and summation and using the result (9) it can be shown that $A(x)$ is given by

$$\begin{aligned}A(x)=\frac{bx^{1+\sigma}}{2\Gamma(-\sigma)\Gamma(1+\mu)}&\left\{\int_0^s y^{\frac{1}{2}(\sigma-\mu)}J_{\mu-\sigma}(bx\sqrt{y})\Psi^*(y)dy\right.\\&+\frac{\sin(\sigma\pi)}{\pi}\int_s^1\frac{y^{\frac{1}{2}(\sigma-\mu)}}{(y-s)^{\sigma}}J_{\mu-\sigma}(bx\sqrt{y})dy\int_0^s\frac{(s-r)^{\sigma}}{r-y}\Psi^*(r)dr\\&\left.+\int_s^1 y^{\frac{1}{2}(\sigma-\mu)}J_{\mu-\sigma}(bx\sqrt{y})G_1^*(y)dy\right\},\end{aligned} \tag{53}$$

where

$$G_1^*(y) = \frac{2^\sigma \Gamma(1+\mu)}{b^{\sigma-1}} \int_s^y \frac{r^{\mu/2} H(b\sqrt{r})}{(y-r)^{1+\sigma}} dr, \quad b^2 s = a^2. \tag{54}$$

The function $\Psi^*(x)$ is the solution of the Fredholm integral equation of the second kind

$$x^{\sigma-\mu}\Psi^*(x) + \int_0^s \Psi^*(y) V^*(x, y) dy = G_2^*(x), \quad 0 < x < s, \tag{55}$$

with the kernel

$$V^*(x, y) = \frac{\sin^2(\sigma\pi)}{\pi^2[(s-x)(s-y)]^{-\sigma}} \int_s^1 \frac{\xi^{\sigma-\mu}(\xi-s)^{-2\sigma}}{(x-\xi)(y-\xi)} d\xi, \quad -1 < \sigma < 0, \tag{56}$$

and the free term

$$G_2^*(x) = -(s-x)^\sigma \frac{\sin(\sigma\pi)}{\pi} \int_s^1 \frac{\xi^{\sigma-\mu}(\xi-s)^{-\sigma}}{x-\xi} G_1^*(\xi) d\xi. \tag{57}$$

Equations (53) and (55) give the solution to the triple integral equations when $\mu > -1$, $-1 < \sigma < 0$, and this is of the same form as that given by Cooke.

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