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WEAK q-RINGS

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Throughout this paper we assume that every ring has unity and all modules are unital right modules. A ring R is called a *(right) q-ring* if every right ideal of R is quasi-injective [**4**]. In this paper we study a generalization of this concept. A ring R is called a *(right) weak q-ring* (in short, *wq-ring*) if every right ideal of R, not isomorphic to R_R, is quasi-injective. A ring R is called a *right pq-ring* if every proper right ideal of R is quasi-injective. Any upper triangular 2×2 matrix ring over a division ring is a wq-ring, which is not a q-ring. In Section 1, some general properties of wq-rings are established and, in particular, it is shown in (1.8) that a semiprime wq-ring has zero singular ideal. In Section 2, wq-rings with zero singular ideals are studied. (2.4) and (2.7) give the structure of such rings. (2.10) shows that any prime wq-ring R, which admits no proper right ideal isomorphic to R_R, is simple artinian. The results (2.11) and (2.13) give some information about general prime wq-rings. It is not clear whether every prime wq-ring with non-zero socle is artinian.

For any ring R, $\hat{R}$, $Z(R)$ and Rad R will stand for the injective hull, the singular submodule and the Jacobson radical of R_R, respectively. A ring R is said to be *semilocal* (*local*) if $R/\text{Rad } R$ is semi-simple artinian (a division ring). A right ideal A of a ring R is said to be *closed* if A_R has no essential extension in R_R. The lattice of closed right ideals of a ring R with $Z(R) = 0$ will be denoted by $L^S(R)$. For any subset X of R, $r(X)(l(X))$ will denote the right (left) annihilator of X in R. The notation, pri-ring (right PID) will stand for principal right ideal ring (principal right ideal domain). For any module M, $N \subset' M$ will denote that N is an essential or large submodule of M.

1. The object of this section is to establish some fundamental properties of wq-rings and pq-rings. For a ring R, K will stand for $\text{Hom}_R(\hat{R}, \hat{R})$. If A is a right ideal of R, we define

$$A^* = \{x \in R | Kx \subset A\}.$$

It is clear that A^* is a right ideal of R contained in A. Also if B and C are right ideals of R such that $B \subset C$, then $B^* \subset C^*$.

(1.1) LEMMA. *Let R be any ring. Let A be a right ideal of R and E a large right ideal of R. Then*

1) *A^* is a left K-module and is a two sided ideal of R;*
2) *A^* is quasi-injective as a right R-module; and*
3) *E is quasi-injective if and only if $E^* = E$.*

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Proof. Using the result of Johnson and Wong [**6**, Theorem (1.1)] the lemma follows.

The following two corollaries are immediate consequences of the above lemma.

(1.2) COROLLARY. *Let R be a right wq-ring. If $A^* \cong R_R$ for some right ideal A of R, then R is a right q-ring.*

(1.3) COROLLARY. (1) *If R is a right wq-ring which is not a right q-ring, then $R^* \subset' R_R$ if and only if there exists a large right ideal E of R such that $E \not\cong R$.*

2) *If R is a right pq-ring, then $R^* \subset' R_R$.*

Johnson and Wong [**5**] have shown that if a ring R has $Z(R) = 0$, then $\hat{R}$ is a right self-injective regular ring having R as a subring. We now have the following.

(1.4) PROPOSITION. *Let R be any ring with $Z(R) = 0$. Then for every right ideal A of R,*

1) *A^* is a left ideal of $\hat{R}$;*

2) *if $A^* \neq 0$, then it contains non-zero idempotents; and*

3) *if A^* contains a right regular element, then R is right self injective.*

Proof. 1) $Z(R) = 0$ yields $K = \mathrm{Hom}_R(\hat{R}, \hat{R}) = \mathrm{Hom}_{\hat{R}}(\hat{R}, \hat{R}) = \hat{R}$. Hence, that A^* is a left ideal of $\hat{R}$ follows by (1.1).

2) Let $0 \neq a \in A^*$. As $\hat{R}$ is a regular ring there exists $x \in \hat{R}$ such that $axa = a$. Hence xa is a nonzero idempotent. By (1), $xa \in A$.

3) Let a be a right regular element in A^*. Since $\hat{R}_R$ is injective, it is divisible by a. We have $\hat{R}a = \hat{R}$. Thus

$$\hat{R} = \hat{R}a \subset A^* \subset R \subset \hat{R}.$$

This completes the proof.

(1.5) LEMMA. *Let R be a wq-ring. If R contains non-trivial central idempotents, then R is a right q-ring.*

Proof. Let $R = A \oplus B$ where A and B are non-zero ideals. Then $A_R \not\cong R_R \not\cong B_R$. Hence A_R and B_R are quasi-injective. As A and B are ideals, we get A and B are right self-injective rings. Hence $R = A \oplus B$ is a right q-ring.

(1.6) PROPOSITION. *Let R be a right wq-ring. If e is an idempotent of R, then either eR or $(1 - e)R$ is quasi-injective.*

Proof. Let $A = eR$, $B = (1 - e)R$. Let B be not quasi-injective. Then $B \cong R_R$ and this gives $B = B_1 \oplus B_2$ where $B_1 \cong A$, $B_2 \cong B$. Again $B_2 = C_1 \oplus C_2$ where $C_1 \cong A$, $C_2 \cong B$. This process can be continued indefinitely. Hence B contains an infinite direct sum

$$E = B_1 \oplus C_1 \oplus \ldots.$$

in which every summand is isomorphic to A. Now $E \not\cong R$ and hence E is quasi-injective. Consequently A is quasi-injective.

(1.7) PROPOSITION. *Let R be a domain. Then R is a right wq-ring if and only if R is a right PID.*

Proof. Suppose that R is a right wq-ring. If $R^* \neq 0$ by (1.4), since R^* contains an idempotent, $R^* = R$ and $R = \hat{R}$, thus R is regular and hence a division ring. If $R^* = 0$, then every right ideal of R is isomorphic to R, and as a consequence R is a right PID. The converse is obvious.

(1.8) PROPOSITION. *A semiprime right wq-ring R has zero right singular ideal.*

Proof. If every large right ideal of R is isomorphic to R_R, then R is right noetherian and hence $Z(R) = 0$ (see [**2**]). So assume that R contains a large right ideal $E \not\cong R$. Let $x \in Z(R)$. Then $r(x) = E'$ for some large right ideal E' of R. If $E' \not\cong R_R$, then E', being quasi-injective, is a two sided ideal; as R is semiprime this implies $x = 0$. Let $E' \cong R$. Then $E' = aR$ for some right regular element $a \in R$. Obviously then $aE \subset' aR = E'$ yields $aE \subset' R$. Then it follows from $aE \cong E$ that $aE \not\cong R$, and that aE is a two sided ideal by (1.1). Now $x(aE) \subset xE' = 0$. Then again $x = 0$, since R is semiprime. This completes the proof.

2. In this section we study rings with $Z(R) = 0$. Johnson and Wong [**5**] proved that $\hat{R}$ is a right self-injective regular ring having R as a subring. This fact will be frequently used in this section without further comments.

(2.1) LEMMA. *Let $R = e_1R \oplus e_2R$ where e_1 and e_2 are orthogonal indecomposable idempotents of R, such that $e_1Re_2 \neq 0$ and $e_2Re_1 = 0$. If $Z(R) = 0$ and e_iR are quasi-injective, $i = 1, 2$, then*

$$R \cong \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$$

where $D = e_1Re_1$, is a division ring.

Proof. Since e_iR is indecomposable and quasi-injective, e_iR is a uniform right ideal. As $\hat{R}$ is a regular ring, the uniformity of $e_i\hat{R}$ implies that $e_i\hat{R}$ is a minimal right ideal of R. Hence $e_i\hat{R}e_i$ is a division ring. By [**6**, Theorem (1.1)] $e_iRe_i = \mathrm{Hom}_R(e_iR, e_iR) = \mathrm{Hom}_R(e_i\hat{R}, e_i\hat{R}) = e_i\hat{R}e_i$. Hence e_iRe_i is a division ring. Again $e_1Re_2 \neq 0$ implies that e_2R is embeddable in e_1R and hence $e_1\hat{R} \cong e_2\hat{R}$. As a result we get $e_1Re_1 \cong e_2Re_2$. So if $D = e_1Re_1$ and $D' = e_2Re_2$, then $D \cong D'$. Now,

$$R = e_1Re_1 + e_1Re_2 + e_2Re_2 = D + D' + G, \quad G = e_1Re_2.$$

G is an ideal of R such that $G^2 = 0$ and $R/G \cong D \oplus D'$. Hence R is a semiprimary ring with G as its Jacobson radical. $G^2 = 0$ implies G is completely reducible as a right R-module. Then as e_1R is uniform and $G \subset e_1R$, we get that G is a minimal right ideal of R. Since e_1R is quasi-injective, $G \subset' e_1R$ and $Z(R) = 0$, and we have

$$\mathrm{Hom}_R(G, G) = \mathrm{Hom}_R(e_1R, e_1R) = e_1Re_1 = D.$$

Now G is a right vector space over $D' = e_2Re_2$. It is clear that every submodule of G_R is a D'-subspace and conversely. Consequently $\dim G_{D'} = 1$. Because every D'-endomorphism of G is an R-endomorphism, $D = \mathrm{Hom}_{D'}(G, G)$. Then $\dim G_{D'} = 1$ yields that $\dim {}_DG = 1$. Hence if x is a fixed non-zero element of G. then for every $d \in D$ there corresponds a unique $d' \in D'$ such that $dx = xd'$. Then the map $g: D \to D'$ given by

$$g(d) = d' \quad \text{if and only if} \quad dx = xd'$$

is a ring isomorphism. Let

$$S = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$$

Define $f: S \to R$ by

$$f\begin{bmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{bmatrix} = d_{11} + d_{12}x + g(d_{22}).$$

This f is a ring isomorphism. This completes the proof.

By the dual of [**8**, Prof. 2.5] established by Wu and Jans we have:

(2.2) Lemma. *Let M be a right R-module and A be a quasi-injective submodule of M. If $M = \sum_{i=1}^{n} \oplus A_i$ where $A_i \cong A$, $1 \leqq i \leqq n$, then M is quasi-injective.*

The following is obvious.

(2.3) Lemma. *Let A and B be non-zero right R-modules such that $A \oplus B$ is quasi-injective. If $0 \to A \xrightarrow{\varphi} B$ is exact, then φ splits. Further if B is indecomposable, φ is an isomorphism.*

(2.4) Theorem. *Let R be a ring such that $Z(R) = 0$ and $L^S(R)$ finite dimensional. Then R is a right wq-ring if and only if:*

I. *R is a right PID, or*

II. *R is semi-simple artinian, or*

III. *$R \cong \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ for some division ring D.*

Proof. Since $L^S(R)$ is finite dimensional, $R = \sum_{i=1}^{n} \oplus e_iR$, where $\{e_i: 1 \leqq i \leqq n\}$ is a set of orthogonal indecomposable idempotents of R. Notice that no proper summand of R_R is isomorphic to R_R. Suppose that R is a wq-ring. We consider two cases.

1) $n = 1$. In this case R_R is uniform and hence R is without zero divisors. If every non-zero right ideal of R is isomorphic to R then R is a right PID and

R is of type (I). So let R have a non-zero right ideal $E \not\cong R$. Then $E \subset' R$ and $E^* = E$. In this case, by (1.4), E contains a non-zero idempotent and hence $R = E$. This is a contradiction. So R is of type (I).

2) $n > 1$. In this case $e_i R \not\cong R$ for every i. As a result each $e_i R$ is quasi-injective and hence uniform. Then

$$\hat{R} = \sum_{i=1}^{n} \oplus e_i \hat{R}$$

being a direct sum of finitely many minimal right ideals is semi-simple artinian. Let H be a homogeneous component of $\hat{R}$. Renumbering if necessary, let

$$H = e_1\hat{R} + \ldots + e_t\hat{R} = (e_1 + e_2 + \ldots + e_t)\hat{R}$$

where $t \leqq n$. If $t < n$, then $e_1 + e_2 + \ldots + e_t$ is a central idempotent of R different from 0 and 1. Hence $R = \hat{R}$ by (1.5) and R is of type (II). Let $t = n$. Then

$$e_i\hat{R} = e_j\hat{R}, \quad 1 \leqq i, j \leqq n.$$

As $e_k R \subset' e_k\hat{R}$, it is easy to see that for all $1 \leqq i \leqq n$, there exist non-zero right ideals $A_i \subset e_i R$ such that $A_i \cong A_j$ for all i, j. We consider two subcases:

a) $n > 2$. As $e_i R \oplus e_j R$ is quasi-injective (for $i \neq j$) and $e_i\hat{R} \cong e_j\hat{R}$, by Harada [**3**], $e_i R \cong e_j R$. Thus R_R is quasi-injective by (2.2). So again $R = \hat{R}$ and R is of type (II).

b) $n = 2$. If none of $A_1 \oplus e_2 R$ and $e_1 R \oplus A_2$ is isomorphic to R, then both of them are quasi-injective and by (2.3), $e_2 R \cong A_1 \cong A_2 \cong e_i R$ so we again get that R is of type (II). If $e_1 R \oplus A_2 \cong R \cong A_1 \oplus e_2 R$, then as the rings of endomorphism of A_1, A_2, $e_1 R$ and $e_2 R$ are all local rings, by the Krull-Schmidt-Azumaya Theorem, $e_1 R \cong A_1$, and $e_2 R \cong A_2$. Hence again $e_1 R \cong e_2 R$ and R is of type (II). So it remains to consider the case when

$$A_1 + e_2 R \not\cong R, \quad e_1 R + A_2 \cong R.$$

In this case, $e_2 R \cong A_1 \subset e_1 R$. Hence $e_1 R e_2 \neq 0$. Let $e_2 R e_1 \neq 0$, then we get a non-zero homomorphism of $e_1 R$ into $e_2 R$, which is a monomorphism, since $e_1 R$ is uniform and $Z(R) = 0$. Consequently $e_1 R$ and $e_2 R$ are embeddable in each other. Then by Bumby [**1**, Corollory 2] $e_1 R \cong e_2 R$. This again yields that R is of type (II). So we get that $e_2 R e_1 = 0$. Then by (2.1)

$$R \cong \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$$

where $D = e_1 R e_1$ is a division ring. Hence R is of type (III).

Conversely, if R is of type (I) or (II) then trivially R is a wq-ring. Finally suppose $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ for some division ring D. The only non-trivial right ideals of R are

$$A_1 = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix} \quad B_2 = \begin{bmatrix} D & D \\ 0 & 0 \end{bmatrix}$$

and A_3, the set of all matrices of the form $\begin{bmatrix} 0 & a\alpha \\ 0 & b\alpha \end{bmatrix}$ where a, b are fixed non-zero elements of D, and $\alpha \in D$. $A_1 \cong A_3$. Thus as A_1, A_2, A_3 are minimal right ideals, and $B_1 = A_1 + A_2 =$ socle (R_R), these four right ideals are trivially quasi-injective. We now prove that B_2 is quasi-injective. The only proper subright ideal of B_2 is A_1. Let $\varphi: A_1 \rightarrow B_2$ be an R-homomorphism. Let

$$\varphi\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}.$$

Define $\eta: B_2 \rightarrow B_2$ by

$$\eta\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} da & db \\ 0 & 0 \end{bmatrix}.$$

Then η is an R-endomorphism of B_2 extending φ. Thus B_2 is also quasi-injective. This completes the proof.

Remark. Notice that the rings of type (II) or (III) are pq-rings. Since a semiprime right Goldie ring R has $Z(R) = 0$ and has $L^S(R)$, finite dimensional, the above theorem gives the following.

(2.5) COROLLARY. *A semiprime right Goldie ring R is a wq-ring if and only if it is either a right PID or a semi-simple artinian ring.*

It can be easily seen that the proof of (2.4) only depends upon $Z(R) = 0$ and that $1 = e_1 + e_2 + \ldots + e_n$ for some indecomposable orthogonal idempotents. Since a semilocal ring R cannot have an infinite set of orthogonal idempotents, we have $1 = e_1 + e_2 + \ldots + e_n$ for some indecomposable orthogonal idempotents e_i in R. As a result, we obtain:

(2.6) THEOREM. *Let R be a semi-local right wq-ring, with $Z(R) = 0$. Then either R is a right PID or R is a right pq-ring.*

(2.7) THEOREM. *Let R be a ring such that $Z(R) = 0$. If R is a right wq-ring, then either R is a right PID, or R is a strongly regular right self-injective ring or R has non-zero right socle.*

Proof. If $R^* \cong R_R$: then $R = \hat{R}$ and hence R is a regular right q-ring. Then by [**4**, Lemma 2.11] either R is strongly regular or R has non-zero right socle. So, assume that $R^* \ncong R_R$. Hence R contains a proper large right ideal. We consider two cases.

1) Every large right ideal of R is isomorphic to R. Then R is right noetherian and hereditary and by (2.5), R is a right PID or R has non-zero socle.

2) R contains a large right ideal which is not isomorphic to R. In this case $R^* \subset' R_R$. Let M be a maximal right ideal of R such that $M \subset' R_R$. If $M \not\cong R$, then $M \subset R^*$ gives $R^* = M$. Let $M \cong R$, whence $M = aR$ for some right regular element a of R. Then $aR^* \subset' R$. As $aR^* \cong R^*$, aR^* is a left ideal in $\hat{R}$ by (1.1) and (1.4). As $\hat{R}$ is a regular ring, and $r(a) = 0$, there exist $x \in \hat{R}$ such that $xa = 1$. Now $R^* = (xa)R^* = x(aR^*) \subset aR^* \subset R^*$. Thus $R^* = aR^* \subset M$. This all shows that any maximal right ideal of R which is large, contains R^*. Hence if every maximal right ideal of R is large, then $R^* \subset \text{Rad } R$. But this contradicts the fact that R^* contains non-zero idempotents by (1.4). So some maximal right ideal of R is not large. Hence R has non-zero right socle.

The above theorem and (1.8) give:

(2.8) COROLLARY. *Let R be a semiprime right wq-ring. If socle $R = 0$, then either R is a right PID or R is strongly regular right self-injective.*

Consequently we have:

(2.9) COROLLARY. *Let R be a prime right wq-ring. If R is not a right PID, then R has non-zero socle and is primitive.*

It is expected that a primitive right wq-ring with non-zero socle must be artinian. In this connection we first of all prove the following theorem.

(2.10) THEOREM. *Let R be a prime right wq-ring such that no proper right ideal of R is isomorphic to R. Then R is simple artinian.*

Proof. The hypothesis on R gives that R is a right pq-ring. So R cannot be a right PID, unless it is a division ring. Hence by (2.9) R is primitive ring with non-zero socle. By [**4**, Theorem (2.13)] we only need to show that R_R is injective. Suppose the contrary. As soc $R \neq 0$, soc R is not finitely generated and hence by [**7**, Theorem (3.1)], $\hat{R} = \text{Hom}_D(V, V)$ for some infinite dimensional right vector space V over a division ring D.

Let e be an indecomposable idempotent in R; such an idempotent exists in R, since soc $R \neq 0$. Then dim $_D(eV) = 1$. Hence $V_D \cong (1 - e)V_D$. Let $V_1 = (1 - e)V$. Then $\hat{R} = \text{Hom}_D(V, V) = \text{Hom}_D(V_1, V_1) = (1 - e)\ \hat{R}\ (1 - e)$. Also $(1 - e)R$ is quasi-injective gives $(1 - e)R\ (1 - e) = \text{Hom}_R((1 - e)R, (1 - e)R) = \text{Hom}_R((1 - e)\hat{R}, (1 - e)\hat{R}) = (1 - e)\hat{R}(1 - e)$. Let $S = (1 - e)R(1 - e)$. Then $S \cong \hat{R}$ gives that S is a right self-injective ring with soc S not finitely generated as an S-module. Further as S is also primitive, we can find countably infinite sets of indecomposable orthogonal idempotents, $\{f_1, f_2, \ldots, g_1, g_2, \ldots\}$ in S such that

$$\sum \oplus f_i S \cong \sum \oplus g_j S = \sum \oplus f_i S \oplus \sum \oplus g_j S.$$

Let fS and gS be the injective hulls in S of the right S-modules $\sum \oplus f_i S$ and $\sum \oplus g_j S$. We can take f and g to be orthogonal. As $fS \cong (f + g)S$, as

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S-modules, this implies the existence of u and v in S such that $uv = f$ and $vu = f + g$. But since $u, v \in R$ we get $fR \cong (f + g)R$. Now $R = (f + g)R \oplus (1 - f - g)R \cong fR \oplus (1 - f - g)R$. This contradicts the hypothesis that no proper right ideal of R is isomorphic to R. Hence R is simple artinian.

We now prove some result on primitive right wq-rings, whose socles are not finitely generated.

(2.11) PROPOSITION. *Let R be a primitive right wq-ring such that* soc R *is not finitely generated. Then for any $0 \neq a \in R$, either aR is completely reducible or every complement of aR is completely reducible and finitely generated.*

Proof. Suppose that aR is not completely reducible, Now $aR \cap \text{soc } R \subset' aR$, since $\text{soc } R \subset' R_R$. If $aR \cap \text{soc } R$ is finitely generated, then $aR \cap \text{Soc } R = eR$ and as a result $aR = \text{soc } R \cap aR$ which is a contradiction. Hence $\text{soc } R \cap aR$ is not finitely generated. Let B be a complement of aR in R. If B is not a finite direct sum of minimal right ideals, then as above, $B \cap \text{soc } R$ is not finitely generated. Hence B contains an infinite direct sum $\sum_{i \in I} \oplus e_i R$ of minimal right ideals of R. We take I to be countable. Hence as $\text{soc } R \cap aR$ is not finitely generated, we get an R-monomorphism

$$f : \sum_{i \in I} \oplus e_i R \to aR.$$

Now $\sum_{i \in I} \oplus e_i R \oplus aR \not\cong R$. So this direct sum is quasi-injective. Consequently by (2.3), f splits. But this implies that $\sum_{i \in I} \oplus e_i R$ is a finite direct sum, which is a contradiction. Hence B is a finite direct sum of minimal right ideals. This completes the proof.

(2.12) COROLLARY. *Let R be as in* (2.11). *Then for any idempotent e of R either eR or $(1 - e)R$ is completely reducible.*

(2.13) PROPOSITION. *Let R be as in* (2.11). *Let $A = \sum_{i \in I} \oplus e_i R$ be an infinite direct sum of minimal right ideals. If A has a complement in R which is not a finite direct sum of minimal right ideals, then A is a closed right ideal of R.*

Proof. On the contrary, let B be a proper essential extension of A in R_R. Then B is not completely reducible. So there exists $b \in B$ such that bR is not completely reducible. Let C be a complement of A in R satisfying the hypothesis. Then $B \cap C = 0 = bR \cap C$. We can have a complement C' of bR containing C. By (2.11) C' is completely reducible and finitely generated; so the same hold for C. This contradicts our hypothesis. Hence $A = B$. This completes the proof.

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