A CLASS OF EXTREME LATTICE-COVERINGS OF n-SPACE BY SPHERES

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All extreme lattice-coverings of n-space by spheres are known for $n \leqq 4$; see for example [3]. Only one class of extreme covering is known for large n, namely that corresponding to the quadratic form

$$\phi_0(\boldsymbol{x}) = n \sum_{i=1}^{n} x_i^2 - 2 \sum_{i<j} x_i x_j; \tag{1.1}$$

this was first shown to be extreme for all $n \geqq 2$ by Bleicher [2].

The object of this paper is to exhibit a new extreme lattice-covering for all odd $n \geqq 5$. The density of the covering is slightly larger than that corresponding to ϕ_0, so that unfortunately no further information is provided on the density of the most economical covering.

We use the notation and some results of Voronoi [5,6] and Barnes and Dickson [1]; a brief description of these follows.

Let $f(\boldsymbol{x}) = f(x_1, x_2, \cdots, x_n) = \boldsymbol{x}'A\boldsymbol{x}$ be positive definite, with determinant $d = d(f)$. Define the inhomogeneous minimum $m(f)$ of f by

$$m(f) = \max_{\boldsymbol{\alpha}} \min_{\boldsymbol{x}} f(\boldsymbol{x} + \boldsymbol{\alpha}) \qquad (\boldsymbol{\alpha} \text{ real}, \boldsymbol{x} \in \Gamma) \tag{1.2}$$

(where Γ denotes the integral lattice) and define

$$\mu(f) = m(f)/d^{1/n}. \tag{1.3}$$

If $A = P'P$ and $\Lambda = \{P\boldsymbol{x} \mid \boldsymbol{x} \in \Gamma\}$, then spheres of radius $(m(f))^{\frac{1}{2}}$ centred at the points of Λ cover space minimally, and the density of the covering is

$$\theta(\Lambda) = J_n(\mu(f))^{n/2}, \tag{1.4}$$

where J_n is the volume of the unit sphere.

We say that f (and the corresponding covering) are *extreme* if $\mu(g) \geqq \mu(f)$ for all forms g sufficiently close to f; if f is extreme, so of course is any from equivalent to a multiple of f, and equivalent forms correspond to the same lattice covering.

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The Voronoi polytope Π corresponding to f is the set of points $\boldsymbol{x}$ satisfying

$$f(\boldsymbol{x}) \leqq f(\boldsymbol{x}-\boldsymbol{l}) \qquad \text{for all} \quad \boldsymbol{l} \in \Gamma .$$

A finite set $\pm \boldsymbol{l}_1, \cdots, \pm \boldsymbol{l}_\sigma$ of integral points suffices to define Π, which has therefore σ pairs of opposite parallel faces, with equations

$$f(\boldsymbol{x}) = f(\boldsymbol{x} \pm \boldsymbol{l}_i) \qquad (i = 1, \cdots, \sigma).$$

A given $\boldsymbol{l} \neq \boldsymbol{0}$ belongs to this set iff

$$f(\boldsymbol{l}) = \min f(\boldsymbol{x})$$

taken over all integral $\boldsymbol{x} \equiv \boldsymbol{l} \pmod{2\Gamma}$ and this minimum is attained only at $\boldsymbol{x} = \pm \boldsymbol{l}$; we call these points the 'modulo 2Γ minima' of f. Always $\sigma \leqq 2^n - 1$, and in general $\sigma = 2^n - 1$, with one pair of faces for each class of $\Gamma/2\Gamma$ other than $\boldsymbol{0}$; such a form we call an *interior* form. From the convexity of Π, it follows easily that

$$(1.5) \qquad m(f) = \max_{\boldsymbol{x} \in \Pi} f(\boldsymbol{x}) = \max_{\boldsymbol{v}} f(\boldsymbol{v})$$

where the maximum is taken over all vertices $\boldsymbol{v}$ of Π. For an interior form, Π is primitive (i.e. each vertex lies on just n faces) and has $(n+1)!$ vertices. A vertex $\boldsymbol{v}$ for which the maximum in (1.5) is attained is said to be maximal.

Two vertices of Π are *congruent* if they are congruent modulo Γ. Each vertex $\boldsymbol{v}$ has $n+1$ congruent vertices; specifically, if $\boldsymbol{v}$ lies on the n planes

$$(1.6) \qquad f(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{l}_i) \qquad (i = 1, \cdots, n)$$

we say that $\boldsymbol{v}$ is determined by the simplex $[\boldsymbol{l}_0, \boldsymbol{l}_1, \cdots, \boldsymbol{l}_n]$ with vertices $\boldsymbol{l}_0 = \boldsymbol{0}, \boldsymbol{l}_1, \cdots, \boldsymbol{l}_n$. Then, for each $j = 0, 1, \cdots, n$, there is a congruent vertex $\boldsymbol{v}_j = \boldsymbol{v} - \boldsymbol{l}_j$ of Π determined by the simplex

$$[\boldsymbol{l}_0 - \boldsymbol{l}_j, \boldsymbol{l}_1 - \boldsymbol{l}_j, \cdots, \boldsymbol{l}_n - \boldsymbol{l}_j].$$

Moreover, f takes the same value at congruent vertices, so that all are maximal if any one is.

If $\boldsymbol{v}$ is determined by the simplex $[\boldsymbol{l}_0, \boldsymbol{l}_1, \cdots, \boldsymbol{l}_n]$, let $(c_0, c_1, \cdots, c_n)$ be its barycentric coordinates with respect to this simplex, so that

$$(1.7) \qquad \boldsymbol{v} = \sum_{i=0}^{n} c_i \boldsymbol{l}_i, \qquad \sum_{i=0}^{n} c_i = 1.$$

We then have (Barnes and Dickson [1]):

THEOREM 1. *If $f(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x}$ is an interior form and $F(\boldsymbol{x}) = \boldsymbol{x}'A^{-1}\boldsymbol{x}$ is its inverse, then f is extreme if and only if F is expressible in the form*

$$(1.8)\qquad F(\boldsymbol{x}) = \sum_{\boldsymbol{v}} \lambda_{\boldsymbol{v}} \left[\sum_{i=1}^{n} c_i(\boldsymbol{l}_i'\boldsymbol{x})^2 - (\boldsymbol{v}'\boldsymbol{x})^2 \right]$$

where the outer sum is over all maximal vertices of Π,

$$(1.9)\qquad \lambda_{\boldsymbol{v}} \geqq 0 \qquad \textit{for all} \quad \boldsymbol{v},$$

and, for each $\boldsymbol{v}$, *the* c_i *are defined by* (1.7).

We note also the corollary, that in (1.8) it suffices to include in the sum only one vertex $\boldsymbol{v}$ from any set of $n+1$ congruent maximal vertices.

In § 2 we describe the from $\phi_2(\boldsymbol{x})$ and prove that it is extreme for all odd $n \geqq 5$. Finally in §3 we examine the density of the corresponding lattice-covering of space, and outline the genesis of the form and its relation to Voronoi's dissection of the space of positive definite quadratic forms into polyhedral cones.

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THEOREM 2. *For odd* $n \geqq 5$, *the form*

$$(2.1)\qquad \phi_2(\boldsymbol{x}) = \sum_{i=2}^{n} x_i^2 + \sum_{2\leqq i<j\leqq n} (x_i - x_j)^2 + b\gamma(\boldsymbol{x})$$

is extreme, where

$$4\gamma(x) = n(x_1^2 + x_2^2) + 4\sum_{i=3}^{n} x_i^2 + 2(n-2)x_1x_2 - 4(x_1 + x_2)\sum_{j=3}^{n} x_j,$$

and $b = b_n$ *is the positive root of*

$$(2.2)\qquad 3(n-3)x^2 + (n^2 - 8n - 6)x - 4n(n+1) = 0\,.$$

It is convenient to transform the coordinates by

$$\boldsymbol{x} = T\boldsymbol{y}: \qquad \begin{aligned} x_1 &= y_1 + y_2 \\ x_2 &= y_1 - y_2 \\ x_i &= y_1 - y_i \qquad (3 \leqq i \leqq n) \end{aligned}$$

and consider the form

$$(2.3)\qquad g(\boldsymbol{y}) = \phi_2(T\boldsymbol{y}) = \sum_{1\leqq i<j\leqq n} (y_i - y_j)^2 + b\sum_{i=1}^{n} y_i^2\,,$$

with

$$(2.4)\qquad d(g) = d(\phi_2)(\det T)^2 = b(n+b)^{n-1}\,.$$

If $\Lambda = T^{-1}\Gamma = \{T^{-1}\boldsymbol{x} \mid \boldsymbol{x}\in\Gamma\}$, then it is easily seen that $\boldsymbol{y}\in\Lambda$ if and only if

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(2.5) $$2y \in \Gamma \quad \text{and} \quad 2y_1 \equiv 2y_2 \equiv \cdots \equiv 2y_n \pmod 2).$$

Thus integral $\boldsymbol{x}$ correspond to points $\boldsymbol{y}$ satisfying (2.5).

The Voronoi polytope Π_{ϕ_2} under this transformation becomes $T^{-1}\Pi_{\phi_2} = \Pi$, which is not to be confused with the polytope Π of the form g. The mod 2Γ minima of ϕ_2 become the mod 2Λ of g, that is, a given $\boldsymbol{m} \neq \boldsymbol{0}$ belongs to the set $\{\pm \boldsymbol{m}_1, \cdots, \pm \boldsymbol{m}_\sigma\}$ of mod 2Λ minima of g if and only if $g(\boldsymbol{m}) = \min g(\boldsymbol{y})$ taken over all $\boldsymbol{y} \equiv \boldsymbol{m} \pmod{2\Lambda}$ and this minimum is attained only at $\boldsymbol{y} = \pm \boldsymbol{m}$. Thus $\boldsymbol{m} = T^{-1}\boldsymbol{l}$, where $\boldsymbol{l}$ is a mod 2Γ minimum of ϕ_2.

LEMMA 1. *Let* e_1 *denote the* ith *unit vector and set* $m = \frac{1}{2}(n-1)$. *The set of* mod 2Λ *minima of* g *consists of the points*

(2.6) $$\tfrac{1}{2}(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \cdots \pm \boldsymbol{e}_n)$$

(2.7) $$\pm(\boldsymbol{e}_{i_1} + \boldsymbol{e}_{i_2} + \cdots + \boldsymbol{e}_{i_r}) \qquad (1 \leqq i_1 < i_2 \cdots < i_r \leqq n,\ 1 \leqq r \leqq m).$$

PROOF. For points of the form (2.6) it is sufficient, after applying a suitable permutation of coordinates, to consider a point of the form

$$\boldsymbol{m} = \tfrac{1}{2}(\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_r - \boldsymbol{e}_{r+1} - \cdots - \boldsymbol{e}_n) = \tfrac{1}{2}(1, \cdots, 1, -1, \cdots, -1).$$

From (2.5), any point congruent to $\boldsymbol{m}$ mod 2Λ is of the form

(2.8) $$\boldsymbol{y} = \tfrac{1}{2}(1 + 4a_1, \cdots, 1 + 4a_r, -1 + 4a_{r+1}, \cdots, -1 + 4a_n)$$

or

(2.9) $$\boldsymbol{y} = \tfrac{1}{2}(-1 + 4a_1, \cdots, -1 + 4a_r, 1 + 4a_{r+1}, \cdots, 1 + 4a_n),$$

with integral $a_1, \cdots, a_n$. Over points of the form (2.8), $\sum y_i^2$ attains its minimum only when $a_1 = \cdots = a_n = 0$ and each $(y_i - y_j)^2$ also takes its minimun value there, so that $g(\boldsymbol{y})$ attains its minimum only at $\boldsymbol{m}$. Similarly, over points of the form (2.9), $g(\boldsymbol{y})$ attains its minimum only at $-\boldsymbol{m}$. Thus all points (2.6) are mod 2Λ minima of g.

For points (2.7) it suffices to consider

$$\boldsymbol{m} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_r = (1, \cdots, 1, 0, \cdots, 0),$$

where $1 \leqq r \leqq m$. From (2.5), any point congruent to $\boldsymbol{m}$ mod 2Λ is of the form

(2.10) $$\boldsymbol{y} = (1 + 2a_1, \cdots, 1 + 2a_r, 2a_{r+1}, \cdots, 2a_n)$$

(2.11) $$\boldsymbol{y} = (2a_1, \cdots, 2a_r, 2a_{r+1} - 1, \cdots, 2a_n - 1),$$

with integral $a_1, \cdots, a_n$. Points of the form (2.10) have $y_i - y_j$ even if $i \leqq r$ and $j \leqq r$ or $i > r$ and $j > r$, and odd if $i \leqq r$ and $j > r$; it is therefore easily seen that $\sum (y_i - y_j)^2$ attains its minimum $r(n-r)$ only when $a_1 = \cdots = a_n = 0$ or $a_1 = \cdots = a_r = -1$ and $a_{r+1} = \cdots = a_n = 0$. $\sum y_i^2$ also attains its minimum r at these points and so $g(\boldsymbol{y})$ attains its minimum $r(n-r) + rb$ over points (2.10) only at $\pm \boldsymbol{m}$. Similarly $g(\boldsymbol{y})$ attains its minimum $r(n-r)+(n-r)b$ over points (2.11) only at $\pm(0, \cdots 0, 1, \cdots, 1)$. Since $1 \leqq r \leqq \frac{1}{2}(n-1)$, $r(n-r)+(n-r)b > r(n-r) + rb$ and so the minimum of $g(\boldsymbol{y})$ over points congruent to $\boldsymbol{m}$ mod 2Λ is attained only at $\pm \boldsymbol{m}$; hence points (2.7) are mod 2Λ minima of g.

There are 2^{n-1} pairs $\pm \boldsymbol{m}$ of the form (2.6) and $2^{n-1}-1$ pairs of the form (2.7), each from a different class of $\Lambda/2\Lambda$. It follows that (2.6) and (2.7) give all the mod 2Λ minima of g.

LEMMA 2. *The set S*

$$\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_r \qquad (1 \leqq r \leqq m) \tag{2.12}$$

$$\tfrac{1}{2}(\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_m + \boldsymbol{e}_{m+1} + \cdots + \boldsymbol{e}_{m+s} - \boldsymbol{e}_{m+s+1} - \cdots - \boldsymbol{e}_n) \qquad (0 \leqq s \leqq m),$$

of mod 2Λ minima of g determines a vertex

$$\begin{aligned} v = \frac{1}{4(n+b)}(&2b+4m+2m-1, 2b+4(m-1)+2m-1, \cdots, \\ &2b+4+2m-1, 2m-1, 2m-1-4, \cdots, \\ &2m-1-4(m-1), 2m-1-4m-b) \end{aligned} \tag{2.13}$$

of Π, and every vertex of Π is equivalent to $\boldsymbol{v}$ or a vertex congruent to $\boldsymbol{v}$.

PROOF. By I we denote the unit matrix and by K the matrix with components $k_{ii} = 0$ $(1 \leqq i \leqq n)$ and $k_{ij} = 1$ $(1 \leqq i, j \leqq n, i \neq j)$, so that

$$aI + bK = \begin{bmatrix} a & b & \cdot & \cdot & \cdot & b \\ b & a & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & b \\ b & \cdot & \cdot & \cdot & b & a \end{bmatrix}.$$

With this notation, the matrix A of g is $(n-1+b)I - K$ and so

$$A^{-1} = \frac{1}{b(n+b)}((1+b)I + K). \tag{2.14}$$

Π is determined by the inequalities

$$g(\boldsymbol{y}) \leqq g(\boldsymbol{y} - \boldsymbol{m}) \qquad (\boldsymbol{m} \in S),$$

that is

$$2\boldsymbol{m}'A\boldsymbol{y} \leqq g(\boldsymbol{m}) \qquad (\boldsymbol{m} \in S).$$

Setting, for convenience,

$$\boldsymbol{z} = A\boldsymbol{y} \tag{2.15}$$

then, since $\boldsymbol{m} \in S$ implies $-\boldsymbol{m} \in S$, the inequalities are

$$2|\boldsymbol{m}'\boldsymbol{z}| \leqq g(\boldsymbol{m}) \qquad (\boldsymbol{m} \in S),$$

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that is

$$(2.16)\quad \begin{aligned} 2|z_{i_1}+\cdots+z_{i_r}| &\leqq r(n-r)+rb \qquad (1\leqq i_1<\cdots<i_r\leqq n, 1\leqq r\leqq m)\\ |z_1\pm\cdots\pm z_n| &\leqq k(n-k)+\frac{n}{b}\,b \qquad (k=\text{number of minus signs}). \end{aligned}$$

The n faces of Π determined by (2.12) are

$$(2.17)\quad \begin{aligned} &2(z_1+\cdots+z_r)=r(n-r)+rb \qquad (1\leqq r\leqq m)\\ &(z_1+\cdots+z_m+\cdots+z_{m+s}-z_{m+s+1}-\cdots-z_n)\\ &\qquad\qquad =(m+1-s)(m+s)+\frac{n}{4}b \qquad (0\leqq s\leqq m). \end{aligned}$$

These intersect at the point

$$z=\tfrac{1}{2}(2m+b, 2m-2+b, \cdots, 2+b, 0, -2, \cdots, -(2m-2), -2m-\tfrac{1}{2}b),$$

which is easily seen to satisfy the inequalities (2.16) with equality only in (2.17). Hence z is a vertex of the region defined by (2.16), and $v=A^{-1}z$ is a vertex of Π.

Since, by (2.15) and (2.14),

$$v=\frac{1}{b(n+b)}((1+b)I+K)z \quad \text{and} \quad \sum_{i=1}^{n} z_i=\tfrac{1}{4}(2m-1)b,$$

$$\begin{aligned} v_i &= \frac{1}{b(n+b)}(bz_i+z_1+\cdots+z_n)\\ &= \frac{1}{b(n+b)}(bz_i+\tfrac{1}{4}(2m-1)b)\\ &= \frac{1}{n+b}(z_i+\tfrac{1}{4}(2m-1)), \end{aligned}$$

and the expression (2.13) results.

Permuting suffixes in (2.12) gives $n!$ distinct vertices of Π equivalent to v. The n vertices congruent to v (other than v) are the points $v-m$ with $m\in S$, which are easily seen to be distinct from these. Hence we have $(n+1)!$ distinct vertices of Π and therefore all vertices of Π.

PROOF OF THEOREM. We shall prove the equivalent result, that $g(v)$ is extreme over Λ, by using the Theorem 1.

Writing, as in (1.7), $v=\sum_{i=1}^{n} c_i m_i$, we deduce from (2.12) that

$$\begin{aligned} c_i &= v_i-v_{i+1} \qquad (1\leqq i\leqq m-1)\\ c_m &= v_m+v_n \end{aligned}$$

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$$c_{m+1} = -v_{m+1} - v_n$$
$$c_{m+j+1} = v_{m+j} - v_{m+j+1} \qquad (1 \leqq j \leqq m)$$

whence

$$c = \frac{1}{4(n+b)}(4, \cdots, 4, 2+b, 4+b, 4, \cdots, 4, 2+b). \tag{2.18}$$

Also write

$$\begin{aligned} \psi(y) &= \sum_{i=1}^{n} c_i(m_i' y)^2 - (v' y)^2 \\ &= \alpha_{11} y_1^2 + \cdots + \alpha_{nn} y_n^2 + 2\alpha_{12} y_1 y_2 + \cdots + 2\alpha_{n-1,n} y_{n-1} y_n. \end{aligned} \tag{2.19}$$

Since all vertices are equivalent or congruent, all vertices are maximal and the sum (1.8) is over all vertices. By the Corollary to Theorem 1 it suffices to sum over all vertices equivalent to v, i.e. to sum over all permutations of coordinates of v.

Summing (2.19) over all permutations of coordinates and counting the number of times terms appear, we obtain

$$\begin{aligned} \sum_{\sigma \in S_n} \sigma(\psi(y)) &= (n-1)! \,(\sum_{i=1}^{n} \alpha_{ii})(y_1^2 + \cdots + y_n^2) \\ &+ 2(n-2)! \,(\sum_{i<j} \alpha_{ij})(2y_1 y_2 + \cdots + 2y_{n-1} y_n) \\ &= 2(n-2)! \{\tfrac{1}{2}(n-1)(\sum_i \alpha_{ii})(y_1^2 + \cdots + y_n^2) \\ &+ (\sum_{i<j} \alpha_{ij})(2y_1 y_2 + \cdots + 2y_{n-1} y_n)\}. \end{aligned} \tag{2.20}$$

We note that

$$\psi(1, \cdots, 1) = \sum_i \alpha_{ii} + 2 \sum_{i<j} \alpha_{ij},$$

so that

$$2 \sum_{i<j} \alpha_{ij} = \psi(1, \cdots, 1) - \sum_i \alpha_{ii}. \tag{2.21}$$

Hence, from (2.20), $\sum_{\sigma \in S_n} \sigma(\psi(y))$ has matrix

$$B = 2(n-2)! \{\tfrac{1}{2}(n-1)(\sum_i \alpha_{ii}) I + \tfrac{1}{2}(\psi(1, \cdots, 1) - \sum_i \alpha_{ii}) K\}.$$

Substituting (2.13) and (2.18) into (2.19) gives

$$\begin{aligned} 48(n+b)^2 \psi(1, \cdots, 1) &= b^2(12m^2 + 3) + b(32m^3 + 24m^2 + 16m + 6) \\ &+ 16m^4 + 32m^3 + 32m^2 + 16m + 3, \end{aligned}$$

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$$C = 48(n+b)^2 \sum_{i=1}^{n} \alpha_{ii} = b^2(12m+3) + b(48m^2+36m+6) + 40m^3+60m^2+26m+3,$$

and, by (2.21),

$$\begin{aligned} 2D &= 48(n+b)^2 2(\alpha_{12} + \cdots + \alpha_{n-1,n}) \\ &= b^2(12m^2-12m) + b(32m^3-24m^2-20m) + 16m^4 - 8m^3 - 28m^2 - 10m, \end{aligned}$$

so that B is a multiple of $mCI + DK$.

If $mC = (1+b)D$ then B is a multiple of $(1+b)I + K$ and so by (2.14) is a multiple of A^{-1}. Thus positive constants

$$\lambda_v = \frac{24(n+b)}{(n-2)!Db}$$

can be chosen so that (1.8), with the sum taken over one vertex from each congruence class, is satisfied, and the theorem is proved.

The relation $mC = (1+b)D$ reduces to

$$b^3(6m-6) + b^2(16m^2-18m-19) + b(8m^3-36m^2-62m-21) - (32m^3+64m^2+40m+8) = 0,$$

that is,

(2.22) $(b+2m+1)(b^2(6m-6) + b(4m^2-12m-13) - (16m^2+24m+8)) = 0.$

b is the positive root of (2.2), which can be written in terms of $m = \frac{1}{2}(n-1)$ as

$$x^2(6m-6) + x(4m^2-12m-13) - (16m^2+24m+8) = 0,$$

and so the relation (2.22) is satisfied.

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Since all vertices of Π are maximal, $m(\varphi_2) = g(\boldsymbol{v})$, where $\boldsymbol{v}$ is given by (2.13); a staightforward computation yields

(3.1) $$m(\varphi_2) = \frac{1}{48(n+b)}\{(32m^3+48m^2+16m) + (36m^2+36m+3)b + (12m+3)b^2\}.$$

From (2.4), $d(\varphi_2) = b(n+b)^{n-1}$ so that $\mu(\varphi_2)/d^{1/n}(\varphi_2)$ may now be calculated. We append a short table, in which we list for comparison the value of μ for the 'principal form' φ_0 defined in (1.1), namely

$$\mu(\varphi_0) = \frac{1}{12} n(n+2)(n+1)^{-1+1/n}.$$

n	b_n	$\mu(\varphi_2)$	$\mu(\varphi_0)$
5	6.552	0.7093	0.6956
7	4.896	0.8981	0.8832
9	4.390	1.0797	1.0655
11	4.162	1.2578	1.2447

As $n \to \infty$, $b_n \to 4$ and $\mu(\varphi_2) \sim \dfrac{1}{12}n$.

The polyhedral cone (see [6], [1, p. 117]), which we denote by Δ_2, of which φ_2 is (apart from multiples) the unique extreme form, has the property that every interior form f has

(i) the set of mod 2 minima specified in Lemma 1;

(ii) a polytope $\Pi(f)$ all of whose vertices are determined by the same simplices as those of $\Pi(\varphi_2)$ (as specified by Lemma 2). It may be shown that, explicitly, Δ_2 is the set of forms $f(x)$ expressible as

$$\sum_{i<j} \lambda_{ij}(y_i - y_j)^2 + \sum_i \mu_i\chi_i(y) + \sum_i v_i\omega_i(y),$$

with

$$\lambda_{ij} \geqq 0 \qquad (1 \leqq i < j \leqq n), \qquad \mu_i \geqq 0, \qquad v_i \geqq 0 \qquad (1 \leqq i \leqq n);$$

where

$$\chi_i(y) = \sum_{j=1}^{n} y_j^2 - y_i^2, \qquad \omega_i(y) = \sum_{j=1}^{n} y_j^2 + y_i^2$$

and the variables x, y are related by the transformation T of §2.

The group of automorphisms of Δ_2 is the full symmetric group $\mathscr{S}_n$, induced by all permutations of the variables $y_1, \cdots, y_n$; and the forms of Δ_2 which are invariant under this group are those of the shape

$$\lambda \sum_{i<j} (y_i - y_j)^2 + \alpha \sum_i y_i^2 . \tag{3.2}$$

According to the theorem of Dickson [4], an interior form f of Δ_2 is extreme if and only if if it is of the shape (3.2) (with $\lambda > 0, \alpha > 0$), and maximizes $\mu(f)$ over the set of such forms. It was this result which led to the consideration of a form of the type (2.3) and the determination of the equation (2.2) satisfied by b_n.

The cone Δ_2 is also of independent interest in providing the first example of a Voronoi cone with more than $\frac{1}{2}n(n+1)$ edges.

References

[1] E. S. Barnes and T. J. Dickson, 'Extreme coverings of n-space by spheres', *J. Aust. Math. Soc.* 7 (1967), 115–127. Corrigendum, *ibid* 8 (1968), 638-640.

[2] M. N. Bleicher, 'Lattice coverings of n-space by spheres', *Can. J. Math.* 14 (1962), 632–650.

[3] T. J. Dickson, 'The extreme coverings of 4-space by spheres', *J. Aust. Math. Soc.* 7 (1967), 490–496.

[4] T. J. Dickson, 'A sufficient condition for an extreme covering of n-space by spheres', *J. Aust. Math. Soc.* 8 (1968), 56–62.

[5] G. Voronoi, 'Recherches sur les paralléloèdres primitifs' (Part 1), *J. reine angew. Math.* 134 (1908), 198–287.

[6] G. Voronoi, 'Recherches sur les paralléloèdres primitifs' (Part 2), *J. reine angew. Math.* 136 (1909), 67–181.

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