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Higher Order Tangents to Analytic Varieties along Curves. II

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Abstract. Let V be an analytic variety in some open set in $\mathbb{C}^n$. For a real analytic curve γ with $\gamma(0) = 0$ and $d \geq 1$, define $V_t = t^{-d}(V - \gamma(t))$. It was shown in a previous paper that the currents of integration over V_t converge to a limit current whose support $T_{\gamma,d}V$ is an algebraic variety as t tends to zero. Here, it is shown that the canonical defining function of the limit current is the suitably normalized limit of the canonical defining functions of the V_t. As a corollary, it is shown that $T_{\gamma,d}V$ is either inhomogeneous or coincides with $T_{\gamma,\delta}V$ for all δ in some neighborhood of d. As another application it is shown that for surfaces only a finite number of curves lead to limit varieties that are interesting for the investigation of Phragmén-Lindelöf conditions. Corresponding results for limit varieties $T_{\sigma,\delta}W$ of algebraic varieties W along real analytic curves tending to infinity are derived by a reduction to the local case.

1 Introduction

This paper continues the study in [4, 6] of limits obtained by blowing up the part of an analytic (resp. algebraic) variety that lies in an algebraic conoid with opening proportional to t^d about an analytic curve γ with tangent vector ξ_0 satisfying $|\xi_0| = 1$. Such a curve γ in $\mathbb{C}^n$ always admits a Puiseux series expansion. If we study local analytic varietes at the origin, then for some natural number q and $\delta > 0$, γ has an expansion of the form

$$\gamma(t) = \sum_{j=q}^{\infty} \xi_j t^{j/q}, \quad 0 < t \leq \delta. \tag{1.1}$$

If we study algebraic varieties near infinity, then for some $R > 0$, γ has the form

$$\gamma(t) = \sum_{j=-\infty}^{q} \xi_j t^{j/q}, \quad R \leq t < \infty.$$

Such curves will be called *simple curves*. The results in the two cases are completely analogous, so for ease of exposition, we will concentrate first on the case of local analytic varieties at the origin which was studied in [4], in which case the parameter d just mentioned satisfies $d \geq 1$. The other case will be discussed in Section 5.

For V an analytic variety of pure dimension $k \geq 1$ defined in a neighborhood of the origin in $\mathbb{C}^n$, the principal object studied in [4] was the family of varieties,

$$V_{\gamma,d,t} := \{w : \gamma(t) + t^d w \in V\}, \quad d \geq 1,\ 0 < t \leq \delta, \tag{1.2}$$

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and its limit, which is defined as follows. Considering the varieties $V_{\gamma,d,t}$ as currents or holomorphic k-chains $[V_{\gamma,d,t}]$, there exists a limiting current or holomorphic k-chain

$$T_{\gamma,d}[V] = \lim_{t \to 0+} [V_{\gamma,d,t}]. \tag{1.3}$$

The support of $T_{\gamma,d}[V]$ is denoted by $T_{\gamma,d}V$ and is called the limit variety of the family $(V_{\gamma,d,t})_t$. In fact, $T_{\gamma,d}V$ is the limit of the sets $V_{\gamma,d,t}$ as t tends to zero in the convergence of closed sets in $\mathbb{C}^n$, *e.g.*, in the Hausdorff metric or in the sense of Meise, Taylor, and Vogt [12, 4.3].

The concept of limit variety extends the notion of the tangent cone T_0V to V at 0, which is the case $\gamma(t) \equiv 0$, $d = 1$. For $d > 1$, limit varieties were shown to exist and to be algebraic varieties in $\mathbb{C}^n$ that are invariant with respect to all translations in the direction of the tangent vector to γ at 0 [4, Theorem 3.2, Lemma 3.6, Proposition 4.1]. As was also pointed out there, the use of these varieties is crucial in the classification of the analytic varieties whose plurisubharmonic functions satisfy estimates in the spirit of the classical Phragmén-Lindelöf estimates for subharmonic functions. References [5, 7] contain applications of this work.

Our previous work proved the existence of the limit currents by studying the limiting behavior of the Whitney canonical defining functions of the varieties, and the work of this paper continues this program. Our main convergence result showed that the canonical defining functions of the varieties $V_{\gamma,d,t}$ converge as $t \to 0$ and that the limit function contains enough information to characterize the limit current $T_{\gamma,d}[V]$. However, we were unable to answer the following basic question (except for varieties of codimension 1).

> Is the canonical defining function of the limit current $T_{\gamma,d}[V]$ equal to the suitably normalized limit of the canonical defining functions of the varieties $V_{\gamma,d,t}$?

For the case $d = 1$, that is, the ordinary tangent cone at the origin, this is true (see Chirka [8, 16.1, Proposition 2]. In this paper we will show that the answer to this question is always "yes" (Theorem 3.1) and then derive consequences from it (Corollary 3.3, Corollary 3.5, and Theorem 4.6) which are important for the characterization of the algebraic surfaces in $\mathbb{C}^n$ satisfying the Phragmén–Lindelöf conditions in [7].

As mentioned above, there are, in fact, two closely related limiting cases to be treated: the local case of the previous discussion and its analogue for the limiting behavior of algebraic varieties along analytic curves which go to infinity, treated in [6]. There are obvious analogues of the theorems in the two cases, and even a natural transformation that relates them, and we give the relationship in Section 5. In this paper, we will first give proofs for the local case, which is somewhat easier to visualize. The results for algebraic varieties studied near infinity are then deduced from the local case by the transformation mentioned above.

Let us also mention here that we have made a careful distinction between the current or holomorphic k-chain, $T_{\gamma,d}[V]$, and the algebraic variety $T_{\gamma,d}V$ which is its support. For our applications, it is important to keep track of the multiplicities of different branches in the limit variety which we do by viewing them and their limits as currents (or equivalently, as holomorphic chains). We will assume that the original analytic variety V is just a set and hence can be identified with its current of integration $[V]$. The same applies to the family of varieties $(V_{\gamma,d,t})_t$. However, the limit current $T_{\gamma,d}[V]$ may have irreducible factors of multiplicity greater than one.

We conclude this introduction with some algebraic questions about limit varieties which seem interesting but which we are unable to answer.

(i) Are there only finitely many limit varieties up to a suitable class of isomorphisms?

(ii) In terms of generators of the ideal of V,

  (a) how can one find generators for the ideals of $T_{\gamma,d}V$ when the degree of $T_{\gamma,d}V$ is greater than 1?

  (b) how can one find all the γ and critical values d_j where $T_{\gamma,d_j}V$ has the degree greater than 1?

An algorithmic approach that might give some information about these questions is part of the proof of Theorem 4.6.

2 Notation

In this section we recall the notation and some results from [4] which are needed to state and prove the main result of this paper, Theorem 3.1, which answers the question formulated in the introduction.

2.1 Varieties and Coordinates

Let V be an analytic variety in a neighborhood of the origin in $\mathbb{C}^n$ of pure dimension k containing the origin. Then we denote by V_{reg} (resp. V_{sing}) the set of all regular (resp. singular) points in V. From Chirka [8] we recall the following definitions.

Let $L \subset \mathbb{C}^n$ be an affine subspace of dimension $n-k$ and z an isolated point of $V \cap L$. Then there is a neighborhood U of z such that the projection $\pi_L \colon U \cap V \to \pi_L(U \cap L) \subset L^{\perp}$ along L is an analytic cover. Its sheet number in z is denoted by $\mu_z(\pi_L|_V)$. The minimum of the sheet numbers $\mu_z(\pi_L|_V)$ when L ranges over all $(n-k)$-dimensional affine subspaces for which z is an isolated point of $V \cap L$ is the *multiplicity* $\mu(V,z)$ of V at z.

If $W = \sum_{j=1}^m n_j[V_j]$ is a holomorphic k-chain, then $\mu(W,z) := \sum_{j=1}^m n_j\, \mu(V_j,z)$. We call $\mu(W,z)$ the *multiplicity* of the holomorphic chain W in z.

If V is a purely k-dimensional algebraic subset or a holomorphic k-chain in $\mathbb{C}^n$ with algebraic support and $L \subset \mathbb{C}^n$ is an affine $(n-k)$-dimensional subspace such that $V \cap L$ is finite and such that the projective closures of V and of L do not have points at infinity in common, then $\sum_{z \in V \cap L} \mu(V,z)$ is the *degree* of V. It does not depend on L.

Next we fix a projection π of rank k that is proper on V and transverse to V at the origin. That is, we have a choice of coordinates $z = (z'', z') \in \mathbb{C}^{n-k} \times \mathbb{C}^k$ such that $\pi(z'', z') = z'$ and an open set $0 \in \mathcal{U} \subset \mathbb{C}^k$ such that $\pi \colon V \to \mathcal{U}$ is proper with discrete fibers and, further, that for some $K > 0$

$$|z| \le K|\pi(z)|, \quad z \in V. \tag{2.1}$$

If B is the branch locus of π, then B and $\pi(B)$ are analytic varieties of dimension at most $k-1$ and $\pi\colon V \setminus B \to \mathcal{U} \setminus \pi(B)$ is a covering map. The number of points in a fiber over $z' \in \mathcal{U} \setminus \pi(B)$ is $m := \mu(V, 0)$ because of (2.1), so we can write

$$\pi^{-1}(z') = \{(\alpha_i(z'), z') : 1 \le i \le m\},$$

where the $\alpha_i(z') = \alpha_i(z'; V)$ are all distinct. We will also use the same notation for $z' \in \mathcal{U} \cap \pi(B)$ by repeating each $\alpha_i(z')$ as many times as indicated by the multiplicity $\mu(V, z)$, where $z := (\alpha_i(z'), z')$.

Following Whitney [14, Appendix V, Section 7], we introduce canonical defining functions for analytic varieties. In doing so we use the dot product on $\mathbb{C}^\nu$, defined by $\langle u, w\rangle := \sum_{j=1}^{\nu} u_j w_j$.

2.2 Canonical Defining Functions

Let V be an analytic variety as in Section 2.1. Using the notation introduced in Section 2.1, the *canonical defining function for V* is defined as

$$P(z, \xi; V, \pi) = \prod_{i=1}^{m} \langle z'' - \alpha_i(z'), \xi\rangle\,. \tag{2.2}$$

We will write $P(z, \xi) = P(z, \xi; V) = P(z, \xi; V, \pi)$ when the missing data are clear from the context. A point z belongs to V if and only if $P(z, \xi) = 0$ for all $\xi \in \mathbb{C}^{n-k}$. Equivalently, one can expand P as a homogeneous polynomial in ξ,

$$P(z, \xi) = \sum_{|\beta|=m} P_\beta(z)\xi^\beta,$$

and then $z \in V$ if and only if $P_\beta(z) = 0$ for all $|\beta| = m$.

Note that P is a polynomial of degree m in z'' and a homogeneous polynomial of degree m in $\xi \in \mathbb{C}^{n-k}$. It is defined at first for $z' \in \mathcal{U} \setminus \pi(B)$ but extends, by the Riemann removable singularity theorem, to be analytic on all of $\mathbb{C}^{n-k} \times \mathcal{U} \times \mathbb{C}^{n-k}$. With the convention made about counting the points $\alpha_i(z')$ with multiplicity when $z' \in \pi(B)$, the formula (2.2) is still valid (see [4, (3.8)], and the discussion that precedes it). The local multiplicity of V at a point (α, z') is equal to the number of times the factor $\langle z'' - \alpha, \xi\rangle$ appears as a factor in $P(z, \xi)$ when the projection π is transverse to V at the given point (otherwise, the multiplicity is smaller).

We will also need canonical defining functions for holomorphic k-chains. If $W = \sum_{j=1}^{p} n_j[W_j]$ is a holomorphic k-chain, *i.e.,* the W_j are the irreducible components of $\operatorname{Supp} W$ in a sufficiently small neighborhood of the origin and $n_j \in \mathbb{N}$, then we choose a projection $\pi : \mathbb{C}^{n-k} \times \mathbb{C}^k$ as in Section 2.1, which satisfies the requirements in Section 2.1 for each of the varieties W_j, and we call

$$P(w, \xi; W, \pi) := \prod_{j=1}^{p} P(w, \xi; W_j, \pi)^{n_j}$$

the *canonical defining function* for W. It is a polynomial in w and ξ which has degree $\nu = \sum_{j=1}^{p} n_j \mu(W_j, 0)$.

For a simple curve γ as in (1.1) and $d \geq 1$, define $V_t = V_{\gamma,d,t}$ as in (1.2). Since V_t is a translated and rescaled version of V, the projection mapping π is also proper on V_t, and the defining function of V_t will also be a translated and scaled version of the canonical function for V. That is, if we write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, where $\gamma_2(t) = \pi(\gamma(t))$, then

$$\begin{aligned} (2.3) \qquad P(\gamma(t) + t^d w, \xi) &= \prod_{j=1}^{m} \langle \gamma_1(t) + t^d w'' - \alpha_j(\gamma_2(t) + t^d w'), \xi \rangle \\ &= t^{md} \prod_{j=1}^{m} \langle w'' - \beta_j(w', t), \xi \rangle = t^{md} P(w, \xi; V_t, \pi), \end{aligned}$$

where

$$(2.4) \qquad \beta_j(w', t) = \beta_j(w', t, d) := \frac{\alpha_j(\gamma_2(t) + t^d w') - \gamma_1(t)}{t^d}, \quad 1 \leq j \leq m.$$

Formula (2.3) gives the canonical functions for the varieties V_t with respect to the projection π onto the z' coordinates up to the scale factor t^{md}.

In [4, Lemma 3.10], it was proved that the V_t converge by showing that after a suitable normalization, the canonical defining functions of the varieties V_t converge as $t \to 0$. This requires a further condition on the projection, namely that it is *transverse* (at infinity) to the variety $T_{\gamma,d}V$. This means that the subspace $L = \{(z'', 0) : z'' \in \mathbb{C}^{n-k}\}$ meets any line in the cone of limiting directions $(T_{\gamma,d}V)_h$ (see Definition 3.4) of $T_{\gamma,d}V$ at infinity only at the origin. (Otherwise, the canonical defining function is not even defined.)

Now we come to the main theorem of [4]. We give a formulation that combines [4, Theorem 3.2, Lemmas 3.1 and 3.10] and the remark following [4, Proposition 3.14].

Theorem 2.1 *Let V be an analytic variety in $\mathbb{C}^n$ that is of pure dimension k and contains the origin, let $m := \mu(V, 0)$, and let γ be a simple curve. Then for each $d \geq 1$, the limit in* (1.3) *exists. Assume, furthermore, that the projection π is transverse to V. As in* (2.1)*, fix $\omega_0 \in \mathbb{R}$ so that $\lim_{t\to 0^+} t^{md-\omega_0} P(w, \xi; V_t, \pi)$ exists and does not vanish identically, and set*

$$Z = \{w \in \mathbb{C}^n : \lim_{t \to 0^+} t^{md-\omega_0} P(w, \xi; V_t, \pi) = 0 \textit{ for all } \xi \in \mathbb{C}^{n-k}\}.$$

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If π is transverse to Z, then

$$\lim_{t\to 0^+} t^{md-\omega_0} P(w,\xi;V_t,\pi) = P(w,\xi;T_{\gamma,d}[V],\pi)\Phi(w',\xi).$$

The canonical defining functions of $T_{\gamma,d}[V]$ and Φ are polynomials. Furthermore, the functions $w \mapsto \Phi(w',\xi)$, $\xi \in \mathbb{C}^{n-k}$, have no common zeros. In particular $Z = T_{\gamma,d}V$.

Thus, the function Φ does not affect the common zeros of the right-hand side as ξ varies over $\mathbb{C}^{n-k}$. However, in [4] we were unable to determine if the function Φ actually depends on w' or not. Also, we were unable to explicitly determine the normalizing constant ω_0. The improvement here, which we will show in Theorem 3.1, is that Φ is independent of w', *i.e.*, Φ is a homogenous polynomial of degree $m-\nu$ in ξ where ν is the degree of $T_{\gamma,d}[V]$. In order to derive formulas for $\Phi(\xi)$ and $\omega_0(d)$, we have to recall some more facts from [4].

2.3 Newton Polygon and Critical Values

For an analytic variety V of pure dimension k, defined in some neighborhood of the origin in $\mathbb{C}^n$, and for a simple curve γ as in (1.1), we call a projection π in $\mathbb{C}^n$ *distinguished* for V and γ if it has rank k, is proper on V, transverse to V at the origin, and transverse to $T_{\gamma,d}V$ for each $d \geq 1$. To prove the existence of such projections, we denote by m the multiplicity of V at the origin and by q the number from (1.1) which is associated with γ. Then

$$M_1 := \{j/b \in [1,\infty[\, : jq \in \mathbb{N}, b \in \mathbb{N}, b \leq m\}$$

is a discrete subset of $\mathbb{R}$. Hence $[1,\infty[\,\setminus M_1 = \bigcup_{j\in\mathbb{N}} I_j$, where the sets $I_j \neq \varnothing$ are open intervals in $\mathbb{R}$ satisfying $I_j \cap I_k = \varnothing$ for $j \neq k$. For each $j \in \mathbb{N}$ we choose $\delta_j \in I_j$ and let $M_2 := \{\delta_j : j \in \mathbb{N}\}$ and $M_0 := M_1 \cup M_2$. By Chirka [8, 38, Proof of Corollary 2], we can choose a projection π in $\mathbb{C}^n$ of rank k which is proper on V and is transverse to V at the origin and to $T_{\gamma,d}V$ for each $d \in M_0$. To show that π is distinguished for V and γ, we need some more preparation.

First, we consider the following expansion of the canonical defining function for V:

$$F(w,t,\xi) := P(\gamma(t)+w,\xi) = P(\gamma(t)+w,\xi;V,\pi) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} t^j w^\beta \xi^\alpha, \tag{2.5}$$

where the sum is the power series expansion of the holomorphic function $F(w,s^q,\xi)$ in $s = t^{1/q}$, w, ξ in a neighborhood of the origin. The *support* M of this series is defined as

$$M := \{(j,l) : qj \in \mathbb{N}_0,\ l \in \mathbb{N}_0,\ a_{j,\beta,\alpha} \neq 0 \text{ for some } \beta \text{ with } |\beta| = l \text{ and } |\alpha| = m\}.$$

For $\theta \in \mathbb{R}^2 \setminus \{0\}$ and $b \in \mathbb{R}$, define the closed half plane

$$H_{\theta,b} := \{x \in \mathbb{R}^2 : \langle x,\theta\rangle \geq b\}.$$

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We call it *admissible* if $\theta \in [0, \infty[^2$ and $M \subset H_{\theta,b}$. The *Newton polygon* N is defined as the intersection of all admissible half planes. Note that all vertices of N are elements of M. In particular, if (j, l) is a vertex of N, then $l \in \mathbb{N}_0$ and $l \leq m$ since $(0, m) \in M$, and $j \geq m$ since $z \mapsto P(z, \xi)$ vanishes to order m at $z = 0$. Hence N has at most $m + 1$ vertices and at most m edges between them (plus two unbounded edges), all of which have slope $s \geq -1$. The *critical values* of γ and V are defined as the rational numbers in the sequence $1 = d_1 < d_2 < \cdots < d_p$, that is, an enumeration of

$$\{1\} \cup \Big\{ -\frac{1}{s} : \ s \text{ is the slope of a bounded edge of } N \Big\}.$$

From the definition, it is clear that $p \leq m + 1$ and if $p = m + 1$, then there is no edge with slope -1. It may seem that the definition of critical values might depend on the choice of the canonical defining function, *i.e.,* on the choice of the projection π, but this is not the case as we will show in Remark 3.6 below.

For a given monomial $t^j w^\beta \xi^\alpha$ in the expansion (2.5) and for $d \geq 1$ we define its *d-degree* by $\omega(d) = j + d|\beta|$ (the exponents of ξ are ignored because these terms all have the same degree). Obviously, each monomial $t^j w^\beta \xi^\alpha$ of d-degree ω is d-quasi-homogeneous in the sense that $t^j (t^d w)^\beta \xi^\alpha = t^\omega w^\beta \xi^\alpha$. Collecting all terms in (2.5) which for a given $d \geq 1$ have the same d-degree, we can regroup the series as

$$F(w, t, \xi) = F_{\omega_0}(w, t, \xi) + \sum_{\omega > \omega_0} F_\omega(w, t, \xi), \tag{2.6}$$

where F_ω is the d-quasihomogeneous part of d-degree ω of the series and

$$\omega_0 = \omega_0(d, V, \pi) = \min\{\omega : F_\omega \text{ does not vanish identically}\}. \tag{2.7}$$

Now note that for $t \in B(0, \epsilon) \setminus]-\infty, 0]$ the quasihomogeneity property implies

$$F_\omega(t^d w, t, \xi) = t^\omega F_\omega(w, 1, \xi),$$

so we have $F(t^d w, t, \xi) = t^{\omega_0} F_{\omega_0}(w, 1, \xi) + \sum_{\omega > \omega_0} t^\omega F_\omega(w, 1, \xi)$. Next note that the critical values $d_1 < d_2 < \cdots < d_p$ all are in the set M_1 defined above. Hence for a given interval I_j we can find $2 \leq k < p$ with $I_j \subset]d_k, d_{k+1}[$ or $I_j \subset]1, d_2[$ or $I_j \subset]d_p, \infty[$. In the first case let (j_1, l_1) be the vertex of the Newton polygon N which is the intersection of the two segments in N with slope $-1/d_k$ and $-1/d_{k+1}$. Then the definition of N implies that $\omega_0(d) = j_1 + dl_1$ and $F_{\omega_0(d)}(w, t, \xi) = \sum_{|\beta| = l_1, |\alpha| = m} a_{j_1, \beta, \alpha} t^{j_1} w^\beta \xi^\alpha$ for each $d \in]d_k, d_{k+1}[$. In particular, we have $F_{\omega_0(d)} = F_{\omega_0(\delta_j)}$ for each $d \in I_j$. Hence we get from (2.3)

$$\begin{aligned} \lim_{t \to 0+} t^{md - \omega_0(d)} P(w, \xi; V_t, \pi) &= \lim_{t \to 0+} t^{-\omega_0(d)} P(\gamma(t) + t^d w, \xi) \\ &= F_{\omega_0(d)}(w, 1, \xi) = F_{\omega_0(\delta_j)}(w, 1, \xi), \ d \in I_j. \end{aligned}$$

Hence the set

$$Z(d) := \{w \in \mathbb{C}^n : \lim_{t \to 0+} t^{md - \omega_0(d)} P(w, \xi; V_t, \pi) = 0 \text{ for all } \xi \in \mathbb{C}^{n-k}\}$$

is constant for $d \in I_j$. Since π is transverse to $T_{\gamma,\delta_j}V$ by its choice, it follows from [4, Proposition 3.12], that

$$T_{\gamma,\delta_j}V = \{w \in \mathbb{C}^n : F_{\omega_0(\delta_j)}(w,1,\xi) = 0 \text{ for all } \xi \in \mathbb{C}^{n-k}\} = Z(\delta_j).$$

Hence π is transverse to $Z(d)$ for each $d \in I_j$. By Theorem 2.1, this implies $Z(d) = T_{\gamma,d}V$. Hence π is transverse to $T_{\gamma,d}V$ for each $d \in I_j$. Since we can argue similarly in the remaining two cases, it follows that π is distinguished for V and γ.

3 Main Theorem

In order to state our main theorem, we introduce the following notation. For V and γ as in Section 2.3, assume that the projection π in $\mathbb{C}^n$ defined in Section 2.1 is distinguished for V and γ and let $(d_j)_{1\le j\le p}$ denote the corresponding critical values. Then for $1 \le j \le p$, let I_j denote the vertical parts of the nonzero points in the fiber of $\pi\colon T_{\gamma,d_j}V \to \mathbb{C}^k$ over $w' = 0$. That is,

$$I_j = \{w'' \in \mathbb{C}^{n-k} : (w'',0) \in T_{\gamma,d_j}V,\ w'' \neq 0\}. \tag{3.1}$$

And for each point of this set, let

$$a(j,w'') = \mu((w'',0), T_{\gamma,d_j}[V]) \tag{3.2}$$

denote the local multiplicity of $T_{\gamma,d_j}[V]$ at this point. That is, the number of times the factor associated to the point is repeated in the canonical defining function of $T_{\gamma,d_j}[V]$. Further let

$$\nu_j = \sum_{w''\in I_j} \mu\big((w'',0), T_{\gamma,d_j}[V]\big) \tag{3.3}$$

denote the sum of these multiplicities.

Theorem 3.1 *Let V be an analytic variety of pure dimension k defined in a neighborhood of the origin in $\mathbb{C}^n$ and let γ be a simple curve in $\mathbb{C}^n$. Assume that $0 \in V$, denote by m the local multiplicity of V at zero, and assume that the projection π defined in 2.1 is distinguished for V and γ. Then for each $d \ge 1$ the following assertions hold:*

(i) *The degree $m(d)$ of the current $T_{\gamma,d}[V]$ is $m(d) = m - \sum_{d_j<d} \nu_j$. In particular, $m(d) = m(d_{j+1})$ if $d_j < d < d_{j+1}$, $1 \le j < p$, and $T_{\gamma,d}V$ is empty for $d > d_p$ if and only if $m = \sum_{j=1}^{p} \nu_j$.*

(ii) $\omega_0(d) = md - \sum_{d_j<d} \nu_j(d-d_j) = dm(d) + \sum_{d_j<d} \nu_j d_j$.

(iii) *The function Φ defined in Theorem 2.3 does not depend on w' and is given by*

$$\Phi(\xi) = \prod_{d_j<d} \prod_{w''\in I_j} \langle -w'', \xi\rangle^{a(j,w'')}.$$

Hence its degree is $m - m(d) = \sum_{d_j<d} \nu_j$.

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Proof For $d = 1$, we have $\omega_0 = m$ and $\Phi \equiv 1$. Since $\deg T_0[V] = m$ by Chirka [8, Proposition 1.16], there is nothing to prove. Hence fix $d > 1$, a neighborhood U of zero, $R > 0$, and define Γ' by

$$\Gamma' := \bigcup_{0<t<R} \pi \circ \gamma(t) + t^d U. \tag{3.4}$$

We may assume that R is so small that the number of connected components of $V \cap \pi^{-1}(\Gamma')$ remains the same when R is replaced by any smaller positive number. Now fix $z' \in U$ and set $C := \{\gamma_2(t) + t^d z' : 0 < t < \epsilon\}$, where $\epsilon > 0$ is suitably small. Since V has multiplicity m at the origin, its inverse image $V \cap \pi^{-1}(C)$ consists of m curves

$$C_j := \{(\alpha_j(\gamma_2(t) + t^d z'), \gamma_2(t) + t^d z') : 0 < t < \epsilon\}, \quad 1 \le j \le m.$$

Set $\beta_{d,j}(z', t) := t^{-d}(\alpha_j(\gamma_2(t) + t^d z') - \gamma_1(t))$. The maps $\beta_{d,j}(z', \cdot)$ are the ones that are denoted by $\beta_j(z', \cdot)$ in (2.4) and appear in the product that defines the canonical function $P(z, \xi; V_{\gamma,d,t}, \pi)$. In particular, it is proved in [4, Lemma 3.7], that if $m(d)$ is the degree of the limit current $T_{\gamma,d}[V]$, then the $\beta_{d,j}(z', t)$ can be split into two groups, one containing $m(d)$ of them which remain bounded as $t \to 0$ and the remaining $m - m(d)$ of them that tend to infinity as $t \to 0$. We assume the points have been relabeled so that $\beta_{d,1}, \dots, \beta_{d,m(d)}$ are the ones that admit a finite limit, while for $m(d) < j \le m$ one has $|\beta_{d,j}(z', t)| \to \infty$ as $t \to 0+$. The convergence proof was based on showing the existence of $\omega_0 = \omega_0(d)$ such that when the basic equation (2.3) is multiplied by $t^{-\omega_0}$, the limit of the left-hand side of the equation exists (uniformly for w in compact sets in $\mathbb{C}^n$). This implies

$$\begin{aligned} t^{md-\omega_0} P(z, \xi, V_t, \pi) = &\Big(\prod_{j=1}^{m(d)} \langle z'' - \beta_{d,j}(z', t), \xi\rangle\Big) \\ &\times \Big(t^{md-\omega_0} \prod_{j=m(d)+1}^{m} \langle z'' - \beta_{d,j}(z', t), \xi\rangle\Big). \end{aligned} \tag{3.5}$$

Then note that the first factor converges to the canonical defining function $P(z, \xi; T_{\gamma,d}[V], \pi)$ as $t \to 0+$. The limit of the second factor is an indeterminate form which we will evaluate next.

To do so, fix a curve C_j. Then there are $a \in \mathbb{R}$ and $w'' \in \mathbb{C}^{n-k} \setminus \{0\}$, which *a priori* depend on z', such that

$$\alpha_j(\gamma_2(t) + t^d z') - \gamma_1(t) = t^a(w'' + o(1)) \quad \text{as } t \to 0+. \tag{3.6}$$

Note that $\beta_{d,j}(z', t)$ remains bounded as $t \to 0+$ if and only if $a \ge d$. Thus, $a < d$ if $j > m(d)$. On the other hand, $T_{\gamma,a}[V] = \lim_{t\to 0^+} t^{-a}(V - \gamma(t))$ so (3.6) implies that $(w'', 0) \in T_{\gamma,a}V$. If we assume that $T_{\gamma,a}V$ is homogeneous, then $\mathbb{C}(w'', 0)$ is in $\ker \pi$ and in $T_{\gamma,a}V = (T_{\gamma,a}V)_h$ in contradiction to the fact that π is distinguished for

V and γ. Now note that by [5, Proposition 4.3 (ii)], $T_{\gamma,\delta}V$ is homogeneous or empty whenever δ is not a critical value. Hence a is one of the critical values d_i with $d_i < d$ and $w'' \in I_i$. This shows that for each j with $m(d) < j \le m$ there are $i(j)$ with $d_{i(j)} < d$ and $w''_j \in I_{i(j)}$ satisfying $\lim_{t\to 0+} t^{d-d_{i(j)}}\beta_{d,j}(z',t) = w''_j$.

On the other hand, for each i with $d_i < d$ and $w'' \in I_i$ there are $a(i,w'')$ many branches. If we let $\widetilde{\gamma}(t) := \gamma(t)+t^d(0,z')$, then it follows from [5, Proposition 4.1 (i)], that $T_{\widetilde{\gamma},d_i}[V] = T_{\gamma,d_i}[V]$, since γ and $\widetilde{\gamma}$ are equivalent modulo d_i. Construct $\widetilde{\beta}_{d_i,j}$ as in (2.4), but with γ replaced by $\widetilde{\gamma}$. Then

$$\begin{aligned}\widetilde{\beta}_{d_i,j}(0,t) &= t^{-d_i}(\alpha_j(\widetilde{\gamma}_2(t)) - \widetilde{\gamma}_1(t)) = t^{-d_i}(\alpha_j(\gamma(t)+t^dz') - \gamma_1(t))\\ &= t^{d-d_i}\beta_{d,j}(z',t).\end{aligned}$$

By [5, Lemma 3.7], $\lim_{t\to 0+}(\widetilde{\beta}_{d_i,j}(0,t),c) = (w'',0)$ for exactly $a(i,w'')$ many indices j. Since for these we have

$$\lim_{t\to 0+} t^{d-d_i}\beta_{d,j}(z',t) = \lim_{t\to 0+}\widetilde{\beta}_{d_i,j}(0,t) = w''.$$

Therefore, the right-hand factor of (3.5) can be written as

$$\begin{aligned}t^{md-\omega_0}\prod_{j=m(d)+1}^{m}\langle z''-\beta_{d,j}(z',t),\xi\rangle &= t^{md-\omega_0}\prod_{d_j<d}\prod_{w''\in I_i}\langle z''-t^{d_i-d}\widetilde{\beta}_{d_i,j}(0,t),\xi\rangle\\ &= t^{md-\omega_0-b}\prod_{d_j<d}\prod_{w''\in I_i}\langle t^{d-d_i}z''-\widetilde{\beta}_{d_i,j}(0,t),\xi\rangle,\end{aligned}$$

for $b := \sum_{d_i<d}\nu_i(d-d_i)$. Since the product converges to a function in z' and ξ which is not identically zero, we must have $md-\omega_0 = \sum_{d_i<d}\nu_i(d-d_i)$ and the limit function is equal to

$$\prod_{d_i<d}\prod_{w''\in I_i}\langle -w'',\xi\rangle^{a(i,w'')}.$$

The remaining assertions about the degrees of the terms are now clear from this formula. ∎

To obtain some corollaries from Theorem 3.1 we apply the expansion (2.5) of the function $F(w,t,\xi) = P(\gamma(t)+w,\xi;V,\pi)$ which we have used already in Section 2.3. From there we recall that for each $d \ge 1$ and $\omega_0 = \omega_0(d)$, defined in (2.7), we have

$$\lim_{t\to 0} t^{-\omega_0}P(\gamma(t)+t^dw,\xi;V,\pi) = \lim_{t\to 0}t^{-\omega_0}F(t^dw,t,\xi) = F_{\omega_0}(w,1,\xi), \tag{3.7}$$

where the convergence is uniform on compact subsets of $\mathbb{C}^n \times \mathbb{C}^{n-k}$. From Theorem 3.1 and (2.3) we therefore have the following formula.

Corollary 3.2 $F_{\omega_0}(w,1,\xi) = P(w,\xi;T_{\gamma,d}[V])\Phi(\xi)$.

We already know from [4, Proposition 3.14] that F_{ω_0} and $P(\,\cdot\,,\cdot\,;\,T_{\gamma,d}[V])$ both describe $T_{\gamma,d}[V]$. However, in general, they cannot coincide since the degree of F_{ω_0} in ξ is m while the degree of $P(\,\cdot\,,\cdot\,;\,T_{\gamma,d}[V])$ in ξ is $m(d)$. The role of Φ is to make up for the difference.

The monomials in the function F_{ω_0} are easily computed from the Newton diagram of $F(w,t,\xi)$ described in Section 2.3. Following the notation introduced there, we have $\omega_0 = \omega_0(d) = \min\{j + dl : (j,l) \in M\} = \min\{j + dl : (j,l) \in N\}$, so the equality in this equation occurs only at the extreme vertices of N where the line of slope $-1/d$ hits the vertex. Thus, unless d is the negative reciprocal of the slope of an edge in the Newton polygon, there can be only one exponent $l = |\beta|$ for the w-terms in the monomials in F_{ω_0}, so it must be homogeneous in w. However, when $d > 1$ is a critical value, the edge will contain at least two vertices, so the polynomial $F_{\omega_0}(w,1,\xi)$ is not homogeneous in w. Therefore, neither is the canonical defining function of $T_{\gamma,d}[V]$, so this variety cannot be homogeneous.

Corollary 3.3 *For $d > 1$ the following are equivalent:*

(i) *d is a critical value for V and γ;*

(ii) *$F_{\omega_0}(w,1,\xi)$ is inhomogeneous;*

(iii) *$T_{\gamma,d}V$ is inhomogeneous.*

From the considerations before Corollary 3.3, it is evident (a more detailed discussion can be found in the proof of Lemma 4.7) that for each critical value d_i for γ and V, the lowest degree homogeneous terms in w that appear in $F_{\omega_0(d_i)}(w,1,\xi)$ are the highest degree homogeneous terms in w that appear in $F_{\omega_0(d_{i+1})}(w,1,\xi)$. To formulate this fact in terms of limit currents, we recall the following notation from [2, §2.4]. It is analogous to the definition of the tangent current $T_0[V]$ of an analytic variety V, which can be found in Chirka [8, §11.6]. Recall that the tangent current $T_0[V]$ is defined as $T_0[V] = \lim_{j\to\infty}[jV]$ where $[jV]$ denotes the current of integration over the variety V, scaled by the factor j. If $W = \sum_{j=1}^{m} n_j[W_j]$ is a holomorphic chain, then its tangent current is defined as $T_0W = \sum_{j=1}^{m} n_j T_0[W_j]$.

Definition 3.4 For an algebraic variety W in $\mathbb{C}^n$ of pure dimension $k \ge 1$ we define its *cone of limiting directions at infinity W_h* by

$$W_h := \{r \lim_{j\to\infty} z_j/|z_j| : r \ge 0,\ z_j \in W, |z_j| \to \infty\}.$$

It has been shown [2] that W_h is an algebraic variety. To formulate Corollary 3.5, it is necessary to consider the cone of limiting directions in the sense of currents or holomorphic chains. In this sense, the cone of limiting directions is defined as

$$[W]_h := \lim_{j\to\infty}\left[\frac{1}{j}W\right],$$

where convergence is in the sense of currents. The existence of the limit follows, *e.g.*, from the more general results in [6] (see Theorem 5.1 of the present paper). If $W = \sum_{j=1}^{m} n_j[W_j]$ is a holomorphic chain, then its cone of limiting directions is defined by additivity, *i.e.*, $W_h = \sum_{j=1}^{m} n_j[W_j]_h$.

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Using this definition, we have the following corollary to Theorem 3.1.

Corollary 3.5 *Let $1 = d_1 < \cdots < d_p$ be the critical values for V and γ. If $d_j < d < d_{j+1}$ for some $j < p$, then $T_0(T_{\gamma,d_j}[V]) = T_{\gamma,d}[V] = (T_{\gamma,d_{j+1}}[V])_h$, and if $d > d_p$ and $0 \in T_{\gamma,d_p}V$, then $T_0(T_{\gamma,d_p}[V]) = T_{\gamma,d}[V]$.*

Remark 3.6 From Corollary 3.5 it follows immediately that for each analytic variety in $\mathbb{C}^n$ that is of pure dimension $k \geq 1$ and contains the origin and for each simple curve γ in $\mathbb{C}^n$, the critical values $1 = d_1 < d_2 < \cdots < d_p$ for γ and V as they are defined in Section 2.3 have the following property.

(3.8) For $d_i < d < d_{i+1}$, $1 \leq i < p$, the variety $T_{\gamma,d}V$ is homogeneous while for $d > d_p$, it is either homogeneous or empty.

Moreover, this set is minimal in this respect by Corollary 3.3. Since the limit varieties $T_{\gamma,d}V$ do not depend on any projection, it follows that the critical values do not depend on the choice of the projection π in their definition in Section 2.3.

3.1 Geometric Interpretation of Critical Values

The critical values measure the rate at which branches of V can approach the curve $\gamma(t)$ as $t \to 0$. This was already used in the proof of Theorem 3.1, equation (3.6), which showed that the leading exponent a in the Puiseux series expansion is equal to a critical value. The degree $\nu(d)$ of the limit current $T_{\gamma,d}[V]$ is a measure as $t \to 0$ of how many zeros V has inside conoids of radius Ct^d about $\gamma(t)$. This is a nonincreasing, integer valued function of d which has jumps exactly at the critical values. A critical value $d = d_j > 1$ is one so that the multiplicity of V in conoids about $\gamma(t)$ of opening $\delta t^{d-\epsilon}$ (δ small) is greater than the multiplicity of V in conoids of opening $Kt^{d+\varepsilon}$ (K large).

4 Applications

We introduce the following notions to derive a further result from Theorem 3.1 which was used in an essential way in [7] to characterize the algebraic surfaces on which the analogue of the classical Phragmén–Lindelöf theorem holds.

Definition 4.1 Let V be an analytic variety in $\mathbb{C}^n$ which contains the origin. A set $(\zeta_j, d_j)_{j=1}^l$ in $(\mathbb{C}^n \times \mathbb{Q})^l$ is called a *critical set* for V of length l if the following conditions are satisfied.

(i) $\zeta_1 \in (T_0V)_{\text{sing}}$, $|\zeta_1| = 1$, and $d_1 = 1$.

(ii) For $1 \leq i \leq l$ define $\gamma_i(t) := \sum_{j=1}^i \zeta_j t^{d_j}$. Then for $1 \leq i \leq l-1$ we have

  (a) d_{i+1} is the smallest critical value for V and γ_i larger than d_i,

  (b) ζ_{i+1} is a singular point of $T_{\gamma_i,d_{i+1}}V$ and $\langle \zeta_{i+1}, \overline{\zeta_1} \rangle = 0$.

If $(\zeta_j, d_j)_{j=1}^l$ is a critical set for a given variety V, then there need not exist a critical set of length $l+1$ for which the first l components coincide with the given one. For example, no such critical set exists if there is no critical value for γ_l and V larger than d_l or, if it does, $T_{\gamma_l,d_{l+1}}V$ has no singular points. However, there are also cases where such extensions of arbitrary length exist, but where we do not want to consider them, *e.g.*, we refer to Example 5.17 below. To describe these cases, we introduce the following definition.

Definition 4.2 Let V be an analytic variety in $\mathbb{C}^n$ which contains the origin, let γ be a simple curve and let d be a critical value for γ and V. A singular point ζ of $T_{\gamma,d}V$ is said to be *terminating* for γ and d if there is a simple curve $\sigma(t) = \gamma(t) + \zeta t^d + o(t^d)$ such that d is the largest critical value of σ and V. A point $\zeta_1 \in (T_0V)_{\text{sing}}$ with $|\zeta_1| = 1$ is called *terminating* if 0 is terminating for $\gamma_1(t) := t\zeta_1$ and $d = 1$.

Terminating singularities are closely associated to the curves in V_{sing} as the next proposition shows.

Proposition 4.3 *Let V be an analytic variety in $\mathbb{C}^n$ which is of pure dimension $k \geq 1$ and which contains the origin. Let γ be a simple curve in $\mathbb{C}^n$, let $d \geq 1$ be a critical value for γ and V, and let $\zeta \in T_{\gamma,d}V$ be a singular point for $T_{\gamma,d}V$. Then ζ is a terminating singularity of $T_{\gamma,d}V$ if and only if there is a singular curve $\sigma(t) = \gamma(t) + \zeta t^d + o(t^d)$ in V such that for small t, the multiplicity of V along the curve $\sigma(t)$ is constant and equal to the multiplicity of $T_{\gamma,d}[V]$ at ζ.*

Remark 4.4 If we recall that any two simple curves whose coefficients agree up to and including order t^d give the same limit currents $T_{\gamma,d'}[V]$ for $d' \leq d$ (see [4, Proposition 4.1,(i)], then this proposition shows that we should have chosen σ for the computation of the limit currents instead of γ. Note, however, that σ is likely to have infinitely many nonzero Puiseux series coefficients while the critical curves constructed by choosing successive critical values and singular points will be finite. In the case of surfaces, where there are only finitely many singular curves, once $\gamma(t)$ has sufficiently high order contact with a singular curve, the successive critical extensions of γ generate successive terms of the Puiseux series expansion of the singular curve.

Proof of Proposition 4.3 We can assume $\zeta = 0$ for the proof, since changing $\gamma(t)$ to $\gamma(t) + \zeta t^d$ has this effect. If σ is any simple curve which satisfies $\sigma(t) = \gamma(t) + o(t^d)$, then we have $T_{\gamma,d}[V] = T_{\sigma,d}[V]$ by [4, Proposition 4.1(i)], and hence $\mu := \mu(T_{\gamma,d}[V], 0) = \mu(T_{\sigma,d}[V], 0)$. From the series expansion of $F(w, t, \xi)$ in (2.5) with γ replaced by σ, the critical values are defined in terms of the support M of the pairs $(j, |\beta|)$ of nonzero terms in the power series and the Newton polygon derived from it. Recall that for a given d with $\omega_0 = \omega_0(d)$, the function F_{ω_0} is the sum over all the monomials in the edge, which may be a vertex, of the Newton polygon of slope $-1/d$. To have no critical values for σ and V larger than d is therefore the same as requiring that the expansion of F_{ω_0} contains at least one monomial with a factor w^β where $|\beta| = \mu$ and that μ is the lowest degree in w of any term that appears in the power series expansion of F.

On the other hand, the function F_{ω_0} is equal to the canonical defining function of $T_{\sigma,d}[V]$ (Corollary 3.2) up to the factor Φ which does not change the multiplicity at any point. Therefore, the multiplicity of $T_{\sigma,d}[V]$ at 0 is equal to the degree of the lowest degree homogeneous polynomial in w in the series expansion of F_{ω_0}. This is characterized by the fact that it is the least integer μ such that

$$\max_{|\xi|=1} |F_{\omega_0}(w, 1, \xi)| = O(|w|^{\mu}), \qquad |w| \to 0.$$

Recalling that

$$(4.1) \qquad t^{-\omega_0} F(t^d w, t, \xi) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} t^{j+d|\beta|-\omega_0} w^{\beta} \xi^{\alpha}$$
$$= F_{\omega_0}(w, 1, \xi) + \sum_{j+d|\beta|>\omega_0} a_{j,\beta,\alpha} t^{j+d|\beta|-\omega_0} w^{\beta} \xi^{\alpha},$$

we therefore see that d being the largest critical value of σ is the same as requiring the $O(|w|^{\mu})$ bound everywhere along γ, *i.e.,* for each small $t > 0$:

$$\max_{|\xi|=1} |t^{-\omega_0} F(t^d w, t, \xi)| = O(|w|^{\mu}), \quad |w| \to 0.$$

Since $t^{-\omega_0} P(\sigma(t) + t^d w, \xi; V) = t^{-\omega_0} F(t^d w, t, \xi)$, this is equivalent to σ being a curve along which the multiplicity of V is μ for all small t. ■

Definition 4.5 Let $(\zeta_j, d_j)_{j=1}^{l}$ be a critical set for the variety V in $\mathbb{C}^n$. It is called a *normal critical set* for V if ζ_1 is not terminating and, if for $i = 2, \ldots, l$, ζ_i is not terminating for γ_{i-1} and d_i.

The main fact about normal critical sets is stated in the following theorem.

Theorem 4.6 *Let V be an analytic surface defined in some neighborhood of the origin satisfying $0 \in V$. Then for each $\zeta_1 \in T_0 V$, $|\zeta_1| = 1$, the set*

$$\{C = (\zeta_j, d_j)_{j=1}^{l} : C \text{ is a normal critical set for } V\}$$

is finite.

For the proof of Theorem 4.6 we will need some preparation. We begin by providing more details about critical sets. Therefore, let $C = (\zeta_j, d_j)_{j=1}^{l}$ be a normal critical set for an analytic variety V in $\mathbb{C}^n$ of pure dimension k and define the curves γ_i by

$$(4.2) \qquad \gamma_i(t) = \zeta_1 t + \zeta_2 t^{d_2} + \cdots + \zeta_i t^{d_i}, \quad 1 \le i \le l.$$

Since the limit currents $T_{\gamma_i, d_i}[V]$, $1 \le i \le l$, will play a crucial role in the analysis, we introduce the following related quantities:

$$(4.3) \qquad m_i = \text{ degree of } T_{\gamma_i, d_i}[V],$$

$$\mu_i := \mu(T_{\gamma_i,d_i}[V],0) = \text{ multiplicity of } T_{\gamma_i,d_i}[V] \text{ at } w=0. \tag{4.4}$$

And recalling the expansion (2.5) with $\gamma(t)$ replaced by $\gamma_i(t)$, let

$$\omega_i := \omega_0(d_i) = \min\{ j + d_i|\beta| : a_{j,\beta,\alpha} \neq 0\}, \tag{4.5}$$

which is the exponent that measures how fast the canonical defining function tends to 0 along the curve $\gamma_i(t)$ (2.6). Also let

$$q_i := \text{ least common denominator of } \{1, d_2, \dots, d_i\}. \tag{4.6}$$

Lemma 4.7 *Let V be an analytic variety of pure dimension $k \geq 1$ defined in some neighborhood of the origin that contains the origin and has local multiplicity m at zero. Let $C = (\zeta_j, d_j)_{j=1}^l$ be a critical set for V. In the notation introduced above, the following statements hold for $1 \leq i \leq l-1$:*

(i) $T_0(T_{\gamma_i,d_i}[V]) = (T_{\gamma_i,d_{i+1}}[V])_h = (T_{\gamma_{i+1},d_{i+1}}[V])_h$;
(ii) $m \geq m_i \geq \mu_i = m_{i+1}$;
(iii) $\omega_{i+1} - \omega_i = \mu_i(d_{i+1} - d_i)$;
(iv) *if $m_i = \mu_i$, then $T_{\gamma_i,d_i}V$ is homogeneous;*
(v) $q_l \leq (m!)^2$.

Proof The first equation of (i) is Corollary 3.5. The second equation follows from the first one and $T_{\gamma_i,d_{i+1}}[V] = T_{\gamma_{i+1},d_{i+1}}[V] + \{\zeta_{i+1}\}$, which is an immediate consequence of the definition of limit varieties.

Assertion (ii) obviously follows from (i) and

$$m_i \leq \deg(T_0[V]) = \mu(T_0[V],0) = m.$$

To prove (iii), assume without loss of generality that the coordinates in $\mathbb{C}^n$ are chosen in such a way that the projection $\pi \colon \mathbb{C}^{n-k} \times \mathbb{C}^k \to \mathbb{C}^k$, $\pi(z'', z) := z'$ is transverse to V at the origin and to all $T_{\gamma_l,d}V$ for $1 \leq d \leq d_l$. This choice is possible, since there are only finitely many varieties of this form by [4, Proposition 4.3]. Let $P(z,\xi,V) = P(z,\xi;V,\pi)$ denote the canonical defining function for V. Then fix $1 \leq i \leq l-1$ and define γ_i by (4.2). Applying (2.5) with γ_i we get

$$F_i(w,t,\xi) := P(\gamma_i(t) + w, \xi, V) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} t^j w^\beta \xi^\alpha. \tag{4.7}$$

By (2.6) we have for $d \geq 1$ and $\omega_0 = \omega_0(d)$

$$t^{-\omega_0} F_i(t^d w, t, \xi) = F_{i,\omega_0}(w,1,\xi) + \sum_{\omega > \omega_0} t^{\omega-\omega_0} F_{i,\omega}(w,1,\xi). \tag{4.8}$$

According to Corollary 3.2, the canonical defining function for $T_{\gamma_i,d}[V]$ satisfies

$$P(w,\xi;T_{\gamma_i,d}[V],\pi)\Phi_d(\xi) = F_{i,w_0}(w,1,\xi). \tag{4.9}$$

By our choice of π, the degree $m_i(d)$ of $T_{\gamma_i,d}[V]$ is equal to the degree of $F_{i,w_0}(w,1,\xi)$ in w, while the local multiplicity $\mu(d) = \mu(T_{\gamma_i,d}[V],0)$ is equal to the degree of the lowest order homogeneous term in w. That is, if we expand $F_{i,\omega_0}(w,1,\xi)$ as a sum of terms homogeneous of degree l in w and m in ξ, then

$$F_{i,\omega_0}(w,1,\xi) = \sum_{l=\mu(d)}^{m(d)} h_l(w,\xi), \quad h_{\mu(d)}(w,\xi) \not\equiv 0,\ h_{m(d)}(w,\xi) \not\equiv 0. \tag{4.10}$$

To determine $m(d_i) = m_i$ and $\mu(d_i) = \mu_i$, we use the fact that C is a critical set for V. Hence, by definition, d_{i+1} is the smallest critical value for γ_i and V larger than d_i. Therefore, there is an edge with slope $-1/d_{i+1}$ in the Newton polygon N of the expansion for F_i. Let (j_0,μ) and (j_1,ν) be the vertices of this edge satisfying $j_0 < j_1$. Then we have $\mu > \nu$ and

$$j_0 + d_{i+1}\mu = \omega_0(d_{i+1}) = j_1 + d_{i+1}\nu.$$

We also have $j_0 + d_i\mu = \omega_0(d_i)$ and consequently $\omega_0(d_{i+1}) - \omega_0(d_i) = \mu(d_{i+1} - d_i)$. This completes the proof of (iii).

To prove (iv), note that if $m_i = \mu_i$, then the degree of $F_{i,\omega_0}(w,1,\xi)$ as a function of w coincides with its vanishing order. Hence $F_{i,\omega_0}(w,1,\xi)$ is homogeneous.

To prove (v), first note that $T_{\gamma_1,d_1}[V] = T_0[V] - \{\zeta_1\}$ and hence

$$m_1 = \deg T_{\gamma_1,d_1}[V] = m.$$

Since C is a critical set for V, for each $1 \le i \le l-1$ the number d_{i+1} is the smallest critical value for V and γ_i which is larger than d_i. By Section 2.3, d_{i+1} can be determined by the Newton polygon N_i of the expansion of $P(\gamma_i(t)+w,\xi;V,\pi)$ according to (2.5). In fact, N_i contains a segment with *a priori* unknown endpoints (j_0,μ) and (j_1,ν) of slope $-1/d_{i+1}$, and $T_{\gamma_i,d_{i+1}}[V]$ can be computed by Corollary 3.2 and formula (4.10) from $F_{i,\omega_0(d_{i+1})}(w,1,\xi) = \sum_{l=\nu}^{\mu} h_l(w,\xi)$. By Corollary 3.5 we have $(T_{\gamma_i,d_{i+1}}[V])_h = T_0(T_{\gamma_i,d_i}[V])$. This implies $\mu = \mu_i$ by the definition of μ_i for $i \ge 2$ and $\mu = m$ for $i = 1$. Hence we have

$$d_{i+1} = \frac{j_1 - j_0}{\mu_i - \nu}.$$

By the definition of q_i and γ_i there are $b_j \in \mathbb{N}$, $1 \le j \le i$, such that $\gamma_i(t) = \sum_{j=1}^{i} \zeta_j t^{b_j/q_i}$. Therefore, there exists $c \in \mathbb{N}$ such that $j_1 - j_0 = c/q_i$. Since $0 \le \nu_{i+1} < \mu_i$, this implies $q_{i+1} \le \mu_i q_i$.

Next assume that for some i with $1 \le i < l$ we have $m_i = \mu_i = \mu_{i+1}$. Then it follows from (ii) that also $\mu_{i+1} = m_{i+1}$. By (iv), this implies that $T_{\gamma_i,d_i}[V]$ and $T_{\gamma_{i+1},d_{i+1}}[V]$ are homogeneous of degree $\mu_i = \mu_{i+1} = m_i = m_{i+1}$. By Corollary 3.3, $T_{\gamma_i,d_{i+1}}[V]$ is not homogeneous and by definition ζ_{i+1} is a singular point of $T_{\gamma_i,d_{i+1}}V$. By Corollary 3.5, we have $(T_{\gamma_i,d_{i+1}}[V])_h = T_0(T_{\gamma_i,d_i}[V]) = T_{\gamma_i,d_i}[V]$. Since

$$T_{\gamma_{i+1},d_{i+1}}[V] = T_{\gamma_i,d_{i+1}}[V] - \{\zeta_{i+1}\}, \tag{4.11}$$

it follows that $\zeta_{i+1} \neq 0$ and that

$$(T_{\gamma_i,d_{i+1}}[V])_h = (T_{\gamma_{i+1},d_{i+1}}[V])_h = T_{\gamma_{i+1},d_{i+1}}[V]$$

and hence $T_{\gamma_i,d_i}[V] = T_{\gamma_{i+1},d_{i+1}}[V]$. Moreover, (4.11) implies that the term $h_{\mu-1}$ in the expansion of $F_{i,\omega_0(d_{i+1})}(w,1,\xi)$ above does not vanish. Since $\mu = \mu_i$, this shows that there is a point $(j_2, \mu_i - 1)$ on the segment of N_i with slope $-1/d_{i+1}$. As before, this implies the existence of $c_0 \in \mathbb{N}$ such that

$$d_{i+1} = \frac{j_2 - j_0}{\mu_i - (\mu_i - 1)} = j_2 - j_0 = \frac{c_0}{q_i}$$

and consequently $q_{i+1} = q_i$ under the present hypothesis.

Now we are ready to prove (v). To do so let

$$m_1 \geq \mu_1 = m_2 \geq \mu_2 = m_3 \geq \mu_3 = \cdots$$

be the sequence of degrees and multiplicities associated to C according to (4.3) and (4.4). As we have proved above, $q_{i+1} = q_i$ whenever $m_i = \mu_i = m_{i+1} = \mu_{i+1}$ and $q_{i+1} \leq \mu_i q_i$ otherwise. Now we define $i_1 := 1$ and, assuming that $i_2, \dots, i_j$ have been defined, we let $i_{j+1} = \min\{i : i > i_j \text{ and } \mu_i < \mu_{i_j}\}$. Finally we get i_J satisfying $\mu_{i_J} = \mu_{i_J+1} = \cdots = \mu_l$. For $1 \leq j < J$ we have by the above estimates $q_{i_j+1} \leq \mu_{i_j} q_{i_j}$. Since the sequence $(\mu_i)_{1\leq i\leq l}$ is not increasing, this implies $q_{i_j+2} \leq \mu_{i_j}^2 q_{i_j}$. Moreover, if $i_{j+1} > i_j + 2$, then $q_i \leq \mu_{i_j}^2 q_{i_j}$ for $i_j + 2 \leq i \leq i_{j+1}$. This implies

$$q_l \leq (\mu_{i_J})^2 q_{i_J} \leq (\mu_{i_J})^2 (\mu_{i_{J-1}})^2 q_{i_{J-1}}.$$

Since $m \geq \mu_{i_1} > \mu_{i_2} > \cdots \geq 1$, it follows from this by induction that $q_l \leq (m!)^2$. ∎

For the next lemma we need to recall the definition of a subanalytic set from Bierstone and Milman [1]. For that purpose let M be a real analytic manifold. For an open subset U of M, denote by $\mathcal{O}(U)$ the ring of all real analytic functions and by $S(\mathcal{O}(U))$ the smallest family of subsets of U which is stable under finite intersections, finite unions and complements and which contains all the sets $\{x \in U : f(x) > 0\}$, $f \in \mathcal{O}(U)$. A subset X of M is called *semianalytic*, if each $a \in M$ has a neighborhood U such that $X \cap U \in S(\mathcal{O}(U))$, and X is called *subanalytic*, if each $a \in M$ has a neighborhood U such that there is a real analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $X \cap U = \pi(A)$, where $\pi\colon M \times N \to M$ is the natural projection.

Lemma 4.8 *Let $F_1(w,s), \dots, F_l(w,s)$ be finitely many holomorphic functions on an open neighborhood of $\{(w,s) \in \mathbb{C}^{n+1} : |w| \leq 1, |s| < 1\}$ with $F_i(0,0) = 0$ for $i = 1, \dots, l$. Set*

$$A := \Big\{ (w,s) : |w| \leq 1,\ s > 0,\ \sum_{i=1}^{l} |F_i(w,s)|^2 = \min_{|z|\leq 1} \sum_{i=1}^{l} |F_i(z,s)|^2 \Big\}.$$

Then $\overline{A}$ is a subanalytic set containing $(0,0)$, and there is an analytic curve $w(s)$ such that $\lim_{s\to 0+}(w(s), s) = (0,0)$ and $(w(s), s) \in A$ for all small positive s.

Proof Define

$$g\colon \{(w,s) \in \mathbb{C}^n \times]0,1[\,:\, |w| \le 1\} \to \mathbb{R}, \quad g(w,s) := \sum_{i=1}^{l} |F_i(w,s)|^2,$$

and

$$h\colon]0,1[\to \mathbb{R}, \quad h(s) := \min_{|z|\le 1} g(z,s)$$

and consider g as a real analytic function in $2n+1$ real variables. We show first that h is a subanalytic function. To so so, write the graph of h as $M_1 \cap M_2$, where

$$M_1 := \left\{ (s,t) \in \mathbb{R}^2 : s > 0,\, g(w,s) = t \text{ for some } |w| \le 1 \right\},$$

$$M_2 := \left\{ (s,t) \in \mathbb{R}^2 : s > 0,\, g(w,s) \ge t \text{ for all } |w| \le 1 \right\}.$$

Define the projection $\pi\colon \mathbb{C}^n \times \mathbb{R}^2 \to \mathbb{R}^2$, $(w,s,t) \mapsto (s,t)$, and set

$$N_1 := \left\{ (w,s,t) \in \mathbb{C}^n \times \mathbb{R}^2 : |w| \le 1,\, 0 < s < 1,\, g(w,s) = t \right\},$$

$$N_2 := \left\{ (w,s,t) \in \mathbb{C}^n \times \mathbb{R}^2 : |w| \le 1,\, 0 < s < 1,\, g(w,s) < t \right\}.$$

Then $M_1 = \pi(N_1)$ and $M_1 \cap M_2 = M_1 \setminus \pi(N_2)$. Hence M_1 and then also $M_1 \cap M_2$ are subanalytic; in the latter case, the Theorem of the Complement [1, Theorem 3.10] is used. Hence we have shown that h is a subanalytic function. Since $A = \{(w,s) : g(w,s) = h(s)\}$, it follows that A is subanalytic. Hence so is its closure by [1, §3]. It is easy to see that $(0,0) \in \overline{A}$.

By the Uniformization Theorem [1, Theorem 0.1], there are a real analytic manifold N and a proper real analytic map $\varphi\colon N \to \overline{A}$ such that $\varphi(N) = \overline{A}$. Fix an arbitrary point $p \in \varphi^{-1}(0,0)$, and let U be a connected open neighborhood of p which is so small that we can think of it as a subset of some $\mathbb{R}^k$. Let π denote the projection $(w,s) \mapsto s$. Since $\pi \circ \varphi$ cannot vanish on all of U, there must be a real line L through p such that $\pi \circ \varphi$ does not vanish identically on $L \cap U$. If we compose φ with a parametrization of L, we get a real analytic map $\alpha\colon]-1,1[\to \overline{A}$ such that $\alpha(0) = (0,0)$ and $\pi \circ \alpha(\tau) \ne 0$ for at least one τ. Since $\pi \circ \alpha$ is an analytic function of one real variable, this implies the existence of $\epsilon > 0$ such that $\pi \circ \alpha(\tau) \ne 0$ whenever $0 < \tau < \epsilon$ and such that $\pi \circ \alpha$ is invertible on $]0,\epsilon[$. The map w from the claim is given by the equation $\alpha((\pi \circ \alpha)^{-1}(s)) = (w(s), s)$. ■

The spirit of the proof of Lemma 4.8 is the same as of the proof of Hörmander [11, Theorem A.2.8]. We thank Pierre Milman for pointing out the relevance of the Theorem of the Complement [1, 3.10].

Corollary 4.9 *Let V be an analytic variety in $\mathbb{C}^n$ of pure dimension k that contains the origin, let γ be a simple curve in $\mathbb{C}^n$, $d \ge 1$ rational, and $2 \le \mu$ the multiplicity of $T_{\gamma,d}[V]$ at $w = 0$. Suppose also that $T_{\gamma,d}V \cap \pi^{-1}\{0\} \cap \{|w''| \le 1\} = \{0\}$. Then there exist constants $c > 0$, $r_0 > 0$, $M > 0$ such that for $0 < t \le r_0$, either*

$$\min_{|w|\le 1} \max_{|\xi|=1} \sum_{|\beta|<\mu} |D_w^\beta P(\gamma(t) + t^d w, \xi; V) t| \ge ct^M \tag{4.12}$$

or else there is a curve $\sigma(t) = \gamma(t) + o(t^d)$ on which we have

$$D_w^\beta P(\gamma(t) + t^d w, \xi; V)|_{w=t^{-d}(\sigma(t)-\gamma(t))} \equiv 0,\ 0 < t < r_0,\ |\xi| = 1,\ |\beta| < \mu. \tag{4.13}$$

Proof Decompose $P(z, \xi; V) = \sum_{|\alpha|=m} P_\alpha(z)\xi^\alpha$ and set

$$\Psi(w, t) = \sum_{|\alpha|=m} \sum_{|\beta|<\mu} |D_w^\beta P_\alpha(\gamma(t) + t^d w)|^2, \quad r(t) = \min_{|w|\le 1} \Psi(w, t).$$

Then the set of points (w, t) in a neighborhood of the origin where $r(t) = \Psi(w, t)$ is a subanalytic set which contains $(0, 0)$. Since $r(0) = 0$, it contains points (w_t, t) with $|w_t|$ small for every $0 < t < \delta$. Hence by Lemma 4.8, there is an analytic curve $w = \kappa(t)$ such that $r(t) = \Psi(\kappa(t), t)$ and $\kappa(t) \to 0$ as $t \to 0+$.

If $r(t) \equiv 0$, then $\sigma(t) := \gamma(t) + t^d\kappa(t)$ satisfies $\sigma(t) = \gamma(t) + o(t^d)$ and σ is a curve on which (4.13) holds. Otherwise $r(t)$ is a positive analytic function with $r(0) = 0$. If $r(t) = c_1 t^{2M} + \cdots$ is the Puiseux series expansion of $r(t)$, then $c_1 > 0, M > 0$, and $r(t) > c_1 t^{2M}/2$ provided t is small and positive. Since the function on the left-hand side of (4.12) is bounded above and below by a constant multiple of $(\Psi(w, t))^{1/2}$, the inequality (4.12) follows. ■

Lemma 4.10 *Let V be an analytic surface in $\mathbb{C}^n$ which contains the origin and has local multiplicity m at 0. Then for each normal critical set $C = (\zeta_j, d_j)_{j=1}^{i-1}$ for V, there exists $p \in \mathbb{N}_0$ such that for each normal critical set $\mathcal{C} = (\zeta_j, d_j)_{j=1}^{l}$ that extends C and satisfies*

$$\mu(T_{\gamma_\nu, d_\nu}[V], 0) = \mu(T_{\gamma_{i-1}, d_{i-1}}[V], 0) \quad \textit{for } i \le \nu \le l, \tag{4.14}$$

the estimate $l - i \le p$ holds.

Proof Note first that the lemma holds trivially if there are no normal critical extensions of C. Otherwise fix an extension $\mathcal{C}$ as above with $l \ge i$ and define the curves γ_k for $1 \le k \le l$ by formula (4.2). Then choose coordinates as in the proof of Lemma 4.7(iii) such that the projection $\pi\colon \mathbb{C}^{n-2} \times \mathbb{C}^2 \to \mathbb{C}^2$, $\pi(z'', z') = z'$ is transverse to V at the origin and to $T_{\gamma_i, d}V$ for $1 \le d \le d_i$, and let $P(z, \xi, V) := P(z, \xi; V, \pi)$ denote the canonical defining function for V. Since $\mathcal{C}$ is a normal critical set, ζ_i is a singular point of $T_{\gamma_{i-1}, d_i}[V]$ which is not terminating for γ_{i-1} and d_i. Hence Proposition 4.3 implies that each simple curve σ which satisfies $\sigma(t) = \gamma_i(t) + o(t^{d_i})$ is not singular for V or the multiplicity of V along $\sigma(t)$ is smaller than $\mu(T_{\gamma_{i-1}, d_i}[V], \zeta_i) = \mu(T_{\gamma_i, d_i}[V], 0) =: \mu$. Applying Corollary 4.9 with $\gamma = \gamma_i$ and $d = d_i$, this shows that the estimate (4.12) must hold under the present hypotheses. Hence there are $r_0, c, M > 0$ such that for $0 < t \le r_0$,

$$\min_{|w|\le 1} \max_{|\xi|=1} \sum_{|\beta|<\mu} \left|D_w^\beta P(\gamma_i(t) + t^{d_i} w, \xi, V)\right| \ge ct^M. \tag{4.15}$$

Now we use the notation that we introduced before Lemma 4.7 and let

$$F_k(w, t, \xi) := t^{-\omega_k} P(\gamma_k(t) + t^{d_k} w, \xi, V), \quad 1 \le k \le l.$$

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For $\beta \in \mathbb{N}_0^n$, $|\beta| < \mu$, the chain-rule gives

$$D_w^\beta P(\gamma_k(t) + t^{d_k} w, \xi, V) = t^{|\beta| d_k} (D_z^\beta P)(\gamma_k(t) + t^{d_k} w, \xi, V), \tag{4.16}$$

and hence

$$D_w^\beta F_k(w, t, \xi) = t^{-\omega_k} t^{|\beta| d_k} (D_z^\beta P)(\gamma_k(t) + t^{d_k} w, \xi, V). \tag{4.17}$$

Since there is nothing to prove if $l \leq i + 1$, we assume from now on that $l > i + 1$. Then for $|\widetilde{w}| \leq 1$ and $t > 0$ small enough,

$$t^{-d_i}(\gamma_l(t) - \gamma_i(t) + t^{d_l}\widetilde{w}) = \sum_{\nu=i+1}^{l} \zeta_\nu t^{d_\nu - d_i} + t^{d_l - d_i}\widetilde{w}$$

tends to zero as t tends to zero. Therefore, there exists $0 < \delta_0 \leq \delta$ such that for $0 < t \leq \delta_0$ we get from (4.15), (4.16), and (4.17) for $w = t^{-d_i}(\gamma_l(t) - \gamma_i(t) + t^{d_l}\widetilde{w})$:

$$\begin{aligned} ct^M &\leq \max_{|\xi|=1} \sum_{|\beta|<\mu} \left| t^{|\beta| d_i} (D_z^\beta P)(\gamma_i(t) + t^{d_i} w, \xi, V) \right| \\ &= \max_{|\xi|=1} \sum_{|\beta|<\mu} \left| t^{|\beta| d_i} (D_z^\beta P)(\gamma_l(t) + t^{d_l} \widetilde{w}, \xi, V) \right| \\ &= \max_{|\xi|=1} \sum_{|\beta|<\mu} \left| t^{|\beta|(d_i - d_l) + \omega_l} D_{\widetilde{w}}^\beta F_l(\widetilde{w}, t, \xi) \right|. \end{aligned} \tag{4.18}$$

Now note that $F_l(\,\cdot\,, t, \,\cdot\,)$ is a polynomial in $\widetilde{w}$ and ξ which converges to $F_{\omega_l}(\,\cdot\,, 1, \,\cdot\,)$ uniformly on compact sets as t tends to zero by [4, Lemma 3.7]. Since $d_i - d_l < 0$, this implies the existence of $D > 0$ such that the the right-hand side of (4.18) can be estimated from above by $Dt^{(d_i - d_l)(\mu - 1) + \omega_l}$ for $0 < t < \delta_0$. From this we conclude

$$M \geq (d_i - d_l)(\mu - 1) + \omega_l. \tag{4.19}$$

By Lemma 4.7 and the hypothesis (4.14) we have

$$\omega_l - \omega_i = \sum_{\nu=i}^{l-1} \omega_{\nu+1} - \omega_\nu = \sum_{\nu=i}^{l-1} \mu(d_{\nu+1} - d_\nu) = \mu(d_l - d_i).$$

Together with (4.19) this implies

$$M \geq \omega_i + d_l - d_i = \omega_i + \sum_{\nu=1}^{l-1} (d_{\nu+1} - d_\nu). \tag{4.20}$$

Now note that by the definition of q_l in (4.6), the positive number $d_{\nu+1} - d_\nu$ is an integer multiple of $1/q_l$. From this, (4.20), and Lemma 4.7(v), we get

$$M \geq \omega_i + (l - i)/q_l \geq \omega_i + (l - i)/(m!)^2$$

and hence $l - i \leq (M - \omega_i)(m!)^2 \leq M(m!)^2$. This proves the statement of the lemma with $p = M(m!)^2$. ∎

Fulton [9, Problem 7.1] states that an arbitrary algebraic curve admits at most a finite number of singularities. Here we present an effective version of this result, which will be needed in the proof of Theorem 4.6. Even in the special case of a plane curve, our estimates are far from optimal, as can be seen from [9, Ch. 5, Theorem 2].

Lemma 4.11 *Let $C \subset \mathbb{C}^n$ be an algebraic curve of degree m. Then C_{sing} consists of not more than $m^2(m-1)$ points.*

Proof Choose a projection π such that the canonical defining function $P(z, \xi; C, \pi)$ exists. We may assume that $\pi(z) = (z_1, 0, \ldots, 0)$. There are $w_1, w_2 \in \mathbb{C}$ and $\xi_0 \in \mathbb{C}^{n-1}$ such that $Q(w_1, w_2) \neq 0$ for $Q(z_1, z_2) := P((z_1, z_2, 0, \ldots, 0), \xi_0; C, \pi)$. Set $C' = \{(z_1, z_2) \in \mathbb{C}^2 : Q(z_1, z_2) = 0\}$. Note that by definition, Q is zero or a polynomial of degree at most m. The first possibility is ruled out by $Q(w_1, w_2) \neq 0$, hence C' is a plane algebraic curve of degree at most m. If w is a singular point of C, then two branches of C meet at w. Hence the discrimininant of C' with respect to z_2 vanishes at w_1. Since the degree of this discrimininant does not exceed $m(m-1)$, there are not more than $m(m-1)$ possible values for w_1. Since the degree of C is m, there are at most m inverse images under π for each zero of the discrimininant. ■

Proof of Theorem 4.6 Obviously, it suffices to consider those $\zeta_1 \in (T_0 V)_{\text{sing}}$ with $|\zeta_1| = 1$ which are not terminating. Fix such a point ζ_1 and note that for any simple curve γ and $d > 1$ the limit varieties $T_{\gamma,d}V$ depend on one fewer variables and hence, in suitable coordinates, they are of the form $C \times \mathbb{C}$ where C is an algebraic curve. By Lemma 4.11, each such curve C can have at most $m^2(m-1)$ singular points. By Lemma 4.10, there is $\nu_1 \in \mathbb{N}_0$, depending only on ζ_1, such that each normal critical set $(\zeta_j, d_j)_{j=1}^{l}$ either has length $l \leq \nu_1$ or satisfies $\mu_{\nu_1} + 1 < \mu_1$. There are at most $(m^2(m-1))^{\nu_1+1}$ such sets. If we apply Lemma 4.10 to each of these, we can find a number $\nu_2 \in \mathbb{N}_0$ such that for each normal critical extension any of them either has length $l \leq \nu_1 + \nu_2 + 1$ or must satisfy $\mu_{\nu_1+\nu_2+1} < \mu_{\nu_1+1}$. Continuing this argument by induction the desired assertion follows, since $m \geq \mu_1$. ■

5 Studying Algebraic Varieties at Infinity

In this section we indicate how the results of the preceding section carry over to algebraic varieties at infinity.

5.1 Canonical Defining Functions for Algebraic Varieties

Let V be an algebraic variety in $\mathbb{C}^n$ which is of pure dimension $k \geq 1$ and has degree m. We choose coordinates in $\mathbb{C}^n$ that are *excellent* for V. This means that the projection $\pi \colon \mathbb{C}^{n-k} \times \mathbb{C}^k \to \mathbb{C}^k$, $\pi(z'', z') := z'$, is proper when restricted to V and satisfies for some $C > 0$ the estimate

$$|z| \leq C(1 + |z'|), \quad z \in V. \tag{5.1}$$

The existence of excellent coordinates is shown, *e.g.*, [8, 7.4, Theorem 2]. Then the branch locus B of $\pi \colon V \to \mathbb{C}^n$ as well as $\pi(B)$ are algebraic varieties of dimension at most $k-1$ and $\pi\colon V \setminus B \to \mathbb{C}^k \setminus \pi(B)$ is a covering map. For $z' \in \mathbb{C}^k \setminus \pi(B)$ there are m points in the fiber over z'. We write $\pi^{-1}(z') = \{(\alpha_i(z'), z') : 1 \le i \le m\}$ where the $\alpha_i(z')$ are all distinct. We will also use the same notation for $z' \in \pi(B)$ by repeating each $\alpha_i(z')$ as many times as indicated by the multiplicity $\mu(V, z)$ for $z = (\alpha_i(z'), z')$. Using this notation, the *canonical defining function* for V is again given by the formula in (2.2) namely:

$$P(z, \xi; V, \pi) := \prod_{i=1}^{m} \langle z'' - \alpha_i(z'), \xi\rangle.$$

It is a polynomial in z and ξ of degree m in z and ξ separately.

If W is a holomorphic k-chain, *i.e.*, $W = n_1[W_1] + \cdots + n_p[W_p]$ where the W_j are the irreducible components of $\operatorname{Supp} W$ and degree $W_j = m_j$, then let $\nu := \sum_{j=1}^{p} n_j m_j$ and define $P(w, \xi; W, \pi) := \prod_{j=1}^{p} P(w, \xi; W_j, \pi)^{n_j}$. Then $P(w, \xi; W, \pi)$ is a polynomial of degree ν in ξ.

5.2 Limit Currents

For $R > 0$ and $q \in \mathbb{N}$ let $\gamma\colon [R, \infty[\to \mathbb{C}^n$ be a curve that has the following convergent expansion

$$\gamma(t) = \sum_{j=-\infty}^{q} \xi_j t^{j/q}, \; R \le t < \infty. \tag{5.2}$$

If $|\xi_q| = 1$, then γ is called a *simple curve*. For an algebraic variety V in $\mathbb{C}^n$ of pure dimension $k \ge 1$, a simple curve γ, and $d \le 1$, we define the algebraic varieties

$$V_t = V_{\gamma,d,t} = \{w \in \mathbb{C} : \gamma(t) + t^d w \in V\}, t \in [R, \infty[.$$

It was shown in [6] that there exists a *limit current* of V of order d along γ, *i.e.*,

$$T_{\gamma,d}[V] = \lim_{t\to\infty} [V_{\gamma,d,t}]. \tag{5.3}$$

Its support is denoted by $T_{\gamma,d}V$ and is called the *limit variety* of V of order d along γ, *i.e.*, $T_{\gamma,d}V = \operatorname{Supp} T_{\gamma,d}[V]$.

The existence of the limit current $T_{\gamma,d}[V]$ is proved in [6] using formula (2.3) with an interpretation that suits the present frame. The following result was derived in [6, Theorem 1, Formula (12), Lemma 4, Lemma 5, Proposition 3].

Theorem 5.1 *Let V be an algebraic variety V of pure dimension $k \ge 1$ and degree m in $\mathbb{C}^n$ and let γ be a simple curve as in* (5.2). *Then for each $d \le 1$ the limit in* (5.3) *exists. Furthermore, fix a projection π transverse to V as in* (5.1), *fix $\omega_0 \in \mathbb{R}$ such that $\lim_{t\to\infty} t^{md-\omega_0} P(w, \xi; V_t, \pi)$ exists and does not vanish identically, and set*

$$Z = \{w \in \mathbb{C}^n : \lim_{t\to\infty} t^{md-\omega_0} P(w, \xi; V_t, \pi) = 0 \textit{ for all } \xi \in \mathbb{C}^{n-k}\}.$$

If π is transverse to Z, then there is a polynomial Φ such that

$$\lim_{t\to\infty} t^{md-\omega_0} P(w,\xi;V_t,\pi) = P(w,\xi;T_{\gamma,d}[V],\pi)\Phi(w',\xi).$$

The canonical defining function of $T_{\gamma,d}[V]$ is a polynomial. Furthermore, the functions $w' \mapsto \Phi(w',\xi)$, $\xi \in \mathbb{C}^{n-k}$, have no common zeros. In particular, $Z = T_{\gamma,d}V$.

We were unable to determine in [6] whether the function Φ in Theorem 5.1 actually depends on w' or not. As in Section 3, it turns out that, in fact, Φ does not depend on w'. This result will be proved by reduction to the local case treated in Section 3. The reduction will be accomplished with the help of the map Ψ of the following lemma.

Lemma 5.2 *Define*

$$\Psi\colon \{z\in\mathbb{C}^n : z_n\neq 0\} \to \{z\in\mathbb{C}^n : z_n\neq 0\},\quad z\mapsto \Big(\frac{z_1}{z_n^2},\dots,\frac{z_{n-1}}{z_n^2},\frac{1}{z_n}\Big). \tag{5.4}$$

Then $\Psi^2 = \mathrm{id}$ and

$$\Psi\big(tB((0,\dots,0,1),\epsilon)\big) \subset \frac{1}{t}B((0,\dots,0,1),6\epsilon) \tag{5.5}$$

whenever $\epsilon < 1/2$.

Proof The proof of $\Psi^2 = \mathrm{id}$ is immediate. To prove the remaining part of the assertion, fix $z := (tu_1,\dots,tu_{n-1},t+tu_n)$ in $tB((0,\dots,0,1),\epsilon)$. Then

$$\begin{aligned}\Psi(z) &= \Big(\frac{u_1}{t(1+u_n)^2},\dots,\frac{u_{n-1}}{t(1+u_n)^2},\frac{1}{t+tu_n}\Big)\\ &= \frac{1}{t}\Big(\frac{u_1}{(1+u_n)^2},\dots,\frac{u_{n-1}}{(1+u_n)^2},1+\frac{-u_n-u_n^2}{(1+u_n)^2}\Big).\end{aligned}$$

Note first that $|(1+u_n)^{-2}| \leq (1-\epsilon)^{-2} \leq (1+2\epsilon)^2$ since $\epsilon \leq \frac{1}{2}$. Hence $\Psi(z) = t^{-1}((0,\dots,0,1)+v)$, where

$$|v| \leq (1+2\epsilon)^2\big(|u|+|u_n^2|\big) \leq (1+2\epsilon)^2(\epsilon+\epsilon^2) = \epsilon+5\epsilon^2+8\epsilon^3+4\epsilon^4 \leq 6\epsilon$$

(here $\epsilon^j \leq 2^{1-j}\epsilon$ for $j\geq 1$ is used). ∎

We will use the following result from Mumford [13]. Its proof contains an acknowledgement to Stolzenfels.

Theorem 5.3 (Mumford [13, Theorem 2.33]) *Let $X \subset \mathbb{P}^n$ be an irreducible algebraic variety and let $X_0 \neq \varnothing$ be a Zariski-open set in X. Then the closure of X_0 in the classical topology is X.*

Proposition 5.4 *Let $V \subset \mathbb{C}^n$ be algebraic. Then the euclidean closure of*

$$\Psi(V \setminus (\mathbb{C}^{n-1} \times \{0\}))$$

is algebraic. We denote it by $\Psi_*(V)$. If $V \not\subset \mathbb{C}^{n-1} \times \{0\}$ is pure k-dimensional, then so is $\Psi_*(V)$.

Proof Let $I = \{f \in \mathbb{C}[z_1, \dots, z_n] : f|_V = 0\}$ be the ideal of V. Fix $f \in I$ and let m be the degree of f. Then f_h, defined by $f_h(z) := z_n^{2m} f(\Psi(z))$, is a polynomial. We define $W := \{w \in \mathbb{C}^n : f_h(w) = 0 \text{ for all } f \in I\}$. It is clear that W is algebraic and that

$$W \setminus (\mathbb{C}^{n-1} \times \{0\}) = \Psi(V \setminus (\mathbb{C}^{n-1} \times \{0\})).$$

Denote by $W_1, \dots, W_N$ the irreducible components of W. By Theorem 5.3, the euclidean closure of $\Psi(V \setminus (\mathbb{C}^{n-1} \times \{0\}))$ is the union of those W_j which have nonempty intersection with $\Psi(V \setminus (\mathbb{C}^{n-1} \times \{0\}))$. In particular, it is algebraic.

To show the second part, recall first that the dimension is constant throughout irreducible components. Fix $w \in V$ with $w_n \neq 0$. Near w, the sets $\Psi_*(V)$ and $\Psi(V)$ coincide. Since Ψ is biholomorphic, the proof is complete. ∎

Lemma 5.5

(i) *If $V \subset \mathbb{C}^n$ is algebraic, then $\Psi^*(V_h) \subset T_0\Psi^*(V)$.*

(ii) *If no irreducible component of $T_0\Psi^*(V)$ is contained in $\mathbb{C}^{n-1} \times \{0\}$, then even $\Psi^*(V_h) = T_0\Psi^*(V)$ is true. This hypothesis is satisfied if the coordinates are excellent in the sense of* (5.1).

Proof (i) Fix $w \in \Psi^*(V_h)$. We must first show that $w \in T_0\Psi^*(V)$. Choose a sequence $(v^j)_{j\in\mathbb{N}}$ in V_h with $v_n^j \neq 0$ for all j and $\lim_{j\to\infty} \Psi(v^j) = w$. For each j there are $r_j \geq 0$ and a sequence $(v^{j,l})_{l\in\mathbb{N}}$ in V such that $\lim_{l\to\infty}|v^{j,l}| = \infty$ and

$$r_j \lim_{l\to\infty} \frac{v^{j,l}}{|v^{j,l}|} = v^j, \quad j \in \mathbb{N}. \tag{5.6}$$

Hence

$$r_j \lim_{l\to\infty} \frac{\Psi(v^{j,l})}{|\Psi(v^{j,l})|} = r_j \lim_{l\to\infty} \frac{(v_n^{j,l})^{-2} v^{j,l}}{|v_n^{j,l}|^{-2}|v^{j,l}|} = \frac{|v_n^j|^2}{(v_n^j)^2} v^j. \tag{5.7}$$

Since $v_n^j \neq 0$, the restriction of equation (5.6) to its n-th coordinate implies the existence of $\delta_j > 0$ such that $|v_n^{j,l}| \geq \delta_j |v^{j,l}|$ for all sufficiently large l. Hence

$$\left|\Psi(v^{j,l})\right| = \frac{|v^{j,l}|}{|v_n^{j,l}|^2} \leq \frac{1}{\delta_j^2 |v^{j,l}|} \to 0 \quad \text{as } l \to \infty.$$

Thus equation (5.7) implies $v^j \in T_0\Psi^*(V)$. Since $T_0\Psi^*(V)$ is closed, we get $w \in T_0\Psi^*(V)$.

Assume now that no irreducible component of $T_0\Psi^*(V)$ is contained in $\mathbb{C}^{n-1} \times \{0\}$. To show that $T_0\Psi^*(V) \subset \Psi^*(V_h)$, it suffices to show that

$$T_0\Psi^*(V) \cap (\mathbb{C}^{n-1} \times \{0\}) \subset \Psi(V_h).$$

To do so, fix $w \in T_0\Psi^*(V)$ with $v_n \neq 0$. There are $r \geq 0$ and a sequence $(w^j)_{j\in\mathbb{N}}$ in $\Psi^*(V)$ such that $\lim_{j\to\infty} w^j = 0$ and

$$r \lim_{j\to\infty} \frac{w^j}{|w^j|} = w.$$

Since $w_n \neq 0$, we may assume that $w_n^j \neq 0$ for all j. Hence $w^j = \Psi(v^j)$ for suitable $v_j \in V$. Note that $|v^j| = |w^j|/|w_n^j|^2 \to \infty$ as $j \to \infty$ since $w^j \to 0$ as $j \to \infty$. Hence

$$r \lim_{j\to\infty} \frac{v^j}{|v^j|} = r \lim_{j\to\infty} \frac{(w_n^j)^{-2} w^j}{|w_n^j|^{-2}|w^j|} = \frac{|w_n|^2}{(w_n)^2} w.$$

Hence $w \in V_h$. Since $w_n \neq 0$, it follows immediately from the definition of Ψ that $w \in \Psi(V_h)$.

To prove (ii), note that (5.1) is equivalent to the existence of $C_1, C_2 > 0$ such that $|z| \leq C_1|\pi(z)|$ whenever $|z| > C_2$. Fix $w \in \Psi(V)$ with $|w| < 1/C_2$ and $w_n \neq 0$. We will show that $|w| \leq C_1|\pi(w)|$. By continuity, this estimate implies the claim. Set $z = \Psi(w)$. Then $|z| \geq 1/|w_n| > C_2$. Hence

$$|w| = \frac{1}{|z_n|^2}|z| \leq \frac{C_1}{|z_n|^2}|\pi(z)| = |\pi(w)|. \qquad \blacksquare$$

Example 5.6 Consider $V = \{(z, z^2) : z \in \mathbb{C}\}$. Then

$$\Psi(V \setminus (\mathbb{C} \times \{0\})) = \{(z^{-3}, z^{-2}) : z \in \mathbb{C}\}$$

and hence $\Psi^*(V) = \{(w^3, w^2) : w \in \mathbb{C}\}$. In this case $T_0\Psi^*(V) = \{0\} \times \mathbb{C}$, while $\Psi^*(V_h)$ is empty.

5.3 Critical Values

For an algebraic variety V of pure dimension $k \geq 1$ and degree m in $\mathbb{C}^n$ and a simple curve γ as in (5.2), we call a projection π in $\mathbb{C}^n$ *distinguished* for V and γ if it has rank k and is transverse to V and to $T_{\gamma,d}V$ for each $d \leq 1$. The existence of distinguished projections is shown similary as in Section 2.3. More precisely, let $q \in \mathbb{N}$ be the number that is associated to γ according to (5.2) and let

$$M_1 := \{ j/b \in\,]-\infty, 1] : qj \in \mathbb{Z},\ \ j \leq q, b \in \mathbb{N}, 1 \leq b \leq m\}.$$

Then M_1 is a discrete subset of $]-\infty, 1]$ and we can choose a sequence $(\delta_j)_{j\in\mathbb{N}}$ so that in each component of $]-\infty, 1] \setminus M_1$ there is exactly one point of this sequence. Then we let $M_0 := M_1 \cup \{\delta_j : j \in \mathbb{N}\}$. By [8, §3.8, proof of Corollary 2], we can choose a projection π in $\mathbb{C}^n$ of rank k which is transverse to V and to $T_{\gamma,d}V$ for each $d \in M_0$. To show that π is distinguished for V and γ, we define

$$F(w,t,\xi) = P(\gamma(t) + w, \xi; V, \pi) = \sum_{j,\beta,\alpha} a_{j,\beta,\alpha} t^j w^\beta \xi^\alpha, \quad w \in \mathbb{C}^n, t \geq R, \xi \in \mathbb{C}^{n-k},$$

and the *support* M of this expansion as well as its *Newton polygon* (see [6, proof of Proposition 5]), similary as in Section2.3. For each $d \leq 1$ we then expand F into a series of d-quasihomogeneous polynomials (in $w, t^{1/q}$, and ξ)

$$F(w,t,\xi) = F_{\omega_0}(w,t,\xi) + \sum_{\omega<\omega_0} F_\omega(w,t,\xi),$$

where F_ω is either zero or d-quasihomogeneous of order ω and where $F_{\omega_0} = F_{\omega_0(d)}$ is not the zero polynomial.

The *critical values* $1 = d_1 > d_2 > \cdots > d_p > -\infty$ for V and γ (with respect to π) are defined as those $d \geq 1$ for which $F_{\omega_0(d)}(w,1,\xi)$ is inhomogeneous as a polynomial in w and ξ. The proof of [6, Proposition 31] contains a constructive method to determine the critical values from the Newton polygon of F. Just as in the local case, it is not *a priori* obvious that critical values do not depend on the choice of the projection π. However, that will become clear in Corollary 5.10. With these preparations it follows as in Section 2.3 that π is distinguished for V and γ.

Definition 5.7 A simple curve γ in $\mathbb{C}^n$, either at infinity or at the origin, is said to be in *standard parametrization* if it has the form $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n-1}(t), t)$ with

$$\lim_{t\to\infty} \frac{\gamma_j(t)}{t} = 0, \quad 1 \leq j < n.$$

Note that by [5, Lemma 2.5], every simple curve can be reparametrized so that it is in standard parametrization.

Proposition 5.8 *Let $V \subset \mathbb{C}^n$ be an algebraic variety of pure dimension k, assume that the standard coordinates are excellent in the sense of* (5.1)*, and set $\pi(z'', z') = z'$ for $z'' \in \mathbb{C}^{n-k}$ and $z' \in \mathbb{C}^k$. Then π is transverse to $\Psi^*(V)$ at the origin,* i.e., *inequality* (2.1) *holds for $\Psi^*(V)$. Fix $d < 1$ and a simple curve γ at infinity in standard parametrization. Set $\sigma := \Psi \circ \gamma$ and consider the expansions*

$$P(\gamma(t) + z, \xi; V, \pi) = \sum_{\omega\leq\omega_0} F_\omega(z,t,\xi), \tag{5.8}$$

$$P(\sigma(s) + z, \xi; \Psi_*(V), \pi) = \sum_{\omega\geq\omega_1} G_\omega(z,s,\xi)$$

where F_ω is d-quasihomogeneous and G_ω is $(2-d)$-quasihomogeneous of degree ω or zero. We also assume $F_{\omega_0} \not\equiv 0$ and $G_{\omega_1} \not\equiv 0$. Then $\omega_0 = 2m - \omega_1$ and $F_{\omega_0}(z,1,\xi) = G_{\omega_1}(z,1,\xi)$ for all z and ξ.

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Proof The statement concerning transversality was shown in Lemma 5.5. To prove (5.8), consider $z \in \mathbb{C}^n$ with $z_n \neq 0$. Then $z \in \Psi_*(V)$ if and only if $\Psi(z) \in V$, *i.e.*, if there is i such that $z_n^{-2}z'' = \alpha_i(z_n^{-2}z')$ for α_i defined in (2.2). Hence

$$P(z,\xi;\Psi_*(V)) = \prod_{i=1}^{m} \langle z'' - z_n^2\alpha_i(z_n^{-2}z'), \xi\rangle.$$

Since $P(\Psi(z),\xi;V) = \prod_{i=1}^{m}\langle z_n^{-2}z'' - \alpha_i(z_n^{-2}z'),\xi\rangle$, this implies $P(\Psi(z),\xi;V) = z_n^{-2m}P(z,\xi;\Psi^*(V))$. Replacing z by $\Psi(z)$ we arrive at

$$P(z,\xi;V) = z_n^{2m}P(\Psi(z),\xi;\Psi_*(V)).$$

By continuity, this equation holds for all $z \in \mathbb{C}^n$. The remainder of the proof of (5.8) consists of calculations which are strictly analogous to the ones in the proof of [6, Proposition 9(d)]. ■

Corollary 5.9 *Let $V \subset \mathbb{C}^n$ be an algebraic variety of pure dimension $k \geq 1$, and assume that the standard coordinates are excellent in the sense of* (5.1)*. Let γ be a simple curve at infinity with limit vector* $(0,\dots,0,1)$ *and set $\sigma := \Psi \circ \gamma$. Then*

$$T_{\gamma,d}[V] = T_{\sigma,2-d}[\Psi^*(V)]$$

for all $d < 1$.

Proof Similary as in [5, Lemma 2.5], it follows that γ can be reparametrized so that it is in standard parametrization. By [6, Proposition 4], $T_{\gamma,d}[V]$ is not affected by reparametrizations. By [4, Proposition 4.1], also $T_{\sigma,2-d}[\Psi^*(V)]$ is not affected by reparametrizations. Hence we may assume that γ is already in standard parametrization. Since π is distinguished for V and γ, it follows from [6, Proposition 1] that $F_{\omega_0(d)}$ determines $T_{\gamma,d}[V]$ for each $d \leq 1$. By Proposition 5.8, π is transverse to

$$Z(2-d) := \{z \in \mathbb{C}^n : G_{\omega_1(2-d)}(z,1,\xi) = 0 \text{for all } \xi \in \mathbb{C}^{n-k}\}$$

for each $d \leq -1$. By Theorem 2.1, this implies that π is transverse to $T_{\sigma,2-d}\Psi^*(V)$ for each $d \leq 1$. Since π is transverse to $\Psi^*(V)$ at the origin, this shows that π is also distinguished for $\Psi^*(V)$ and σ. Since $F_{\omega_0(d)} = G_{\omega_1(2-d)}$, the result now follows from Corollary 3.2. ■

Corollary 5.10 *Let V be an algebraic variety of pure dimension $k \geq 1$ and let γ be a simple curve at infinity with limit vector* $(0,\dots,0,1)$*. Denote by $1 = \delta_1 > \delta_2 > \cdots > \delta_q$ the critical values for V and γ. Set $\sigma := \Psi\circ\gamma$ and denote by $1 = d_1 < d_2 < \cdots < d_p$ the critical values for $\Psi^*(V)$ and σ. Then $p = q$ and $d_j = 2 - \delta_j$ for $j = 1,\dots,q$.*

In particular, critical values do not depend on the choice of the distinguished projection π.

To state the main theorem of the present section, analogous definitions to (3.1) and (3.3) are needed. For a simple curve γ as in (5.2) let $1 = d_1 > d_2 > \cdots > d_p$ denote the critical values for V and γ. Then set for $1 \le j \le p$

$$I_j := \{w'' \in \mathbb{C}^{n-k} : (w'', 0) \in T_{\gamma,d_j}V, w'' \neq 0\}$$

and

$$\nu_j := \sum_{(w'',0)\in I_j} \mu((w'', 0), T_{\gamma,d_j}[V]). \tag{5.9}$$

(For the definition of the multiplicities $\mu((w'', 0), W)$, see Section2.1).

Theorem 5.11 *Let V be an algebraic variety in $\mathbb{C}^n$ that is of pure dimension $k \ge 1$ and of degree m, let γ be a simple curve, and let π be a projection which is distinguished for V and γ. Then for each $d \le 1$ we have the following.*

(i) *The degree $m(d)$ of the current $T_{\gamma,d}[V]$ is $m - \sum_{d_j>d} \nu_j$. In particular, $m(d) = m(d_{j+1})$ if $d_{j+1} < d < d_j$, $1 \le j < p$, and $T_{\gamma,d}V$ is empty for $d < d_p$ if and only if $m = \sum_{j=1}^{p} \nu_j$.*
(ii) $\omega_0(d) = md - \sum_{d_j>d} \nu_j(d - d_j) = dm(d) + \sum_{d_j>d} \nu_j d_j$.
(iii) *The function Φ defined in Theorem* 5.1 *does not depend on w' and is given by*

$$\Phi(\xi) = \prod_{d_j>d} \prod_{(w'',0)\in I_j} \langle -w'', \xi\rangle^{a(j,w'')}.$$

Hence its degree is $m - m(d) = \sum_{d_j>d} \nu_j$.
(iv) *For Φ as in part* (iii) *we have $F_{\omega_0}(w, 1, \xi) = P(w, \xi; T_{\gamma,d}[V], \pi) \cdot \Phi(\xi)$.*
(v) *If $d < 1$ is a critical value for γ and V, then $T_{\gamma,d}V$ is inhomogeneous.*

Proof The part of the claim that pertains to $d = 1$ is clear. Hence we may assume $d < 1$. Then $T_{\gamma,d}V$ is nonvoid only when the limit vector of γ is in V_h as has been shown [6, Proposition 4(iv)]. So we may assume that γ is in standard parametrization with respect to excellent coordinates. Set $\sigma = \Psi \circ \gamma$. The present result will be proved by applying Theorem 3.1 to $\Psi^*(V)$ and σ.

Statement (i) follows immediately, since the numbers ν_j for V and γ defined in (5.9) coincide with the numbers ν_j for $\Psi^*(V)$ and σ defined in (3.3). Statement (ii) follows in the same way, since in Proposition 5.8 it is shown that $\omega_0 = 2m - \omega_1$ if ω_0 and ω_1 are as in that proposition.

To see (iii), denote the polynomial Φ of Theorem 5.1 for a moment by Φ^∞. Then Proposition 5.8 and Theorems 2.1 and 3.1 imply

$$\begin{aligned} P(w, \xi; T_{\gamma,d}[V])\Phi^\infty(w', \xi) &= F_{\omega_0}(w, 1, \xi) = G_{\omega_1}(w, 1, \xi) \\ &= P(w, \xi; T_{\sigma,2-d}[\Psi^*(V)])\Phi(\xi), \end{aligned}$$

for all $w = (w'', w') \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^{n-k}$. Now let $Q(w', \xi)$ be an irreducible component of $\Phi^\infty(w', \xi)$. By Theorem 5.1, there is no w' such that $Q(w', \xi) = 0$

for all ξ. Hence Q cannot be a factor of $P(w,\xi;T_{\sigma,2-d}[\Psi^*(V)])$ since the latter is a canonical defining function. So Q must be a factor of Φ, which implies that it does not depend on w'.

(iv) is now clear, and (v) is already known from Corollaries 5.9 and 5.10. ■

Corollary 5.12 *For V and γ as in Theorem 5.11, let $1 = d_1 > d_2 > \cdots > d_p$ be the critical values for γ and V. If $d_{j+1} < d < d_j$ for some $j < p$, then*

$$T_0(T_{\gamma,d_j}[V]) = T_{\gamma,d}[V] = (T_{\gamma,d_{j+1}}[V])_h,$$

and if $d < d_p$ and $T_{\gamma,d_p}V \neq \varnothing$, then $T_0(T_{\gamma,d_p}[V]) = T_{\gamma,d}[V]$.

Definition 5.13 The notions of *critical set*, *terminating singularity*, and *normal critical set* have natural analogues in the present situation. The definitions are the same as the ones in Section 4, except that inequalities are reversed and T_0V is replaced by V_h.

If γ is a simple curve at infinity in standard parametrization with respect to excellent coordinates, and $\rho(t) := \gamma(t) + \zeta t^d + o(t^d)$ as $t \to \infty$, then $\sigma := \Psi \circ \rho$ satisfies $\rho(s) = \Psi \circ \sigma(s) + \zeta s^{2-d} + o(s^{2-d})$ as $s \to 0$. Hence the method used to prove Theorem 5.11 can be used to derive the next two results from Proposition 4.3 and Theorem 4.6.

Proposition 5.14 *Let V be an algebraic variety in $\mathbb{C}^n$ which is of pure dimension $k \geq 1$. Let γ be a simple curve at infinity, let $d \leq 1$ be a critical value for γ and V, and let $\zeta \in T_{\gamma,d}V$ be a singular point. Then ζ is a terminating singularity of $T_{\gamma,d}V$ if and only if there is a singular curve $\rho(t) = \gamma(t) + \zeta t^d + o(t^d)$ in V such that, for large t, the multiplicity of V along the curve ρ is constant and equal to the multiplicity of $T_{\gamma,d}[V]$ at ζ.*

Theorem 5.15 *Let V be an algebraic surface in $\mathbb{C}^n$. Then for each $\zeta_1 \in V_h$ with $|\zeta_1| = 1$, the set $\{C = (\zeta_j, d_j)_{j=1}^l : C$ is a normal critical set for $V\}$ is finite.*

Example 5.16 Let $P \in \mathbb{R}[x,y,z]$ be defined as $P(x,y,z) := (y-x)(y^2-z-1)$. and let $V := \{\zeta \in \mathbb{C}^3 : P(\zeta) = 0\}$. Then define the sequence $(a_k)_{k\in\mathbb{N}_0}$ by the Taylor series expansion of the function $(1+s)^{1/2}$ at the origin, *i.e.*, $(1+s)^{1/2} = \sum_{k=0}^{\infty} a_k s^k$. Next define the sequences $(\zeta_j)_{j\in\mathbb{N}}$ and $(d_j)_{j\in\mathbb{N}}$ by

$$\zeta_1 := (0,0,1), \quad d_1 := 1, \quad \zeta_j := (a_{j-2}, a_{j-2}, 0), \quad d_j := \frac{3-2(j-1)}{2},\ j \geq 2.$$

We claim that for each $l \in \mathbb{N}$ the set $C_l := (\zeta_j, d_j)_{j=1}^l$ is critical for V.

To prove our claim by induction, note that $V_h = \{(x,y,z) \in \mathbb{C}^3 : (y-x)y^2 = 0\}$. This shows that $\zeta_1 = (0,0,1)$ is in $(V_h)_{\text{sing}} \cap S^2$. Hence C_1 is a critical set for V. Assume next that for some $l \geq 1$, the set $C_l = (\zeta_j, d_j)_{j=1}^l$ is critical. Then define

$$\sigma_1(t) := 0, \quad \sigma_l(t) := \sum_{j=0}^{l-2} \frac{a_j}{t^j}, \quad \text{and} \quad \gamma_l(t) := \sum_{j=1}^{l} \zeta_j t^{d_j} = (\sqrt{t}\sigma_l(t), \sqrt{t}\sigma_l(t), t).$$

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Then

$$\begin{aligned}
P(\gamma_l(t)+(x,y,z)) &= (y-x)\Big((y+\sqrt{t}\sigma_l(t))^2 - t(1+\frac{1}{t}) - z\Big)\\
&= (y-x)\Big(y^2+2y\sqrt{t}\sigma_l(t)+t\sigma_l(t)^2 - t\Big(\sigma_l(t)+\sum_{j=l-1}^{\infty}\frac{a_j}{t^j}\Big)^2 - z\Big)\\
&= (y-x)\Big(y^2+2y\sqrt{t}\sum_{j=0}^{l-2}\frac{a_j}{t^j} - 2t\Big(\sum_{j=0}^{l-2}\frac{a_j}{t^j}\Big)\Big(\sum_{j=l-1}^{\infty}\frac{a_j}{t^j}\Big)\\
&\qquad - t\Big(\sum_{j=l-1}^{\infty}\frac{a_j}{t^j}\Big)^2 - z\Big).
\end{aligned}$$

Using this expansion, we can compute the Newton diagram. By results from [6] it follows that the largest critical value for γ_l and V which is smaller than d_l is d, where d is determined by the fact that $2y\sqrt{t}$ and $-2ta_{l-1}/t^{l-1}$ have the same d-degree, namely $2-l$. This implies $d = 3/2 - l$ and hence $d_{l+1} = 3/2 - l = (3-2l)/2$. Moreover, we get

$$\begin{aligned}
T_{\gamma_1,d_2}V &= \{(x,y,z)\in\mathbb{C}^3 : (y-x)(y^2-1)=0\},\\
T_{\gamma_l,d_{l+1}}V &= \{(x,y,z)\in\mathbb{C}^3 : (y-x)(y-a_{l-1})=0\} \text{ for } l\ge 2.
\end{aligned}$$

Thus $\zeta_{l+1} := (a_{l-1}, a_{l-1}, 0)$ is a singular point of $T_{\gamma_l,d_{l+1}}V$ which satisfies $\langle\zeta_{l+1},\zeta_1\rangle = 0$. Hence our claim is proved by induction.

Note that the curve $\sigma\colon [0,\infty[\to\mathbb{C}^3, \sigma(t) = (\sqrt{t+1}, \sqrt{t+1}, t)$ satisfies $\operatorname{tr}(\sigma)\subset V_{\text{sing}}$. Therefore, the curves $(\gamma_j)_{j\in\mathbb{N}}$ are just partial sums of the Puiseux series expansion of the curve σ.

Example 5.17 Let $Q\in\mathbb{R}[x,y,z]$ be defined as $Q(x,y,z) := (y-x)(y^2-z^3-z^4)$ and let $V := \{\zeta\in\mathbb{C}^3 : Q(\zeta)=0\}$. Using the sequence $(a_k)_{k\in\mathbb{N}_0}$ from the previous example, we define the sequences $(\xi_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^3$ and $(\delta_j)_{j\in\mathbb{N}}$ in $\mathbb{Q}$ by

$$\xi_1 := (0,0,1),\quad \delta_1 := 1,\quad \xi_j := (a_{j-2}, a_{j-2}, 0),\quad \delta_j := \frac{2(j-1)+1}{2},\ j\ge 2.$$

Then for each $l\in\mathbb{N}$, the set $\mathcal{C}_l := (\xi_j,\delta_j)_{j=1}^{l}$ is critical for V. This follows easily from the previous example by [6, Proposition 9].

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