A method of finding (i) the double points of a unicursal curve, (ii) unicursal quartics with three given double points.

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The equations

$$x=\frac{f_1(t)}{\phi(t)},\ y=\frac{f_2(t)}{\phi(t)},$$

where $f_1(t)$, $f_2(t)$, $\phi(t)$ are polynomials in t, determine a unicursal curve. If α and β can be found so that

$$x-\alpha=\frac{(at^2+2bt+c)\psi_1(t)}{\phi(t)} \text{ and } y-\beta=\frac{(at^2+2bt+c)\psi_2(t)}{\phi(t)},$$

then the point (α, β) is on the curve and is given by two values of t, t_1 and t_2, which are the roots of the equation $at^2+2bt+c=0$. The gradients of the tangents through (α, β) are

$$\underset{t\to t_1}{\mathrm{L}}\ \frac{y-\beta}{x-a} \text{ and } \underset{t\to t_2}{\mathrm{L}}\ \frac{y-\beta}{x-a},$$

or

$$\underset{t\to t_1}{\mathrm{L}}\ \frac{\psi_2(t)}{\psi_1(t)} \text{ and } \underset{t\to t_2}{\mathrm{L}}\ \frac{\psi_2(t)}{\psi_1(t)}.$$

Hence if t_1 and t_2 are real and distinct, there are two distinct tangents through (α, β), and (α, β) is a node. If $t_1=t_2$, the tangents coincide and (α, β) is a cusp. If t_1 and t_2 are imaginary, the tangents are imaginary and (α, β) is a conjugate point.

We have assumed that $\psi_1(t)$ and $\psi_2(t)$ have no common factor. If they had, (α, β) would be a multiple point of higher order than the second and the tangents through it could be found as above.

Since, when (x, y) is a double point,

$$f_1(t)-x\phi(t) \text{ and } f_2(t)-y\phi(t)$$

have a common factor of the form $at^2+2bt+c$, we may find the double points as follows :—regard $f_1(t)-x\phi(t)$ and $f_2(t)-y\phi(t)$ as polynomials in t, and proceed to find their H.C.F. At a certain stage of the process the remainders will be of the form

$$ut^2+vt+w,\ u't^2+v't+w',$$

where u, v, w, u', v', w' are functions of x and y. These remainders must both be multiples of the common factor $at^2+2bt+c$, and therefore

$$\frac{u}{u'}=\frac{v}{v'}=\frac{w}{w'}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Solve these equations for x and y. Any values of x and y which satisfy all three equations give double points. (Of course, we may take instead of $f_1(t)-x\phi(t)$ and $f_2(t)-y\phi(t)$ any legitimate combination of these expressions which will simplify the process of finding the H.C.F.).

Let us consider as an example the curve

$$\frac{x}{a}=\frac{t^2}{t^3+t^2-1},\quad \frac{y}{a}=\frac{-t^4-t^3}{t^3+t^2-1}.$$

Equating the values of t^3+t^2-1, we have

$$xt^2+xt+y=0.$$

We have also

$$\left.\begin{aligned} &xt^2+xt+y=0.\\ &xt^3+(x-a)t-x=0.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Apply the rule for finding the H.C.F. to the expressions in equations (2).

$$\begin{array}{r|c|l} xt^2+xt+y & xt^3+(x-a)t-x & t\\ & xt^3\ +\ xt^2\ +\ yt & \\ \cline{2-2} & -at^2-yt-x & \end{array}$$

Hence for double points

$$\frac{x}{a}=\frac{x}{y}=\frac{y}{x},$$

or

$$x(y-a)=0,\quad x^2=y^2,\quad x^2=ay.$$

The three equations are satisfied at (0, 0), (a, a), $(-a, a)$, and these points are therefore double points. Since x and y have a common factor t^2, (0, 0) is a cusp. The gradient of the tangent there is

$$\underset{t\to 0}{\mathrm{L}}\frac{y}{x}=\underset{t\to 0}{\mathrm{L}}(-t^2-t)=0,$$

hence OX is the tangent at the origin.

Again,
$$x-a=\frac{-a(t-1)(t^2+t+1)}{t^3+t^2-1},$$

$$x+a=\frac{a(t+1)(t^2+t-1)}{t^3+t^2-1},$$

$$y-a=\frac{-a(t^2+t-1)(t^2+t+1)}{t^3+t^2-1}.$$

Hence (a, a) is a conjugate point, being given by $t^2+t+1=0$, and $(-a, a)$ is a node, being given by $t^2+t-1=0$. The gradient of a tangent at the node is

$$\mathrm{L}\frac{y-a}{x+a} = \mathrm{L}\left(-\frac{t^2+t+1}{t+1}\right) \text{ when } t^2+t-1=0,$$

$$= \mathrm{L}\frac{-2}{t+1} = \mathrm{L}(-2t) = 3\cdot2 \text{ or } -1\cdot2.$$

The form of the curve is shown in Fig. 1. The numbers affixed are the values of t, and it is interesting to note in this and the

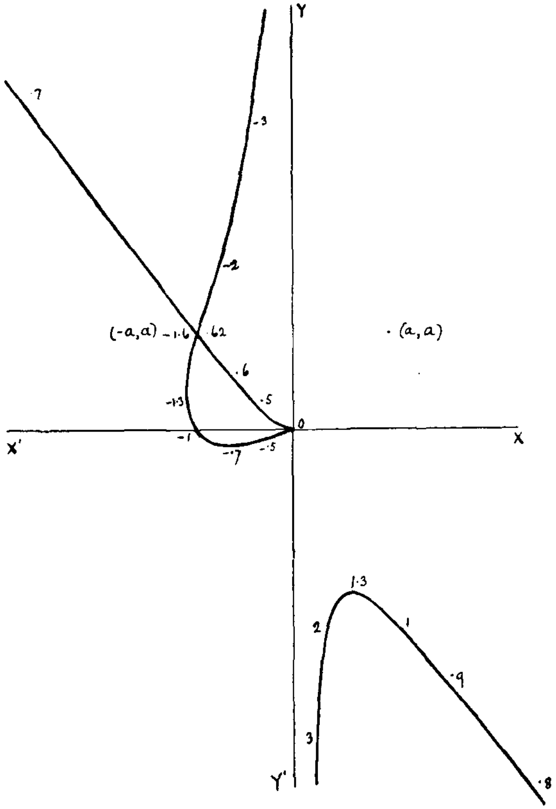

Fig. 1.

following examples how rapidly t varies at some parts of the curve and how slowly at others.

The constraint equation of the curve is found by eliminating t between the freedom equations, or between the equations (2). But since we may combine the expressions in these equations as we did in finding the H.C.F., the above elimination reduces to the elimination of t between the equations

$$xt^2+xt+y=0,\ at^2+yt+x=0, \quad \dots\dots\dots\dots\dots\dots(3)$$

formed by equating to zero the remainders of the second degree in the H.C.F. process. From (3)

$$\frac{t^2}{x^2-y^2}=\frac{t}{x^2-ay}=\frac{1}{x(y-a)},$$

whence the constraint equation is

$$(x^2-ay)^2=x(y-a)(x^2-y^2).$$

It is easy to verify generally that any solutions of *all* the three equations (1) give double points on the curve. As above, the constraint equation of the curve is found by eliminating t from the equations

$$ut^2+vt+w=0,\ u't^2+v't+w'=0,$$

and is therefore

$$(wu'-w'u)^2=(uv'-u'v)(vw'-v'w).$$

Hence any values of x and y which satisfy

$$wu'-w'u=0,\ uv'-u'v=0, \text{ and } vw'-v'w=0$$

give double points. It should be noted that though the three equations (1) are not independent it is possible to find values of x and y which satisfy two of them and do not satisfy the third. For example, any solutions of $u=0$ and $u'=0$ would satisfy the first two and might not satisfy the third. The constraint equation shows that if none of the expressions $wu'-w'u$, $uv'-u'v$, $vw'-v'w$ is a perfect square, (α, β) is a double point only if $x=\alpha$, $y=\beta$ satisfy all three equations (1).

The nature of the double point can be decided by an examination of the discriminants of the factors ut^2+vt+w, $u't^2+v't+w'$. If v^2-4wu is positive when the coordinates of the double point are substituted in u, v, w, the double point is a node; if it is negative, a conjugate point. But if $v^2 - 4wu = 0$ we cannot at once infer that the double point is a cusp. For the coordinates (α, β) of the double point may satisfy the equations $u = v = w = 0$, and in this case the factor $ut^2 + vt + w$ disappears on substituting the coordinates. Hence

$$ut^2 + vt + w \text{ is of the form } \mathrm{F} . (pt^2 + qt + r)$$

where F is a factor which is zero when $x = \alpha$ and $y = \beta$, and the double point is a node, cusp, or conjugate point according as the roots of $pt^2 + qt + r = 0$ are real, equal, or imaginary. (If *both* of the discriminants $v^2 - 4wu$, $v'^2 - 4w'u'$ vanish when $x = \alpha$, $y = \beta$, (α, β) is generally a cusp.)

An example may help to make this clearer. Take the curve

$$x = a(t^3 - 2t),\ y = a(t^4 - 3t^2 + 1).$$

We have for the H.C.F.

$$\begin{array}{r|l|l|l}
t & at^3 - 2at - x & at^4 - 3at^2 + (a - y) & t \\
 & at^3 - xt^2 + t(y - a) & at^4 - 2at^2 - xt & \\
\cline{2-3}
 & t^2x - (y + a)t - x & - at^2 + tx + (a - y) &
\end{array}$$

therefore for double points $\dfrac{x}{a} = \dfrac{a + y}{x} = \dfrac{x}{a - y}$,

or $$x^2 + y^2 = a^2,\ xy = 0, \quad x^2 = a(y + a).$$

Whence we find $(a, 0)$, $(-a, 0)$, $(0, -a)$ are double points; [$(0, a)$ is not a double point, though its coordinates satisfy two of the equations].

The discriminants of the remainders are

$$(a + y)^2 + 4x^2,\ x^2 + 4a(a - y).$$

The first of these vanishes when $x = 0$, $y = -a$, but the second is positive. Since $x^2 = a(y + a)$ we may write the first remainder in the form

$$t^2x - t\frac{x^2}{a} - x$$

and we see that it vanishes on account of the factor x. Divide out by x/a and we get $at^2 - xt - a$, whose discriminant, $x^2 + 4a^2$ is always positive, and hence $(0, -a)$ is a node, and the other double points are also nodes. Fig. 2 shows the form of the curve.

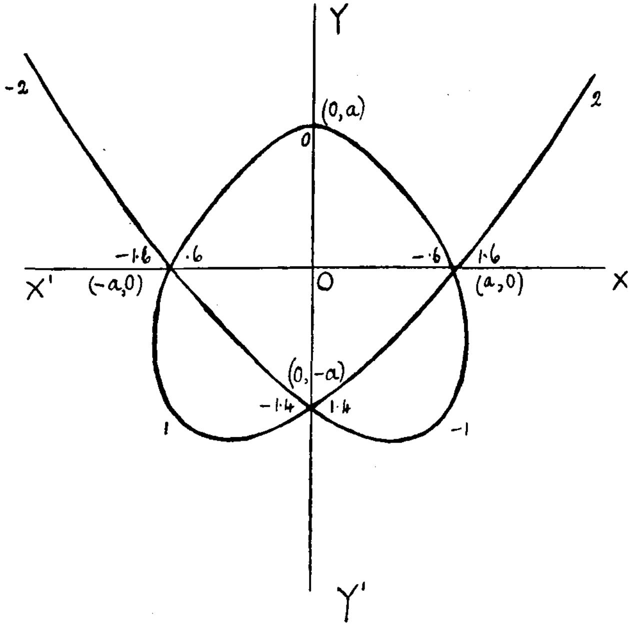

Fig. 2.

The foregoing discussion suggested a means of obtaining unicursal quartic curves with three given points as double points. If we can find two conics through the three points and their equations can be put in the forms $\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'}$, where u, v, w, u', v', w', are linear functions of x and y, then the coordinates of the three given points will generally satisfy all the three equations

$$\frac{u}{u'} = \frac{v}{v'} = \frac{w}{w'}.$$

If therefore we eliminate t from the equations

$$ut^2 + vt + w \;= 0,$$
$$u't^2 + v't + w' = 0,$$

we shall have the constraint equation of a quartic with the given points as double points. If we arrange these equations as equations in x and y and solve, we express x and y as functions of t, and obtain the freedom equations of the curve.

It is easy to find the two conics. We may determine the constants in the equations

$$(x+a)(y+b)=c \qquad \alpha y = x^2+\beta x+\gamma,$$

so that the conics they represent may pass through three given points. The second equation may then be written

$$\alpha(y-\delta)=(x+a)(x+\beta-a),$$

where
$$\alpha\delta=\gamma-a\beta+a^2,$$

and the two equations therefore give

$$\frac{x+a}{c}=\frac{1}{y+b}=\frac{a(y-\delta)}{c(x+\beta-a)}.$$

Thus if we take the points $(a, 0)$, $(-a, 0)$, $(0, -a)$ to be the double points, we may choose $x^2+y^2=a^2$, $x^2=a(y+a)$ as the conics. These may be written $\frac{x}{a+y}=\frac{a-y}{x}=\frac{a}{x}$, which gives the third equation $xy=0$. This is satisfied by the coordinates of the given points.

Consider now

$$xt^2+2kt(a-y)+la=0, \;\ldots\ldots\ldots\ldots\text{(i)}$$
$$(a+y)t^2+2ktx+lx=0, \;\ldots\ldots\ldots\ldots\text{(ii)}$$

where k and l are arbitrary constants.

Eliminate t from these equations and we have

$$l(x^2-ay-a^2)+4k^2xy(x^2+y^2-a^2)=0.$$

Solve for x and y and we get

$$\frac{x}{a}=-\frac{4kt^3+lt^2}{t^4+4kt^2+2klt},\quad \frac{y}{a}=\frac{(l+2kt)^2-t^4}{t^4+4kt^2+2klt}.$$

We have here the constraint and the freedom equations of a unicursal quartic with the three given points as double points.

The discriminant of (i) is $k^2(a-y)^2-lax\equiv D_1$, say,

„ „ „ (ii) „ $k^2x^2-lx(a+y)\equiv D_2$ „

When $x=a,\ y=0,\quad D_1=(k^2-l)a^2,\quad D_2=(k^2-l)a^2$;

„ $x=-a,\ y=0,\quad D_1=(k^2+l)a^2,\quad D_2=(k^2+l)a^2$;

„ $x=0,\ y=-a,\quad D_1=4k^2a^2,\quad D_2=0$.

Hence $(0, -a)$ must be a node, but by choosing $l=k^2$ we can make $(a, 0)$ a cusp, or by $l>k^2$, a conjugate point.

By rearranging the coefficients of t^2, t, and the constant terms in (i) and (ii) in the cyclic order we obtain three curves.

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Greater control over the nature of the double points may be obtained by the introduction of more arbitrary constants. Thus the equations of any conics through the three points $(a, 0)$, $(-a, 0)$, $(0, -a)$ may be written

$$x^2 + lxy = (a+y)(a+my),\ x^2 + \lambda xy = (a+y)(a+\mu y).$$

Whence
$$\frac{x}{a+y} = \frac{a+my}{x+ly} = \frac{a+\mu y}{x+\lambda y}.$$

The third conic is therefore

$$(a+my)(x+\lambda y) = (x+ly)(a+\mu y),$$

or
$$y[x(m-\mu) + y(m\lambda - l\mu) + a(\lambda - l)] = 0.$$

It passes through all three points if

$$l - \lambda + m\lambda - l\mu = 0.$$

Provided that l, m, λ, μ, are chosen to satisfy this equation, they are otherwise at our disposal and we may assign values to them so that the discriminants have given signs, or that the curves may satisfy other conditions.

The following examples have been constructed by the above methods:—

(i)
$$\frac{x}{a} = \frac{3t - 2t^3}{t^4 - 2t^2 + 2},\ \frac{y}{a} = \frac{t^4 - 3t^2 + 1}{t^4 - 2t^2 + 2};$$

$$x^2y^2 = (x^2 + y^2 - a^2)(x^2 + 2y^2 + ay - a^2);\quad \text{(Fig. 3).}$$

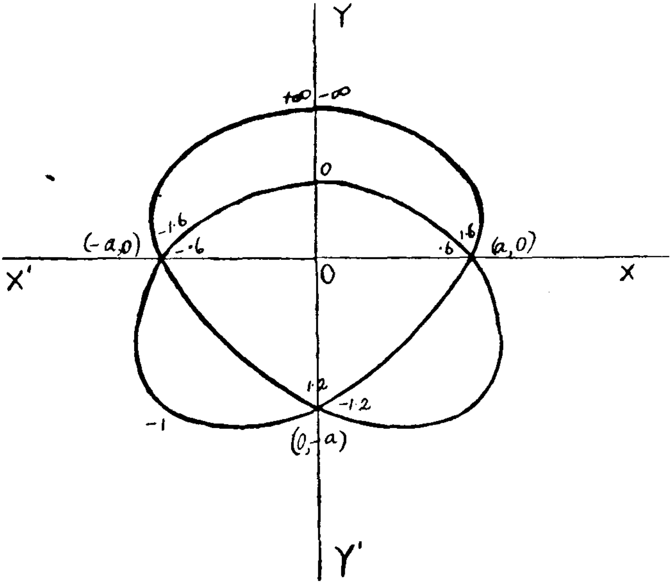

Fig. 3.

(ii) $$\frac{x}{a}=\frac{4t}{t^4+3},\quad \frac{y}{a}=\frac{-2(t^2+1)}{t^4+3}\,;$$

$$x^2(y+a)^2+(x^2-3y^2-2ay)(x^2-y^2)=0\,;\qquad \text{(Fig. 4).}$$

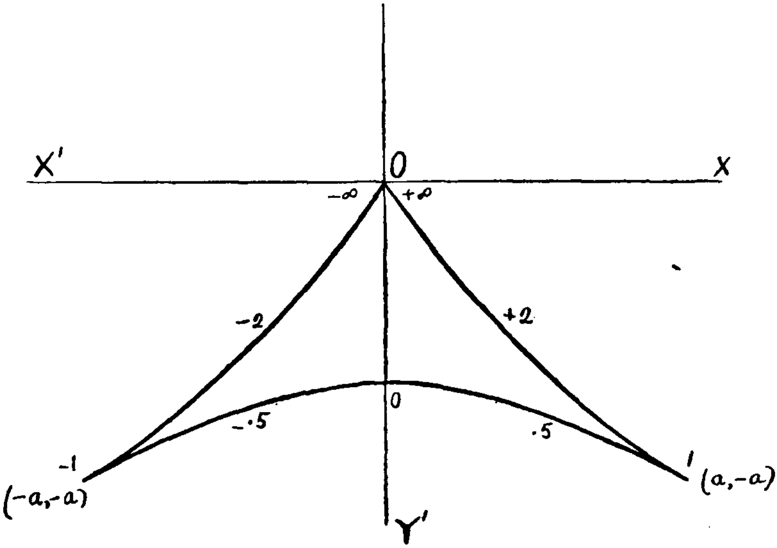

Fig. 4.

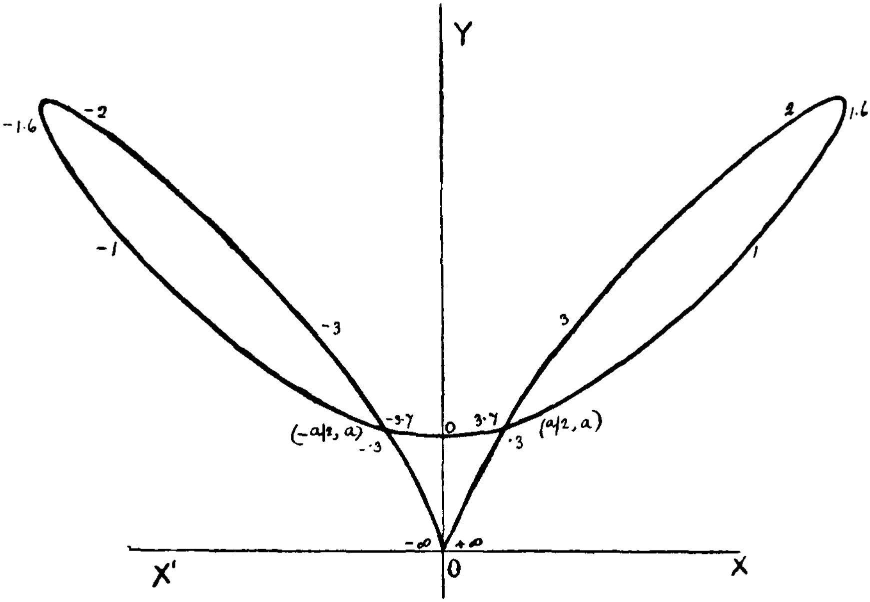

Fig. 5.

(iii) $$\frac{x}{a}=\frac{22t}{t^4-3t^2+12},\quad \frac{y}{a}=\frac{11(t^2+1)}{t^4-3t^2+12};$$

$$121x^2(y-a)^2+4(4x^2-12y^2+11ay)(4x^2-y^2)=0;\quad \text{(Fig. 5).}$$

(iv) $$\frac{x}{a}=\frac{2t(1-2t^2)}{4t^4-3t^2+1},\quad \frac{y}{a}=\frac{-(1+t^2)(1-2t^2)}{4t^4-3t^2+1};$$

$$9x^2(y-a)^2+8(2x^2-y^2-ay)(x^2-2y^2+ay)=0;\quad \text{(Fig. 6).}$$

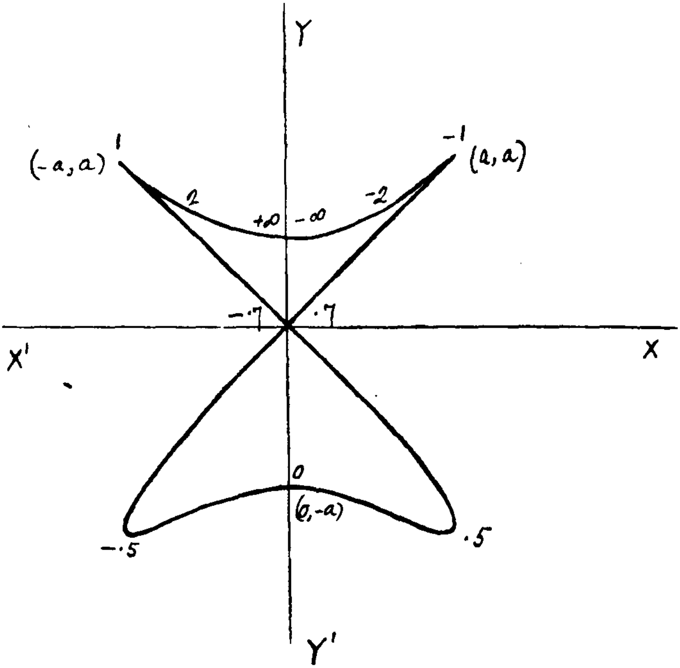

Fig. 6.

(v)
$$\frac{x}{a}=\frac{t-2t^2}{t^4-t^3+1},\quad \frac{y}{a}=\frac{t^4-2t^3+t^2-1}{t^4-t^3+1};$$

$$(x^2+y^2-a^2)^2=xy(x^2-ay-a^2);$$

($(a, 0)$ is a conjugate point on this curve); (Fig. 7).

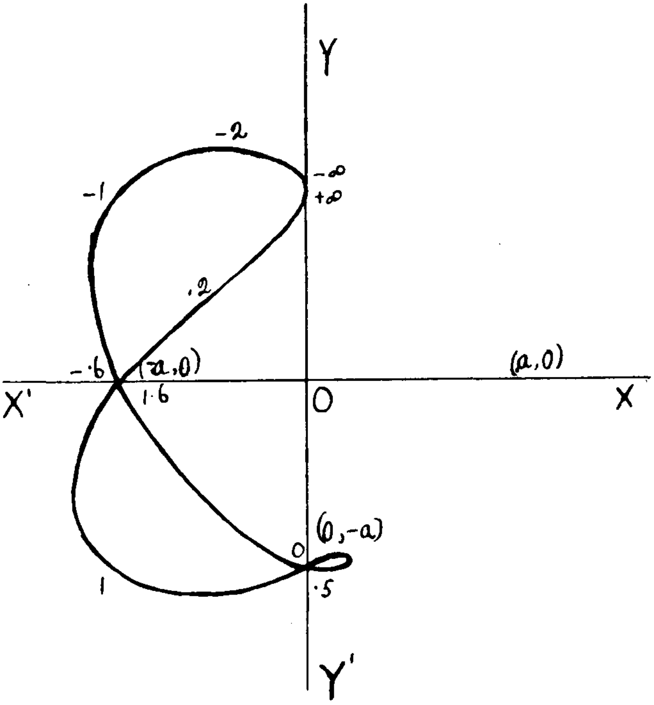

Fig. 7.

(vi)
$$\frac{x}{a}=\frac{-t(3t-2)}{2t^4-4t^3+1},\quad \frac{y}{a}=\frac{t^2(t-2)^2-1}{2t^4-4t^3+1};$$

$$(x^2+2y^2+ay-a^2)^2=8xy(x^2-ay-a^2);\quad \text{(Fig. 8).}$$

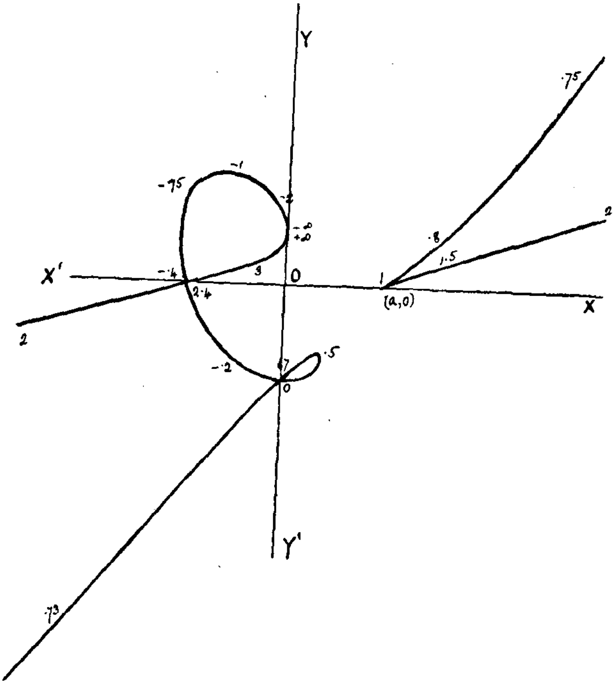

Fig. 8.