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A NOTE ON UNCONDITIONAL BASES

BY
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A sequence (x_i) in a Banach space X is a *Schauder basis* for X provided for each $x \in X$ there is a unique sequence of scalars (a_i) such that

$$\sum_{i=1}^{\infty} a_i x_i = x, \tag{1.1}$$

convergence in the norm topology. It is well known [1] that if (x_i) is a (Schauder) basis for X and (f_i) is defined by

$$f_i(x) = a_i \tag{1.2}$$

where $x = \sum_{i=1}^{\infty} a_i x_i$, then $f_i(x_j) = \delta_{ij}$ and $f_i \in X^*$ for each positive integer i.

A sequence (x_i) is a *basic sequence* in X if (x_i) is a basis for $[x_i]$, where the bracketed expression denotes the closed linear span of (x_i).

A series $\sum_{i=1}^{\infty} y_i$ in a Banach space X is *unconditionally convergent* to y provided

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} y_{p(n)} = y \tag{1.3}$$

for each permutation p of the positive integers.

It is known that each $f \in [f_i]$ has a representation $\sum_{i=1}^{\infty} f(x_i) f_i$ (i.e. (f_i) is a basic sequence).

By $(n_i) \nearrow$ we mean "(n_i) is an increasing sequence of positive integers" and by $\overline{(n_i)}$ we mean $\omega \backslash (n_i)$. Here ω denotes the positive integers.

Also if (x_i) is a basis for X with coefficient functionals (f_i) and if $x = \sum_{i=1}^{\infty} f_i(x) x_i$ we denote by $\bar{x}$, the formal series $\sum_{i=1}^{\infty} |f_i(x)|\, x_i$.

Our first theorem is known. Proofs of the various implications can be found in the references cited below. An excellent general reference is [9].

THEOREM 1. *The following are equivalent:*

(a) *(x_i) is an unconditional basis for X;*

(b) *$\bar{x}$ exists for each $x \in X$ and there is a $K_1 > 0$ such that $\|x\| \le K_1 \|\bar{x}\|$ for each $x \in X$;*

(c) *For each $x \in X$ and $(n_i) \nearrow$, the series $\sum_{i=1}^{\infty} f_{n_i}(x) x_{n_i}$ converges (and there is a $K_2 > 0$ such that*

$$\left\| \sum_{i=1}^{\infty} f_{n_i}(x) x_{n_i} \right\| \le K_2 \left\| \sum_{i=1}^{\infty} f_i(x) x_i \right\| \quad \textit{for each} \quad x \in X$$

and each $(n_i) \nearrow$) and,

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(d) (f_i) *is an unconditional basis for* $[f_i]$.
See [1], [2], [5], [6], [10], [11] and [12].

The purpose of this note is to add an equivalence to the above which appears to be new.

THEOREM 2. *(a) is equivalent to (e): for each subsequence* (x_{n_i}) *of* (x_i) *the associated sequence of coefficient functionals* (g_{n_i}) *in* $[x_{n_i}]^*$ *is similar to* (f_{n_i}) *in* X^* (i.e. $\sum_{i=1}^{\infty} a_i f_{n_i}$ *converges if and only if* $\sum_{i=1}^{\infty} a_i g_{n_i}$ *converges*).

Proof. Clearly g_{n_i} is the restriction of f_{n_i} to $[x_{n_j}]$ for each i. Thus $\sum_{i=1}^{\infty} \alpha_i f_{n_i}$ converges, $\sum_{i=1} \alpha_i g_{n_i}$ converges, since restriction is norm decreasing.

Conversely, if $\sum_{i=1}^{\infty} \alpha_i g_{n_i}$ converges, the convergence is uniform over

$$\left\{ \sum_{i=1}^{\infty} b_i x_{n_i} \in [x_{n_i}] : \| \sum_{i=1}^{\infty} b_i x_{n_i} \| \leq K \right\}$$

for any $K>0$. Since (x_i) is unconditional there is a $K>0$ such that if $\|\sum_{i=1}^{\infty} b_i x_i\| \leq 1$ then $\|\sum_{i=1}^{\infty} b_{n_i} x_{n_i}\| \leq K$ (by (c)). Thus $\sum_{i=1}^{\infty} \alpha_i f_{n_i}$ converges and (f_{n_i}) and (g_{n_i}) are similar.

Conversely if (f_{n_i}) and (g_{n_i}) are similar for $(n_i)\nearrow$ then for $f = \sum_{i=1}^{\infty} \alpha_i f_i \in [f_i]$, there is an $M>0$ such that for all p, q,

$$\left\| \sum_{i=p}^{q} \alpha_{ni} f_{ni} \right\| \leq M \left\| \sum_{i=p}^{q} \alpha_{ni} g_{ni} \right\|$$

(Since the mapping $T(\sum_{i=1}^{\infty} \alpha_{n_i} g_{n_i}) = \sum_{i=1}^{\infty} \alpha_{n_i} f_{n_i}$ is continuous.) But

$$\left\| \sum_{i=p}^{q} \alpha_{ni} g_{ni} \right\| = \operatorname{Sup}\left\{ \left| \sum_{j=p}^{q} \alpha_{ni} f_{ni}(y) \right| : y \in [x_{ni}],\ \|y\| \leq 1 \right\} \leq \left\| \sum_{i=n_p}^{n_q} \alpha_i f_i \right\|$$

and this goes to zero as $p, q \to \infty$. Thus (by (c) and (d)) (x_i) is unconditional.

As mentioned previously, (a)$\Leftrightarrow$(e) seems to be new. However, our proof of the implication (a)$\Rightarrow$(e) has as a corollary a lemma of Kadec and Pelczynski [8]. Our proof is similar to theirs.

We now make some comments on the conditions (b) and (e).

We first consider (b). If (x_i) is an unconditional basis for X it follows from (b) that $(*)\tilde{x} \in X$ for $x \in X$. In the same paper [10] Veic asserts (see [11, p. 437] for the translation) that the non-unconditional basis (x_n) for c_0 (the sequences tending to 0 with the sup norm) given by $\sum_{i=1}^{n} e_i$ where $e_i = (\delta_{ij})_{j=1}^{\infty}$ has property (*), and thus (*) alone does not characterize unconditional bases. While it is true that (*) does not characterize unconditional bases, the example does *not* have property (*). For, clearly there is an $h \in l^1 = c_0^* = \{\xi = (\xi_i) : \|\xi\| = \sum_{i=1}^{\infty} |\xi_i| < +\infty\}$ such that $h(x_n) = 1$ for all n (e.g. $h = e_1$). Let $x = ((-1)^{n+1} n^{-1}) \in c_0$. Then observe that if

$f_n(x_m)=\delta_{nm}, f_n=e_n-e_{n+1}$ and so $|f_n(x)|\ x_n=1/n+1/n+1$. Thus

$$h\left(\sum_{i=1}^{N}|f_n(x)|\ x_n\right)=\sum_{i=1}^{N}\left(\frac{1}{n}+\frac{1}{n+1}\right),$$

so $\sum_{i=1}^{\infty}|f_n(x)|\ x_n$ cannot possibly converge, i.e. $\bar{x}\notin c_0$.

Consider the following example. In l^1 let $y_1=e_1$ and for $n\geq 2$, let $y_n=e_{n-1}-e_n$. Also, let $f_1=(1,1,1,\ldots)\in l^\infty=(l^1)^*$, the bounded sequences with the sup norm. For $n>2$ let $f_n=(0,0,\ldots,0,-1,-1,-1,\ldots)$, where all terms after the $(n-1)$-st are -1. Then (y_n, f_n) is a basis for l^1. Since (x_n, y_{n+1}) are biorthogonal ((x_n) the basis for c_0 above), it follows from (a)$\Leftrightarrow$(d) that (y_n) is a non-unconditional basis for l^1. We show that (*) is satisfied. If $a=(a_i)\in l^1$ then $|f_n(a)|=|\sum_{i=n}^{\infty}a_i|$ if $n\geq 2$ and $|f_1(a)|=|\sum_{i=1}^{\infty}a_i|$. Thus,

$$\left\|\sum_{i=1}^{\infty}|f_i(a)|\ y_i\right\|=\left|\left|\sum_{i=1}^{\infty}a_i\right|+\left|\sum_{i=2}^{\infty}a_i\right|+\left|\sum_{i=2}^{\infty}\left|\left|\sum_{j=i+1}^{\infty}a_j\right|-\left|\sum_{j=i}^{\infty}a_i\right|\right|\right|$$

$$\leq 3\sum_{i=1}^{\infty}|a_i|=3\ \|a\|;$$

$$\text{i.e. } \sum_{i=1}^{\infty}|f_i(a)|\ y_i\in l^1.$$

Finally we consider (e). It is interesting that in the case where (x_i) is not necessarily an unconditional basis the following result is valid.

REMARK. Let (x_i, f_i) be a basis for X with $\inf_n \|x_n\|>0$ such that there is a subsequence (x_{n_i}) which is similar to the unit vector basis (e_i) of c_0. Then (f_{n_i}) is similar to the unit vector basis for l^1.

Proof. If (g_{n_i}) is as in (*e*), then if $\sum_{i=1}^{\infty}a_i g_{n_i}$ converges then $\sum_{i=1}^{\infty}|a_i|<+\infty$. Hence if $\sum_{i=1}^{\infty}a_i f_n$ converges then $\sum_{i=1}^{\infty}|a_i|<+\infty$. Since $\sup_n\|f_n\|<+\infty$ (since $\inf_n\|x_n\|>0$), it follows that (f_{n_i}) is similar to (e_i) in l^1.

It is somewhat surprising that the result is false if the roles of c_0 and l^1 are interchanged. For, if (x_i, h_i) denotes the universal basis [7] of Pelczynski for $C[0, 1]$, the continuous functions on $[0, 1]$ with the sup norm, then there is a subsequence (z_{n_i}) similar to (e_i) in l^1. However (h_{n_i}) cannot be similar to (e_1) in c_0 since c_0 would then be embedded in the weakly sequentially complete space $(C[0, 1])^*$, a contradiction [2].

Also, using the fact that $[h_i]=l^1$, the unit vector basis (e_i) of c_0 may not be replaced by a basis (ω_i) for any reflexive space.

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