PRIMARY IDEALS IN PRÜFER DOMAINS

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Introduction. A Prüfer domain is an integral domain D with the property that for every proper prime ideal P of D the quotient ring D_P is a valuation ring. Examples of such domains are valuation rings and Dedekind domains, a Dedekind domain being merely a noetherian Prüfer domain. The integral closure of the integers in an infinite algebraic extension of the rationals is another example of a Prüfer domain (**5**, p. 555, Theorem 8). This third example has been studied initially by Krull (**4**) and then by Nakano (**8**). Nakano makes free use in his work of the following two properties of such domains D:

(a) D can be expressed as a union of domains, each of which is an integral closure of the integers in a finite algebraic extension of the rationals.

(b) For every proper prime ideal P of D, the quotient ring D_P is a rank 1 valuation ring.

Using Nakano's papers as a guide, we extend most of his results to the case of an arbitrary Prüfer domain, thus showing that property (a) above and the rank 1 assumption of property (b) are not at all essential to much of his work. Some of our results are more complete than Nakano's even in the special case treated by him, but probably more important is the conceptual simplicity of our approach. In fact, we show in §2 that the ideal theory for primary ideals in a Prüfer domain can be reduced to consideration of the ideal theory in a rank 1 valuation ring; and the ideal theory in such a valuation ring is particularly transparent, usually involving nothing more than an examination of the real line.

In §3, we divide the P-primary ideals of D into four classes, and we then apply the reduction process of §2 to study what happens when one performs various operations on the ideals of these classes.

1. Preliminary remarks. We adhere in all respects to the terminology of Zariski–Samuel (**10**), except that we use "ideal" to mean ideal other than the whole ring. In particular, if D is an integral domain and P a prime ideal of D, then D_P denotes the quotient ring of D with respect to P. Also, we use $\subset$ to denote containment and $<$ to denote proper containment.

Let A be an ideal of a domain D, and let P be a minimal prime divisor of A. The ideal $Q = AD_P \cap D$ is then primary (since AD_P has maximal radical) and is called the *isolated primary P-component of A* (**6**, p. 9). Q is thus characterized by the property that it is a primary ideal such that $QD_P = AD_P$.

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As stated in the Introduction, a *Prüfer domain* D is an integral domain such that D_P is a valuation ring for every proper prime ideal P of D. There are many equivalent definitions, e.g. Krull **(5)** or Bourbaki **(1**, exercise 12, p. 93**)**. (Krull uses the name "Multiplikationsring (im weiteren Sinne)" for what we call a Prüfer domain; **(5** p. 554 and footnote 3, p. 546**)**.) From this point on *we shall consistently use the letter D to denote a Prüfer domain.* It follows from the linear ordering of the ideals of D_P that the primary ideals of D which are contained in a given prime ideal are also linearly ordered. By a *P-primary ideal* of D, we mean any primary ideal having radical P, *including P itself.* We use the letters $Q, Q_1, Q_2, \ldots$ to denote P-primary ideals of D.

We also make use of the fact that if R is a valuation ring and P a prime ideal of R, then every non-unit of R_P is actually in R. Thus, if $A \subset P$ is an ideal of R, then $AR_P = AR_P \cap R$; and if $Q \subset P$ is a primary ideal of R, then $QR_P = QR_P \cap R = Q$.

2. Reduction to a rank 1 valuation ring.

2.1. Lemma. *Let R be an arbitrary integral domain, and let A be an ideal of R with prime radical. If AR_M is primary for every maximal ideal M of R containing A, then A is primary.*

Proof. Let $P = \sqrt{A}$ and let $Q = AR_P \cap R$. Then Q is an isolated primary component of A, and hence QR_M is an isolated primary component of AR_M. Therefore, since AR_M is primary by hypothesis, $QR_M = AR_M$. But this is true for every maximal ideal M of R, so $Q = A$ by **(7**, p. 23, Theorem 8.9**)**.

2.2. Theorem. *Let $Q_1, \ldots, Q_n$ be P-primary ideals of D. Then $Q_1 \cdots Q_n$ is also P-primary.*

Proof. If M is a maximal ideal of D containing $Q_1 \cdots Q_n$, then it is sufficient to prove that $(Q_1 \cdots Q_n)D_M = (Q_1 D_M) \cdots (Q_n D_M)$ is primary, by 2.1. Thus, we may assume that D is a valuation ring. Then $Q_i D_P = Q_i$ by the remarks of §1. Therefore

$$Q_1 \cdots Q_n)D_P = (Q_1 D_P) \cdots (Q_n D_P) = Q_1 \cdots Q_n.$$

Since $(Q_1 \cdots Q_n)D_P$ has maximal radical, $(Q_1 \cdots Q_n)D_P$ is primary in D_P. Then

$$Q_1 \cdots Q_n = (Q_1 \cdots Q_n)D_P \cap D$$

is primary in D.

We now introduce the reduction process. To begin, we fix a proper prime ideal P of D, and *we assume that there exists a P-primary ideal $Q < P$.* If P_1 denotes the union of the prime ideals $<P$, then P_1 is a prime ideal; and it follows from the linear ordering of the primary ideals contained in P that P_1 is contained in every P-primary ideal.

Let $i: D \to D_P$ be the injection map; let $h: D_P \to D^*$, where $D^* = D_P/(P_1 D_P)$, be the canonical homomorphism; and let $\tau = h \circ i$. Since the PD_P-primary ideals of D_P all contain $P_1 D_P$, it follows that h maps in a 1–1 way the set of PD_P-primary ideals of D_P onto the set of all non-zero ideals of D^*. Thus, if Q^* denotes the ideal generated by $\tau(Q)$ in D^*, then the correspondence $Q \to Q^*$ is a 1–1 correspondence between the P-primary ideals of D and the P^*-primary ideals of D^*. Moreover, by well-known theorems (**10**, p. 219), this correspondence is an isomorphism with respect to the operation: . By Theorem 2.2 the correspondence is also an isomorphism with respect to formation of products. Thus, $Q_1{}^*Q_2{}^* = (Q_1 Q_2)^*$ and $Q_1{}^*: Q_2{}^* = (Q_1: Q_2)^*$ for any P-primary ideals Q_1, Q_2.

The importance of these observations lies in the fact that D^* is a rank 1 valuation ring. This means that D^* is the ring of a valuation v having value group G (i.e. v maps *onto* G) where G is an additive subgroup of the real numbers (**11**, p. 45). We fix once and for all the valuation v andthe group G.

To prove a theorem involving P-primary ideals, it is thus sufficient to prove the theorem for D^* and then pull back to D via the correspondence described above.

If $v(A^*)$ denotes the image of a non-zero ideal A^* of D^* under the map v, then $v(A^*)$ is an upper class in G. If $A_1{}^*, A_2{}^*$ are two non-zero ideals of D^*, then

$$v(A_1{}^*A_2{}^*) = v(A_1{}^*) + v(A_2{}^*),$$

where

$$v(A_1{}^*) + v(A_2{}^*) = \{v(a_1) + v(a_2) \mid a_i \in A_i{}^*\}.$$

Moreover, A^* is principal if and only if $v(A^*)$ contains a least element. Denoting the greatest lower bound of the elements of $v(A^*)$ by $\inf v(A^*)$, we then have that A^* is principal if and only if $\inf v(A^*) \in v(A^*)$. In particular, it is easily seen that $\inf v(P^*)$ is always in G, so P^* is principal if and only if $\inf v(P^*) > 0$. Further, since P^* is the only proper prime ideal of D^*, P^* is principal if and only if D^* is noetherian, and hence if and only if v is discrete. Equivalently, $(P^*)^2 < P^*$ if and only if v is discrete; cf. (**9**, pp. 10–11). Finally, it is easily seen that the value group G is dense in the reals if and only if $\inf v(P^*) = 0$.

The following lemma will frequently prove useful.

2.3. Lemma. $\inf v(Q_1{}^*Q_2{}^*) = \inf v(Q_1{}^*) + \inf v(Q_2{}^*)$. *If* $Q_1 < Q_2$, then $\inf v(Q_1{}^*: Q_2{}^*) = \inf v(Q_1{}^*) - \inf v(Q_2{}^*)$.

Proof. The first assertion follows from the above remarks. If $(P^*)^2 < P^*$, then $Q_1{}^* = (x_1)$, $Q_2{}^* = (x_2)$ for some $x_i \in D^*$; so $Q_1{}^*: Q_2{}^* = (x_1/x_2)$, and the second assertion is immediate. Assume then that $(P^*)^2 = P^*$. Since $Q_2{}^*(Q_1{}^*: Q_2{}^*) \subset Q_1{}^*$,

$$\inf v(Q_2{}^*) + \inf v(Q_1{}^* : Q_2{}^*) \geqslant \inf v(Q_1{}^*).$$

Therefore

$$\inf v(Q_1^*: Q_2^*) \geqslant \inf v(Q_1^*) - \inf v(Q_2^*).$$

The value group G in this case is dense in the reals. Therefore for every real number $\epsilon > 0$, there exist $x_1, x_2 \in D^*$ such that

$$\inf v(Q_2^*) - \epsilon/2 < v(x_2) < \inf v(Q_2^*),$$

$$\inf v(Q_1^*) < v(x_1) < \inf v(Q_1^*) + \epsilon/2.$$

Then

$$\inf v(x_1/x_2 \cdot Q_2^*) = v(x_1) - v(x_2) + \inf v(Q_2^*) > \inf v(Q_1^*).$$

Thus, $x_1/x_2 \cdot Q_2^* \subset Q_1^*$; so $x_1/x_2 \in Q_1^*: Q_2^*$. (Note that $x_1/x_2 \in D^*$, since

$$v(x_1) > \inf v(Q_1^*) \geqslant \inf v(Q_2^*) > v(x_2).)$$

But also

$$v(x_1/x_2) = v(x_1) - v(x_2) < \inf v(Q_1^*) + \epsilon/2 - (\inf v(Q_2^*) - \epsilon/2)$$
$$= \inf v(Q_1^*) - \inf v(Q_2^*) + \epsilon.$$

This proves the lemma.

As an application of the reduction process, one immediately obtains the following corollary after first checking its validity in the case where D is a rank 1 valuation ring (using 2.3 for the rank 1 case when necessary).

2.4. COROLLARY. *Let Q, Q_1, Q_2 be P-primary ideals of the Prüfer domain D. Then*

(a) $\bigcap_{i=1}^{\infty} Q^i$ *is prime.*
(b) $Q^n = Q^{n+1}$ *for some $n > 0$ implies that $Q = Q^2 = P$.*
(c) $Q \subset Q_1 < P$ *implies that $Q_1^n \subset Q$ for some n.*
(d) $P^2 < P$ *implies that $Q = P^n$ for some n.*
(e) $Q < P$ *implies that $Q^2 < QP$.*
(f) *If $Q_1 < Q_2$, then $Q_1 : Q_2 = Q_1$ implies that $Q_2 = P = P^2$.*

We conclude this section by mentioning some criteria for the existence of a P-primary ideal $Q < P$. If $P^2 < P$, then $Q = P^2$ is a primary ideal with this property (by 2.2). In the case that $P^2 = P$, it is not always true that there exists a P-primary ideal $Q < P$. For example, in **(3**, §5**)**, an example is given of a valuation ring R having a maximal ideal P such that P is the only P-primary ideal. In general, if $P \neq 0$ is a prime ideal of a Prüfer domain, then it is easily seen that there exists a P-primary ideal $Q < P$ if and only if $P \neq \bigcup P_\alpha$, P_α a prime ideal $<P$. From this it follows that if D is a Prüfer domain with ascending chain condition for prime ideals, then for any prime ideal $P \neq 0$, there exists a P-primary ideal $Q < P$.

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3. Classification of primary ideals. If Q is a P-primary ideal of D, then (obviously) one of the following four possibilities occurs:

$$(3.1) \qquad \begin{aligned} &\text{I. } QP < Q < Q : P, \\ &\text{II. } QP < Q = Q : P, \\ &\text{III. } QP = Q < Q : P, \\ &\text{IV. } QP = Q = Q : P. \end{aligned}$$

By the *class* of Q is meant one of these four possibilities. We shall use the notation $Q \in \text{III}$, for example, to mean Q is of class III. Note that P itself is in either class III or I, depending on whether $P^2 = P$ or $P^2 < P$. The existence of primaries of each class in the case of a rank 1 valuation ring follows immediately from the next theorem. Furthermore, using the reduction process of §2, one can easily check that there does not exist a P-primary ideal Q_1 such that $QP < Q_1 < Q$; and similarly, if $Q \neq P$, then there does not exist a Q_1 such that $Q < Q_1 < Q : P$.

The next theorem interprets the above classification in terms of the remarks of §2. Let then Q^* denote an arbitrary P^*-primary ideal of D^* (i.e. Q^* is any non-zero ideal of D^*).

3.2. THEOREM.

$$\begin{aligned} &Q^* \in \text{I} \Leftrightarrow \inf v(Q^*) \in v(Q^*) \quad \textit{and} \quad \inf v(P^*) > 0, \\ &Q^* \in \text{II} \Leftrightarrow \inf v(Q^*) \in v(Q^*) \quad \textit{and} \quad \inf v(P^*) = 0, \\ &Q^* \in \text{III} \Leftrightarrow \inf v(Q^*) \in G \quad \textit{but} \quad \inf v(Q^*) \notin v(Q^*), \\ &Q^* \in \text{IV} \Leftrightarrow \inf v(Q^*) \notin G. \end{aligned}$$

Proof. Apply 2.3.

In the remainder of the paper we list a number of results, most of which can be obtained by reducing to the case where D is a rank 1 valuation ring and then using 2.3 or 3.2. *Where this is merely a routine exercise we omit the proof entirely.*

3.3. COROLLARY. *Q is an isolated primary component of a principal ideal if and only if $Q \in I$.*

3.4. THEOREM. *The following are equivalent:*

(a) *P is an isolated primary component of a principal ideal.*

(b) $P^2 < P$.

(c) *If $\{Q_\lambda\}$ is the set of P-primary ideals $\neq P$, then $\bigcup Q_\lambda < P$.*

(d) $Q \in I$.

(e) *Q is an isolated primary component of a principal ideal and $P = Q : x$ for some $x \in D$.*

Proof.

(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e): routine exercise.

(e) $\Rightarrow$ (a): $QD_P = yD_P$ for some $y \in D$. But $x \notin Q$, so $yD_P < xD_P$; and hence $y/x \in PD_P$. On the other hand, since $P = Q : x$, $x P \subset Q$. Therefore $(xP) D_P \subset QD_P = yD_P$. Hence $PD_P = (y/x) D_P$.

Note that the equivalence of $Q \in I$ and $P^2 < P$ ensures that all P-primary ideals are in class I when one of them is. We shall see in the next theorems that this is not at all the case for primaries of the other classes.

If Q_1, Q_2 are P-primary ideals, then $Q_1 Q_2$ is also P-primary by 2.2; and if $Q_1 < Q_2$, then $Q_1 : Q_2$ is also P-primary. In the next theorem we use the notation "(a) · (b) = (c)" to mean "If Q_1 is of class (a) and Q_2 is of class (b), then $Q_1 Q_2$ is of class (c)." We use similar notation for the operation : in Theorem 3.6.

3.5. THEOREM.

II · II = II, IV · II = IV,
II · III = III, IV · III = IV,
III · III = III, IV · IV = III *or* IV.

3.6. THEOREM. *Assume* $Q_1 < Q_2$. *Then*

II : II = II, III : II = III, IV : II = IV,
II : III = II, III : III = II, IV : III = IV,
II : IV = IV, III : IV = IV, IV : IV = II *or* IV.

Using 3.2, it is an easy matter to construct examples in a rank 1 valuation ring to show that IV · IV = III or IV and IV: IV = II or IV can all actually occur.

3.7. THEOREM. *If* $Q^n \in$ I, II, *or* IV, *then* $Q^{n+k}: Q^k = Q^n$. *If* $Q^n \in$ III, *then* $Q^n: P = Q^{n+k}: Q^k > Q^n$ *(n, k integers* $\geqslant 1$*)*.

Proof. $Q^n: P \supset Q^{n+k}: Q^k \supset Q^n$ is immediate (in the rank 1 case). The first assertion now follows from 3.1 and the second from 3.6.

We conclude with an application to factors. If $Q < Q_1$, then Q_1 is said to be a *factor* of Q if there exists a P-primary ideal Q_2 such that $Q = Q_1 Q_2$.

3.8. LEMMA. *Let A, B, C be ideals of a ring R. Then $A = BC$ implies $A = B(A : B)$.*

Proof. $A = BC$ implies $C \subset A : B$. Therefore

$$A = BC \subset B(A : B) \subset A.$$

It is not necessarily true, even for a Prüfer domain, that $C = A : B$, as we shall see. In fact, it is easily seen that the statement "$A = BC$ implies $C = A : B$" holds for all ideals A, $B \neq 0$, C of a ring R if and only if the cancellation law for ideals is valid in R; and Gilmer **(2)** has shown that when R

is a domain, the cancellation law is equivalent to the assertion that every R_P is a noetherian valuation ring.

3.9. THEOREM. *Suppose that* $QQ_1 = QQ_2$. *If* $Q_1 < Q_2$, *then* $Q_2 \in$ II, $Q_1 \in$ III, *and* $Q_1 = Q_2P$.

3.10. COROLLARY. *If* $Q = Q_1 Q_2$, *then* $Q_2 = Q : Q_1$ *or* $Q_2 = (Q : Q_1)P$.

Proof. Clearly $Q_2 \subset Q : Q_1$. On the other hand, $Q = Q_1 Q_2$ implies that $Q = Q_1(Q : Q_1)$ by 3.8. Therefore, $Q_1 Q_2 = Q_1(Q : Q_1)$. If $Q_2 < Q : Q_1$, then $Q_2 = (Q : Q_1)P$ by 3.9.

3.11. THEOREM. *Assume that* $Q < Q_1$. *If* $Q \in$ I, III, *or* IV, *then* Q_1 *is a factor of* Q. *If* $Q \in$ II, *then* Q_1 *is a factor of* Q *if and only if* $Q_1 \in$ II.

In Theorem 3.11, one can actually say more about the missing factor Q_2 by using the product theorem (3.5). For example, if $Q = Q_1 Q_2$ and $Q \in$ II, $Q_1 \in$ II, then $Q_2 \in$ II also, etc. Also 3.10 tells us that if $Q = Q_1 Q_2$, then the factor Q_2 is either $Q : Q_1$ or $(Q : Q_1)P$; and 3.8 tells us that Q_2 may always be chosen to be $Q : Q_1$. If $Q \in$ I, it follows from reduction to the rank 1 case that $Q_2 \neq (Q : Q_1)P$. Similarly, by using the product and colon theorems (3.5 and 3.6), one sees immediately that when $Q \in$ II, then $Q : Q_1$ is again the only possibility for Q_2. On the other hand, it again follows from reduction to the rank 1 case that when $Q \in$ III or IV, then $(Q : Q_1)P$ is always a possible choice for Q_2.

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