Proceedings of the Edinburgh Mathematical Society (2005) **48**, 743–755 ©
DOI:10.1017/S0013091502000020 Printed in the United Kingdom

WEIGHTED LIPSCHITZ CONTINUITY AND HARMONIC BLOCH AND BESOV SPACES IN THE REAL UNIT BALL

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(Received 10 January 2002)

Abstract The characterization by weighted Lipschitz continuity is given for the Bloch space on the unit ball of $\mathbb{R}^n$. Similar results are obtained for little Bloch and Besov spaces.

Keywords: harmonic function theory; Bloch space; Berov space; weighted Lipschitz continuity

2000 *Mathematics subject classification:* Primary 42B35
Secondary 31B05; 32A50; 32C50

1. Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$ with $n \geqslant 2$, where $\mathrm{d}\nu$ is the normalized volume measure on $\mathbb{B}$ and $\mathrm{d}\sigma$ is the normalized surface measure on the unit sphere $S = \partial\mathbb{B}$.

In [**9**], Krantz gave the following extension of a result of Hardy and Littlewood for holomorphic functions on the unit disc (see also [**13**] and [**8**]).

Theorem 1.1. *Let f be a harmonic function on $\mathbb{B}$ and $0 < \alpha \leqslant 1$. Then the following are equivalent.*

(i) $|\nabla f(x)| \leqslant C(1-|x|^2)^{\alpha-1}$ *for any* $x \in \mathbb{B}$.

(ii) $|f(x) - f(y)| \leqslant C|x-y|^{\alpha}$ *for any* $x \in \mathbb{B}$ *and* $y \in \mathbb{B}$.

In light of this result, it is natural to consider the limit case $\alpha = 0$.

The *harmonic Bloch space* $\mathcal{B}$ is the space of all harmonic functions on $\mathbb{B}$ for which

$$\sup_{x\in\mathbb{B}}(1-|x|^2)|\nabla f(x)| < \infty,$$

and the *harmonic little Bloch space* $\mathcal{B}_0$ consists of the functions $f \in \mathcal{B}$ such that

$$\lim_{|x|\to 1}(1-|x|^2)|\nabla f(x)| = 0.$$

The *harmonic Besov space* $\mathcal{B}_p$ is the space of all harmonic functions on $\mathbb{B}$ for which

$$\int_{\mathbb{B}} (1-|x|^2)^p |\nabla f(x)|^p \, \mathrm{d}\tau(x) < \infty,$$

where

$$\mathrm{d}\tau(x) = (1-|x|^2)^{-n} \, \mathrm{d}\nu(x)$$

is the invariant measure on $\mathbb{B}$.

Let f be a continuous function in $\mathbb{B}$. If there exists a constant C such that

$$(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}|f(x)-f(y)| \leqslant C|x-y| \tag{1.1}$$

for any $x, y \in \mathbb{B}$, then we say that f satisfies *a weighted Lipschitz condition*.

The main purpose of this paper is to give some characterizations of $\mathcal{B}$, $\mathcal{B}_0$ and $\mathcal{B}_p$ by means of a weighted Lipschitz condition. We refer to [**3**, **4**, **7**, **10**, **14**] for the corresponding results in the complex unit ball for holomorphic or $\mathcal{M}$-harmonic functions. See [**6**, **11**, **15**, **17**–**19**] for the various characterizations of the Bloch, little Bloch and Besov spaces in the unit ball of $\mathbb{C}^n$. Our main results are the following theorems.

Theorem 1.2. *Let f be a harmonic function on $\mathbb{B}$. Then the following are equivalent.*

(i) *$f \in \mathcal{B}$.*

(ii) *f satisfies the weighted Lipschitz condition.*

Theorem 1.3. *Let f be harmonic on $\mathbb{B}$. Then $f \in \mathcal{B}_0$ if and only if*

$$\lim_{|x|\to 1^-} \sup\left\{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|} : y \in \mathbb{B},\ y \neq x\right\} = 0.$$

Theorem 1.4. *Let $p \in (2(n-1), \infty)$. For any harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}} (1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^p \mathrm{d}\tau(x)\,\mathrm{d}\tau(y) < \infty.$$

Remark 1.5. Let us remark on the validity of theorem 1.4 that if $p \in (1, 2(n-1))$, then the integral condition forces the function to be constant, a fact which is already known in the complex case ($p \in (1,2)$) (see [**16**]).

2. Preliminaries

We shall be using the following notation: we will write $x, y \in \mathbb{R}^n$ in polar coordinates by $x = |x|x'$ and $y = |y|y'$.

For any $y, w \in \mathbb{R}^n$, the symmetry lemma shows that (see [**2**])

$$||y|w - y'| = ||w|y - w'|. \tag{2.1}$$

https://doi.org/10.1017/S0013091502000020 Published online by Cambridge University Press

The same deduction yields

$$||y|w - (1 - |w|^2)y'| = ||w|y - (1 - |w|^2)w'|,$$

so that

$$||y|^2 w - (1 - |w|^2)y| = |y|\,||w|y - (1 - |w|^2)w'|. \tag{2.2}$$

For any $a \in \mathbb{B}$, denote by φ_a the Möbius transformation in $\mathbb{B}$. It is an involutive automorphism of $\mathbb{B}$ such that $\varphi_a(0) = \lim_{x\to 0} \varphi_a(x) = a$ and $\varphi_a(a) = 0$, which is of the form (see [**1**])

$$\varphi_a(x) = \frac{|x - a|^2 a - (1 - |a|^2)(x - a)}{||x|a - x'|^2}, \quad a, x \in \mathbb{B}. \tag{2.3}$$

From (2.2) with $w = a$ and $y = x - a$, we have

$$|\varphi_a(x)| = \frac{|x - a|}{||a|x - a'|}, \tag{2.4}$$

whence

$$1 - |\varphi_a(x)|^2 = \frac{(1 - |x|^2)(1 - |a|^2)}{||a|x - a'|^2}. \tag{2.5}$$

For any $a \in \mathbb{B}$ and $\delta \in (0, 1)$, we define

$$E(a, \delta) = \{x \in \mathbb{B} : |\varphi_a(x)| < \delta\},$$
$$B(a, \delta) = \{x \in \mathbb{B} : |x - a| < \delta\}.$$

Clearly, $E(a, \delta) = \varphi_a(B(0, \delta))$.

Lemma 2.1. *Let $x, w \in \mathbb{B}$ and $y \in E(w, \delta)$. Then*

$$\frac{1 - \delta}{1 + \delta}||x|w - x'| \leqslant ||x|y - x'| \leqslant \frac{1 + \delta}{1 - \delta}||x|w - x'|.$$

Proof. From (2.4) and (2.1) we have $|\varphi_y(w)| = |\varphi_w(y)|$, so that $y \in E(w, \delta)$ is equivalent to $w \in E(y, \delta)$. By symmetry, we only have to prove the right-hand inequality. Since $||x|y - x'| \leqslant ||x|(y - w)| + ||x|w - x'|$, it is enough to show that

$$|y - w| \leqslant \frac{2\delta}{1 - \delta}||x|w - x'|$$

for any $y \in E(w, \delta)$.

Define $\eta = \varphi_w(y)$, then $y = \varphi_w(\eta)$ and $|\eta| < \delta$. From (2.3), a direct computation yields

$$|\varphi_w(\eta) - w| = \frac{|\eta|}{||w|\eta - w'|}(1 - |w|^2).$$

Therefore, by the simple inequality $1 - |w| \leqslant ||x|w - x'|$ we get

$$|y - w| = |\varphi_w(\eta) - w| \leqslant \frac{\delta}{1 - \delta}(1 - |w|^2) \leqslant \frac{\delta}{1 - \delta}2||x|w - x'|,$$

as desired. □

As a direct corollary, we have

$$1-|x|^2 \simeq 1-|y|^2, \quad x \in E(y,\delta). \tag{2.6}$$

In fact, taking $w = x$ in Lemma 2.1 we get $||x|y - x'| \simeq 1 - |y|^2$. The assertion now follows from (2.1).

Lemma 2.2. *Let $0 < \delta < \lambda$. Then, for any points x and y in $\mathbb{B}$,*

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|t|y|x-y'|^{\lambda}}\,\mathrm{d}t \leqslant \frac{8^{\lambda}}{\delta(\lambda-\delta)}\frac{1}{||y|x-y'|^{\lambda-\delta}}.$$

Proof. Note that for any $t \in [0,1]$ and $x, y \in \mathbb{B}$,

$$||y|x-y'| \leqslant 2|t|y|x-y'|. \tag{2.7}$$

Indeed, from the triangle inequality we have

$$\left.\begin{aligned} |t|y|x-y'| &\geqslant 1-t, \\ |t|y|x-y'| &\geqslant ||y|x-y'|-(1-t), \end{aligned}\right\} \tag{2.8}$$

so that summing up yields (2.7).

If $||y|x-y'| \geqslant 1$, then we have $|t|y|x-y'| \geqslant \frac{1}{2}$ from (2.7). Combining this with the inequality $||y|x-y'| \leqslant 2$, we get

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|t|y|x-y'|^{\lambda}}\,\mathrm{d}t \leqslant \frac{2^{\lambda}}{\delta} \leqslant \frac{2^{2\lambda-\delta}}{\delta}\frac{1}{||y|x-y'|^{\lambda-\delta}}.$$

Now assume that $||y|x-y'| < 1$ and define $r = 1-||y|x-y'|$, then $0 < r < 1$ and $1-r = ||y|x-y'|$. From (2.7) and (2.8) we have

$$1-rt = 1-t+t||y|x-y'| \leqslant 3|t|y|x-y'|.$$

It leads to

$$\begin{aligned} \int_0^1 \frac{(1-t)^{\delta-1}}{|t|y|x-y'|^{\lambda}}\,\mathrm{d}t &\leqslant 3^{\lambda}\int_0^1 \frac{(1-t)^{\delta-1}}{(1-tr)^{\lambda}}\,\mathrm{d}t \\ &\leqslant 3^{\lambda}\frac{\lambda}{\delta(\lambda-\delta)}\frac{1}{(1-r)^{\lambda-\delta}} \\ &\leqslant C\frac{1}{||y|x-y'|^{\lambda-\delta}}. \end{aligned}$$

This completes the proof. □

Let F be the hypergeometric function (see [**5, 12**])

$$F(a,b;c;s) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}s^k$$

for $a, b, c \in \mathbb{R}$ and c neither zero nor a negative integer, where the Pochhammer symbol $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k \in \mathbb{N}$. We need some known properties of hypergeometric functions:

(i) Bateman's integral formula (see [**12**])

$$F(a,b;c+\mu;s) = \frac{\Gamma(c+\mu)}{\Gamma(c)\Gamma(\mu)} \int_0^1 t^{c-1}(1-t)^{\mu-1} F(a,b;c;ts)\,\mathrm{d}t \tag{2.9}$$

with $c, \mu > 0$ and $s \in (-1,1)$; and

(ii) for any integer m [**12**, p. 69],

$$\left.\begin{aligned} F(-m,b;c;1) &= \frac{(c-b)_m}{(c)_m}, \\ F(-m,a+m;c;1) &= \frac{(-1)^m(1+a-c)_m}{(c)_m}. \end{aligned}\right\} \tag{2.10}$$

The following identity furnishes the hypergeometric function with an integral representation.

Lemma 2.3. *Let $t > 1$, $\lambda \in \mathbb{R}$ and $r \in (-1,1)$, then*

$$\int_{-1}^1 \frac{(1-u^2)^{(t-3)/2}}{(1-2ru+r^2)^\lambda}\,\mathrm{d}u = \frac{\Gamma(\frac{1}{2}(t-1))\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}t)} F(\lambda, \lambda+1-\tfrac{1}{2}t; \tfrac{1}{2}t; r^2). \tag{2.11}$$

Proof. Let $C_m^\lambda(u)$ be the Gegenbauer polynomials. They can be defined by the generating function

$$(1-2ru+r^2)^{-\lambda} = \sum_{m=0}^\infty C_m^\lambda(u) r^m, \tag{2.12}$$

where

$$\left.\begin{aligned} C_{2m}^\lambda(u) &= (-1)^m \frac{(\lambda)_m}{m!} F(-m, m+\lambda; \tfrac{1}{2}; u^2), \\ C_{2m+1}^\lambda(u) &= (-1)^m \frac{(\lambda)_m}{m!} 2uF(-m, m+\lambda+1; \tfrac{3}{2}; u^2). \end{aligned}\right\} \tag{2.13}$$

To calculate the integral in (2.11), we apply (2.12) and (2.13). Then we deduce that it is only needed to evaluate the integral

$$\int_{-1}^1 (1-u^2)^{(t-3)/2} F(-m, m+\lambda; \tfrac{1}{2}; u^2)\,\mathrm{d}u,$$

or, rather, an integral over the interval $(0,1)$ by the simple change of variables $t = u^2$. For this integral, we first use Bateman's integral formula (2.9) with $s = 1$ and apply (2.10), so that it can be represented by Pochhammer symbols. The calculation of the integral in (2.11) then leads to a series which by definition is the desired hypergeometric function. □

Lemma 2.4. *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then, for any $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{||x|y - x'|^{n+\alpha+\beta}}\,\mathrm{d}y \approx \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log \dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}$$

The notation $a(x) \approx b(x)$ means that the ratio $a(x)/b(x)$ has a positive finite limit as $|x| \to 1$.

Proof. Denote the above integral by $J_{\alpha,\beta}(x)$. From Stirling's formula we need only show that

$$J_{\alpha,\beta}(x) = \frac{\Gamma(\frac{1}{2}n+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2}n+1)} F(\tfrac{1}{2}(n+\alpha+\beta), \tfrac{1}{2}(2+\alpha+\beta); \alpha+\tfrac{1}{2}n+1; |x|^2).$$

For any continuous function f of one variable and any $\eta \in \partial\mathbb{B}$, we have the formula (see [**2**, p. 216])

$$\int_{\partial\mathbb{B}} f(\langle\zeta,\eta\rangle)\,\mathrm{d}\sigma(\zeta) = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}(n-1))\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-u^2)^{(n-3)/2} f(u)\,\mathrm{d}u,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. Taking

$$f(u) = (1-2ru+r^2)^{-(n+\alpha+\beta)/2}$$

for fixed $r \in (0,1)$ and combining with Lemma 2.3 we have

$$\begin{aligned} \int_{\partial\mathbb{B}} (1-2r\langle\zeta,\eta\rangle+r^2)^{-(n+\alpha+\beta)/2}\,\mathrm{d}\sigma(\zeta) \\ &= \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}(n-1))\Gamma(\frac{1}{2})} \int_{-1}^{1} \frac{(1-u^2)^{(n-3)/2}}{(1-2ru+r^2)^{(n+\alpha+\beta)/2}}\,\mathrm{d}u \\ &= F(\tfrac{1}{2}(n+\alpha+\beta), \tfrac{1}{2}(2+\alpha+\beta); \tfrac{1}{2}n; r^2). \end{aligned}$$

Consequently, from the polar coordinates formula we get

$$\begin{aligned} J_{\alpha,\beta}(x) &= n\int_0^1 r^{n-1}(1-r^2)^\alpha \int_S (1-2r|x|\langle x',\eta\rangle + r^2|x|^2)^{-(n+\alpha+\beta)/2}\,\mathrm{d}\sigma(\zeta) \\ &= C\int_0^1 r^{n-1}(1-r^2)^\alpha F(\tfrac{1}{2}(n+\alpha+\beta), \tfrac{1}{2}(2+\alpha+\beta); \tfrac{1}{2}n; r^2|x|^2)\,\mathrm{d}r. \end{aligned}$$

The assertion now follows from Bateman's integral formula (2.9). □

3. Bloch space

In this section we give the proof of Theorems 1.2 and 1.3. Theorem 1.2 can be rewritten in the following form.

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Theorem 3.1. *For any harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}$ if and only if*

$$\sup\left\{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|} : x, y \in \mathbb{B},\ x \neq y\right\} < \infty. \tag{3.1}$$

Proof. Assume that $f \in \mathcal{B}$. For any $x, y \in \mathbb{B}$, we have

$$\begin{aligned} f(x)-f(y) &= \int_0^1 \frac{\mathrm{d}f}{\mathrm{d}t}(tx+(1-t)y)\,\mathrm{d}t \\ &= \sum_{k=1}^{n}(x_k-y_k)\int_0^1 \frac{\partial f}{\partial x_k}(tx+(1-t)y)\,\mathrm{d}t. \end{aligned}$$

By the Cauchy–Schwarz inequality and the simple inequality

$$\left|\frac{\partial f}{\partial x_k}\right| \leqslant |\nabla f|,$$

we have

$$\begin{aligned} |f(x)-f(y)| &\leqslant \sqrt{\sum_{k=1}^{n}|x_k-y_k|^2}\sqrt{\sum_{k=1}^{n}\left(\int_0^1\left|\frac{\partial f}{\partial x_k}(tx+(1-t)y)\right|\mathrm{d}t\right)^2} \\ &\leqslant |x-y|\sqrt{n}\int_0^1 |(\nabla f)(tx+(1-t)y)|\,\mathrm{d}t. \end{aligned}$$

Since $(1-|tx+(1-t)y|^2)|(\nabla f)(tx+(1-t)y)| \leqslant \|f\|_{\mathcal{B}}$, it follows that

$$\frac{|f(x)-f(y)|}{|x-y|} \leqslant \sqrt{n}C\|f\|_{\mathcal{B}}\int_0^1 \frac{1}{1-|tx+(1-t)y|^2}\,\mathrm{d}t.$$

Now, by the triangle inequality $|tx+(1-t)y| \leqslant t|x|+(1-t)|y|$, we have

$$1-|tx+(1-t)y| \geqslant (1-t)(1-|x|)$$

and

$$1-|tx+(1-t)y| \geqslant t(1-|y|),$$

so that

$$\begin{aligned} \int_0^1 \frac{1}{1-|tx+(1-t)y|}\,\mathrm{d}t &\leqslant \int_0^1 \frac{1}{\{(1-t)(|1-|y|)\}^{1/2}}\frac{1}{\{t(|1-|x|)\}^{1/2}}\,\mathrm{d}t \\ &= \pi\frac{1}{(1-|x|)^{1/2}(1-|y|)^{1/2}}. \end{aligned}$$

Thus

$$(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|z-w|} \leqslant 2\pi\sqrt{n}C\|f\|_{\mathcal{B}}. \tag{3.2}$$

This proves the necessity.

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Conversely, suppose that f is harmonic and that (3.1) is satisfied. We will show that $f \in \mathcal{B}$.

Fix $\delta \in (0,1)$. Since f is harmonic, combining the result in [**11**, p. 504] with (2.7) we get

$$(1-|x|^2)|\nabla f(x)| \leqslant C \int_{E(x,\delta)} |f(y)| \,\mathrm{d}\tau(y).$$

Fixing $x \in \mathbb{B}$ and replacing f by $f - f(x)$, we have

$$(1-|x|^2)|\nabla f(x)| \leqslant C \int_{E(x,\delta)} |f(y) - f(x)| \,\mathrm{d}\tau(y). \tag{3.3}$$

Therefore,

$$(1-|x|^2)|\nabla f(x)| \leqslant C \sup\{|f(y) - f(x)| : w \in E(x,\delta)\}.$$

Note that for any $y \in E(x,\delta)$ we have $|\varphi_x(y)| \leqslant \delta$, and thus

$$\frac{\sqrt{1-|\varphi_x(y)|^2}}{|\varphi_x(y)|} \geqslant C.$$

It follows from Lemma 2.1 that

$$\frac{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}{|y-x|} \geqslant C, \quad y \in E(x,\delta). \tag{3.4}$$

Consequently,

$$(1-|x|^2)|\nabla f(x)| \leqslant C \sup\left\{\frac{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}{|y-x|}|f(y)-f(x)| : y \in E(x,\delta)\right\}, \tag{3.5}$$

in which the right-hand side can be controlled by the condition in (3.1). This implies $f \in \mathcal{B}$ and completes the proof of Theorem 3.1. □

Theorem 3.2. *For any harmonic function f on $\mathbb{B}$, $f \in \mathcal{B}_0$ if and only if*

$$\lim_{|x|\to 1^-} \sup\left\{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|} : y \in \mathbb{B},\ y \neq x\right\} = 0. \tag{3.6}$$

Proof. Assume that $f \in \mathcal{B}_0$. Let $f_t(x) = f(tx)$, $t \in (0,1)$. By (3.1), we have

$$(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|(f-f_t)(x)-(f-f_t)(y)|}{|x-y|} \leqslant C\|f-f_t\|_{\mathcal{B}}$$

and

$$\begin{aligned}(1-|x|^2)^{1/2}&(1-|y|^2)^{1/2}\frac{|f_t(x)-f_t(y)|}{|x-y|} \\ &= t\frac{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}{(1-|tx|^2)^{1/2}(1-|ty|^2)^{1/2}}(1-|tx|^2)^{1/2}(1-|ty|^2)^{1/2}\frac{|f(tx)-f(ty)|}{|tx-ty|} \\ &\leqslant C\frac{t}{(1-t^2)^{1/2}}(1-|x|^2)^{1/2}\|f\|_{\mathcal{B}}.\end{aligned}$$

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By the triangle inequality, we obtain

$$\sup\left\{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|} : y \in \mathbb{B},\ y \neq x\right\}$$
$$\leqslant C\frac{t}{1-t^2}(1-|x|^2)^{1/2}\|f\|_{\mathcal{B}} + \|f-f_t\|_{\mathcal{B}}.$$

In the above inequality, $t \to 0$ as $|x| \to 1^-$ the first term on the right-hand side converges to 0, and afterwards if $t \to 1^-$, then the second term on the right-hand side also converges to 0, which is similar to the case $n = 2$ which can be found in [**20**].

Now suppose that f is harmonic and (3.6) is satisfied. We will show that $f \in \mathcal{B}_0$.

We remind the reader that (the *invariant gradient*) $|\tilde{\nabla} f(x)|$ is defined as

$$|\tilde{\nabla} f(x)| = (1-|x|^2)|\nabla f(x)|.$$

From (3.3) and (3.4) we have

$$|\tilde{\nabla} f(x)| \leqslant C(n,r)\int_{E(x,r)}(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|}\,\mathrm{d}\tau(y).$$

By the assumption (3.6), for any given $\epsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\sup\left\{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|} : y \in \mathbb{B},\ y \neq x\right\} < \epsilon,$$

whenever $|x| > \delta$. Since

$$\int_{E(x,r)}\mathrm{d}\tau = \tau(E(a,\delta)) = \tau(B(0,\delta)) = n\int_0^\delta t^{n-1}(1-t^2)^{-n}\,\mathrm{d}t,$$

we have

$$|\tilde{\nabla} f(x)| < C\epsilon$$

for any $|z| > \delta$, which means that $|\tilde{\nabla} f(x)| \to 0$ as $|z| \to 1^-$. This completes the proof. □

4. Besov spaces

In this section, we give a higher-dimensional form of the Holland–Walsh characterization for Besov spaces. When $p \to \infty$, it also reveals the weighted Lipschitz characterization of the Bloch space.

Theorem 4.1. *Let $p \in (2(n-1),\infty)$ and let f be harmonic on $\mathbb{B}$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}}(1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^p\mathrm{d}\tau(x)\,\mathrm{d}\tau(y) < \infty. \tag{4.1}$$

To prove Theorem 4.1, we need the following lemma.

For this lemma we remind the reader that for $x \in \mathbb{B}$ we have

$$\tilde{\nabla} f(x) = \nabla(f \circ \varphi_x)(0).$$

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Lemma 4.2. *Let $p \geqslant 1$ and let f be harmonic on $\mathbb{B}$. Then*

$$\int_{\mathbb{B}} \biggl(\int_0^1 \frac{|\tilde{\nabla} f(ta)|}{1 - t|a|} \, \mathrm{d}t \biggr)^p \mathrm{d}\nu_\alpha(a) \leqslant C \int_{\mathbb{B}} |\tilde{\nabla} f(a)|^p \, \mathrm{d}\nu_\alpha(a), \tag{4.2}$$

where $\mathrm{d}\nu_\alpha(a) = (1 - |a|^2)^\alpha \, \mathrm{d}\nu(a)$, $\alpha > -1$.

For the proof we remind the reader that the Hardy–Littlewood integral means that $M_p(r, |f|)$ is defined as

$$M_p(r, |f|) = \int_{\partial\mathbb{B}} |f(r\zeta)|^p \, \mathrm{d}\sigma(\zeta).$$

Proof of Lemma 4.2. Fix $\epsilon \in (0, 1)$. Let $t \in [0, 1]$, $a \in \mathbb{B}$. If at least one of t and $|a|$ is less than ϵ, then $|ta| = t|a| < \epsilon$, so

$$\frac{1}{1 - t|a|} \leqslant \frac{1}{1 - \epsilon},$$

and the left-hand side of (4.2) can be controlled by

$$\int_{\mathbb{B}\setminus\epsilon\mathbb{B}} \biggl(\int_\epsilon^1 \frac{|\tilde{\nabla} f(ta)|}{1 - t|a|} \, \mathrm{d}t \biggr)^p \mathrm{d}\nu_\alpha(a) + C \sup_{x \in \epsilon\mathbb{B}} |\tilde{\nabla} f(x)|^p.$$

Denote the first summand above by I. From the polar coordinates integral formula and Minkowski's inequality we have

$$\begin{aligned} I &= n \int_\epsilon^1 \int_{\partial\mathbb{B}} \biggl(\int_\epsilon^1 \frac{|\tilde{\nabla} f(ts\zeta)|}{1 - ts} \, \mathrm{d}t \biggr)^p \mathrm{d}\sigma(\zeta) s^{n-1} (1 - s^2)^\alpha \, \mathrm{d}s \\ &\leqslant C \int_\epsilon^1 \biggl(\int_\epsilon^1 \frac{M_p(ts, |\tilde{\nabla} f|)}{1 - ts} \, \mathrm{d}t \biggr)^p s^{n-1} (1 - s^2)^\alpha \, \mathrm{d}s \\ &\leqslant C \int_\epsilon^1 \biggl(\int_{\epsilon^2}^s h(\rho) \, \mathrm{d}\rho \biggr)^p (1 - s^2)^\alpha \, \mathrm{d}s, \end{aligned}$$

where

$$h(\rho) = \frac{\rho^{(n-1)/p} M_p(\rho, |\tilde{\nabla} f|)}{1 - \rho}.$$

Applying Hardy's inequality

$$\int_0^1 \biggl(\int_0^s h(\rho) \, \mathrm{d}\rho \biggr)^p (1 - s)^\alpha \, \mathrm{d}s \leqslant C \int_0^1 h^p(t) (1 - t)^{\alpha+1} \, \mathrm{d}t,$$

which follows from Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned} I &\leqslant C \int_0^1 \biggl(\int_0^s h(\rho) \, \mathrm{d}\rho \biggr)^p (1 - s)^\alpha \, \mathrm{d}s \\ &\leqslant C \int_0^1 t^{n-1} (1 - t)^\alpha M_p^p(t, |\tilde{\nabla} f|) \, \mathrm{d}t \\ &= C \int_{\mathbb{B}} |\tilde{\nabla} f(a)|^p \, \mathrm{d}\nu_\alpha(a). \end{aligned}$$

It remains to show that

$$\sup_{x\in\mathbb{B}} |\tilde{\nabla} f(x)|^p \leqslant C \int_{\mathbb{B}} |\tilde{\nabla} f(a)|^p \, \mathrm{d}\nu_\alpha(a). \tag{4.3}$$

The starting point to prove this is the following inequality:

$$|g(x)| \leqslant C \int_{E(x,\delta)} |g(w)| \, \mathrm{d}\tau(w),$$

which holds for any harmonic function g. Since each partial derivative of a harmonic function remains harmonic, we have

$$|\nabla f(x)| \leqslant C \int_{E(x,\delta)} |\nabla f(w)| \, \mathrm{d}\tau(w).$$

Recall that $|\tilde{\nabla} f(x)| = (1-|x|^2)|\nabla f(x)|$ and $1-|w| \simeq 1-|x|$ for any $w \in E(x,\delta)$ and $a \in \mathbb{B}$, hence we have

$$|\tilde{\nabla} f(x)| \leqslant C \int_{E(x,\delta)} |\tilde{\nabla} f(w)| \, \mathrm{d}\tau(w).$$

Because $1-|w| \simeq 1-|x| \simeq 1$ for any $w \in E(x,\delta)$ and $x \in \epsilon\mathbb{B}$, applying Hölder's inequality we get

$$\begin{aligned} |\tilde{\nabla} f(x)|^p &\leqslant C \int_{E(x,\delta)} |\tilde{\nabla} f(w)|^p \, \mathrm{d}\tau(w) \\ &\leqslant C \int_{E(x,\delta)} |\tilde{\nabla} f(w)|^p \, \mathrm{d}\nu_\alpha(w). \end{aligned}$$

The assertion (4.3) now follows. This finishes the proof of Lemma 4.2. □

Proof of Theorem 4.1. Assume that $f \in C^\infty(\mathbb{B})$. For any $a \in \mathbb{B}$, we have

$$\frac{|f(a)-f(0)|}{|a|} = \left| \int_0^1 \nabla f(ta) \frac{a}{|a|} \, \mathrm{d}t \right| \leqslant \int_0^1 \frac{|\tilde{\nabla} f(ta)|}{1-t|a|} \, \mathrm{d}t.$$

It follows from Lemma 4.2 that

$$\begin{aligned} \int_{\mathbb{B}} \frac{|f(a)-f(0)|^p}{|a|^p} \, \mathrm{d}\nu_\alpha(a) &\leqslant C \int_{\mathbb{B}} \left(\int_0^1 \frac{|\tilde{\nabla} f(a)|}{1-t|a|} \, \mathrm{d}t \right)^p \mathrm{d}\nu_\alpha(a) \\ &\leqslant C \int_{\mathbb{B}} |\tilde{\nabla} f(a)|^p \, \mathrm{d}\nu_\alpha(a). \end{aligned}$$

Replacing f with $f \circ \varphi_x$, integrating with respect to $\mathrm{d}\tau(x)$, taking $y = \varphi_x(a)$, and setting $\alpha = p/2 - n$, we have

$$\begin{aligned}\int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(y) - f(x)|^p}{|\varphi_x(y)|^p} (1 - |\varphi_x(y)|^2)^{p/2} \, \mathrm{d}\tau(x) \, \mathrm{d}\tau(y) &\leqslant C \int_{\mathbb{B}} \int_{\mathbb{B}} |\tilde{\nabla} f(y)|^p (1 - |\varphi_x(y)|^2)^{p/2} \, \mathrm{d}\tau(x) \, \mathrm{d}\tau(y) \\ &\leqslant C \int_{\mathbb{B}} |\tilde{\nabla} f(y)|^p \, \mathrm{d}\tau(y) \int_{\mathbb{B}} (1 - |\varphi_x(y)|^2)^{p/2} \, \mathrm{d}\tau(x) \\ &\leqslant C \int_{\mathbb{B}} |\tilde{\nabla} f(y)|^p \, \mathrm{d}\tau(y).\end{aligned}$$

In the last step, we used the estimate

$$\int_{\mathbb{B}} (1 - |\varphi_x(y)|^2)^{p/2} \, \mathrm{d}\tau(x) \leqslant C$$

for $p > 2(n-1)$, which follows from (2.5) and Lemma 2.4. Since

$$\frac{(1 - |\varphi_x(y)|^2)^{p/2}}{|\varphi_x(y)|^p} = \frac{(1 - |x|^2)^{p/2}(1 - |y|^2)^{p/2}}{|x - y|^p},$$

we get (4.1).

Conversely, suppose that f is harmonic and that it satisfies (4.1), we will show that $f \in B_p$. For any fixed $\delta \in (0, 1)$,

$$|\tilde{\nabla} f(x)| \leqslant C \int_{E(x,\delta)} |f(x) - f(y)| \, \mathrm{d}\tau(y).$$

Then, by applying Hölder's inequality and (3.4) we obtain

$$\begin{aligned}|\tilde{\nabla} f(x)|^p &\leqslant C \int_{E(x,\delta)} |f(x) - f(y)|^p \, \mathrm{d}\tau(y) \\ &\leqslant \int_{E(x,\delta)} |f(x) - f(y)|^p \frac{(1 - |x|^2)^{p/2}(1 - |y|^2)^{p/2}}{|x - y|^p} \, \mathrm{d}\tau(y).\end{aligned}$$

Thus, (4.1) implies $f \in B_p$. This completes the proof. □

Acknowledgements. G.R. was supported by the NNSF of China (under grant nos 10001030, 19871081) and by a postdoctoral fellowship from the University of Aveiro, UI&D 'Mathematics and Applications'.

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