Peaucellier's Cell and other linkages in non-euclidean Geometry.

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1. There are certain well-known linkages for effecting the transformation of inversion, and, incidentally, by inverting a suitably situated circle, for producing straight-line motion. Reference may be made to the classic lecture by A. B. Kempe, "How to draw a straight line" (London: Macmillan, 1877). It is the object of this paper to call attention to the fact that these linkages have the corresponding property in non-euclidean geometry.

2. We shall consider first Peaucellier's Cell. This consists of two links OA, OB, each equal to a, and four links AP, AP', BP, BP', each equal to b $(a > b)$. Then it is readily proved that OP and OP' each bisect the interior angle AOB, and therefore O, P, P' are collinear.

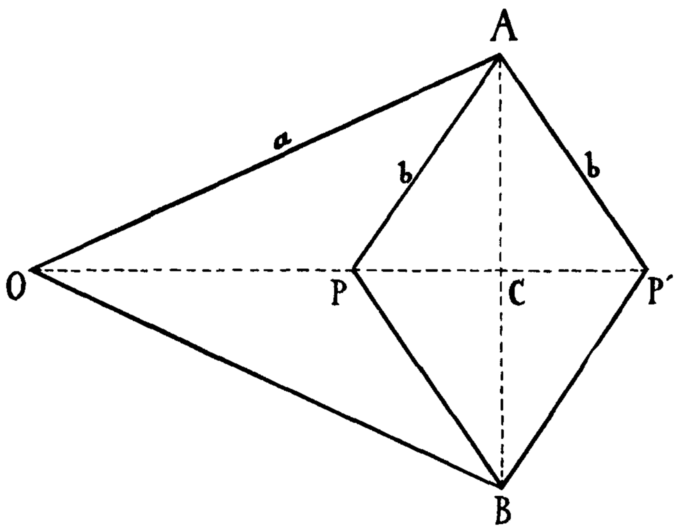

Fig. 1

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Let $OP = r$, $OP' = r'$, $\angle AOP = \angle BOP = \theta$. Then, for the case of elliptic geometry

$$\begin{aligned}\cos b &= \cos r \cos a + \sin r \sin a \cos \theta \\ &= \cos r' \cos a + \sin r' \sin a \cos \theta,\end{aligned}$$

i.e. r, r' are two distinct solutions of the equation

$$\cos r \cos a + \sin r \sin a \cos \theta = \cos b.$$

Put $\tan \frac{1}{2}r = t$, and the equation becomes

$$(\cos a + \cos b)\, t^2 - 2t \sin a \cos \theta + (\cos b - \cos a) = 0.$$

Hence, if the roots of this equation are t, t' we have

$$tt' = \frac{\cos b - \cos a}{\cos b + \cos a} = \tan \tfrac{1}{2}(a - b) \tan \tfrac{1}{2}(a + b) = \text{const} = k \text{.......(1)}$$

P, P' are therefore inverse points * with respect to a circle of radius ρ such that $\tan^2 \frac{1}{2}\rho = \tan \frac{1}{2}(a - b) \tan \frac{1}{2}(a + b)$

or $\cos \rho = \cos a / \cos b$.

The latter result follows at once by considering the case where P, P' coincide at C, and then $AC = b$, and $\angle ACO$ is right,

3. In non-euclidean geometry, however, inversion is not a (1, 1), but a (2, 2) point transformation. In euclidean geometry the relation between r and r' is $rr' = a^2 - b^2$, which is (1, 1). In elliptic geometry the relation (1) is (1, 1) as between $\tan \frac{1}{2}r$ and $\tan \frac{1}{2}r'$, but between the parameter $\tan \frac{1}{2}r$ and the point P on the fixed line OP the correspondence is (2, 1), as we see from the following. The value of r which corresponds to a given point P is indeterminate to an added multiple of π. Hence there is a (1, 1) correspondence between the points P and the values of $\tan r$, and therefore a (2, 1) correspondence between the values of $\tan \frac{1}{2}r$ and the points P, viz. to the point P correspond the two values of $\tan \frac{1}{2}r$:— t and $-t^{-1}$. The correspondence between $\tan \frac{1}{2}r$ and $\tan \frac{1}{2}r'$ being (1, 1), that between P and P' is (2, 2). The (2, 2) relation between $\tan r$ ($= T$) and $\tan r'$ ($= TT'$) is found to be

$$(k^2 - 1)^2 T^2 T'^2 = 4k(kT + T')(T + kT') \text{.............(2)}$$

When the equation is expressed conversely in terms of t and t' it gives

$$(tt' - k)(ktt' - 1)(kt + t')(t + kt') = 0 \text{................(3)}$$

* See the author's Non-euclidean Geometry," p. 241.

The four relations got by equating these four factors to zero are obtainable one from another by changing t or t' into its negative reciprocal. We have thus the four relations

$$\left.\begin{array}{l} t_1 = \quad kt^{-1} \\ t_3 = -k^{-1}t \end{array}\right\} \qquad \left.\begin{array}{l} t_2 = -kt \\ t_4 = k^{-1}t^{-1} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Of these the first pair give coincident points P_1 and P_3, and the second two give another pair of coincident points P_2 and P_4. The two points P_1 and P_2 are the two inverses of P.

Since $t_1 t_2 = -k^2$, P_1 and P_2 are inverses with regard to a (virtual) circle whose radius R is given by

$$\tan^2 \tfrac{1}{2}R = -\tan^4 \tfrac{1}{2}\rho.$$

In hyperbolic geometry this equation becomes

$$\tanh^2 \tfrac{1}{2}R = \tanh^4 \tfrac{1}{2}\rho = k^2,$$

and the circle is real. The inverse of any point on this circle, with respect to the circle of radius ρ is given by

$$\tanh \tfrac{1}{2}R \tanh \tfrac{1}{2}R' = \tanh^2 \tfrac{1}{2}\rho = \tanh \tfrac{1}{2}R.$$

Hence R' is infinite, and this circle is the inverse of the absolute.

4. *The contraparallelogram.*

In this linkwork we have

$$AB' = A'B = a$$
$$AB = A'B' = b$$

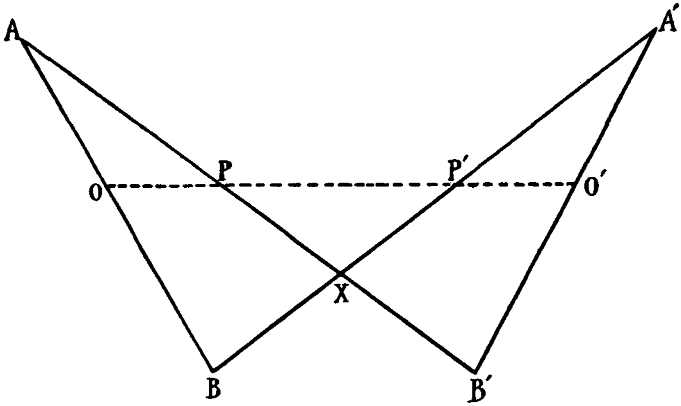

Fig. 2

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$(a > b)$. Let AB' and $A'B$ intersect in X. Take $XP = XP'$ on XA and XA' respectively, and let PP' cut AB and $A'B'$ in O and O'.

Then since OPP' is a transversal of the triangle ABX,

$$\frac{\sin BP'}{\sin XP'} \cdot \frac{\sin XP}{\sin PA} \cdot \frac{\sin AO}{\sin OB} = 1$$

therefore
$$\frac{\sin AO}{\sin OB} = \frac{\sin AP}{\sin PB'}$$

i.e. O is independent of the angle BAB' and is therefore a fixed point on AB. As the linkwork is moved, the four points O, P, P', O' will therefore remain in line.

Let $AP = a_1$, $PB' = a_2$, $AO = b_1$, $OB = b_2$, so that

$$\frac{\sin a_1}{\sin a_2} = \frac{\sin b_1}{\sin b_2} \quad\text{.............................}(5)$$

Let $OP = r$, $OP' = r'$, and $\angle BOP = \theta$; then from the triangles AOP and BOP'

$$\cos a_1 = \cos b_1 \cos r - \sin b_1 \sin r \cos\theta,$$
$$\cos a_2 = \cos b_2 \cos r' + \sin b_2 \sin r' \cos\theta.$$

Eliminating $\cos\theta$, writing t and t' for $\tan\frac{1}{2}r$ and $\tan\frac{1}{2}r'$, and simplifying by means of the relation

$$\frac{\tan\frac{1}{2}(a_1 + b_1)}{\tan\frac{1}{2}(a_1 - b_1)} = \frac{\tan\frac{1}{2}(a_2 + b_2)}{\tan\frac{1}{2}(a_2 - b_2)}$$

which is derived from (5), we obtain

$$\begin{aligned} tt' &= \tan\tfrac{1}{2}(a_1 - b_1)\tan\tfrac{1}{2}(a_2 + b_2) \\ &= \tan\tfrac{1}{2}(a_2 - b_2)\tan\tfrac{1}{2}(a_1 + b_1). \end{aligned}$$

Hence when O is fixed, P and P' are inverse points with regard to a circle of radius ρ where

$$\tan^2\tfrac{1}{2}\rho = \tan\tfrac{1}{2}(a_1 - b_1)\tan\tfrac{1}{2}(a_2 + b_2).$$

5. Relations of the same form hold between the angles of the contraparallelogram and also for the pseudoparallelogram (the simple, not crossed, quadrilateral with opposite sides equal). If

the angles are denoted by ϕ and ϕ', and $\tan \frac{1}{2}\phi = t$, $\tan \frac{1}{2}\phi' = t'$, we find

$$\text{for the contraparallelogram: } tt' = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}$$

$$\text{for the pseudoparallelogram: } tt' = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

On the other hand the diagonals $AA' (= r)$ and $BB' (= r')$ are connected by the relations.

$$\sin \tfrac{1}{2}r \sin \tfrac{1}{2}r' = \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b) \quad \text{(Contraparallelogram)}$$
$$\cos \tfrac{1}{2}r \cos \tfrac{1}{2}r' = \cos \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b) \quad \text{(Pseudoparallelogram)}.$$

6 In the contraparallelogram (Fig. 2), if O is fixed and P, P' are otherwise constrained to move in a straight line the only links required are PA, AB and BP', connected by the relation

$$\frac{\sin AP}{\sin BP'} = \frac{\sin AO}{\sin OB}.$$

The constraint may be produced by rendering this linkwork symmetrical in another way (see Fig. 3) so that this linkage also makes $\tan \frac{1}{2} OP \tan \frac{1}{2} OP'$ constant (*cf.* Kempe, p. 17).

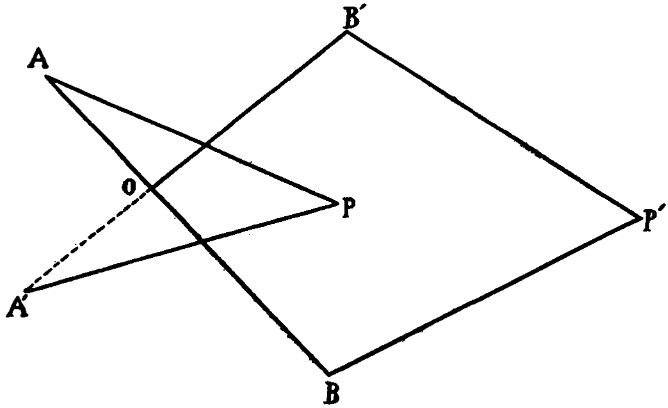

Fig. 3

7. In conclusion it may be mentioned that the apparatus for trisecting angles or dividing into any number of equal parts (Kempe, p. 42) can also be applied in non-euclidean geometry. This is a combination of contraparallelograms $OA_1 B_1 C_1$, $OA_2 (C_1) B_2 C_2$, $OA_3 (C_2) B_3 C_3$ (Fig. 4).

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Let

$$\left.\begin{aligned} OA_1 &= B_1C_1 = a_1 \\ OC_1 &= A_1B_1 = b_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} OA_2 &= B_2C_2 = a_2 = b_1 \\ OC_2 &= A_2B_2 = b_2 \end{aligned}\right\}$$

$$\left.\begin{aligned} OA_3 &= B_3C_3 = a_3 = b_2 \\ OC_3 &= A_3B_3 = b_3 \end{aligned}\right\}$$

$$\angle A_1OA_2 = A_2OA_3 = A_3OC_3 = \theta$$

$$\angle OC_1B_1 = OC_2B_2 = OC_3B_3 = \phi.$$

Then
$$\tan\tfrac{1}{2}\theta\ \tan\tfrac{1}{2}\phi = \frac{\sin\frac{1}{2}(a_1 - b_1)}{\sin\frac{1}{2}(a_1 + b_1)}$$
$$= \frac{\sin\frac{1}{2}(a_2 - b_2)}{\sin\frac{1}{2}(a_2 + b_2)} = \frac{\sin\frac{1}{2}(a_3 - b_3)}{\sin\frac{1}{2}(a_3 + b_3)}.$$

From these relations we obtain

$$\frac{\tan\frac{1}{2}a_1}{\tan\frac{1}{2}b_1} = \frac{\tan\frac{1}{2}a_2}{\tan\frac{1}{2}b_2} = \frac{\tan\frac{1}{2}a_3}{\tan\frac{1}{2}b_3},$$

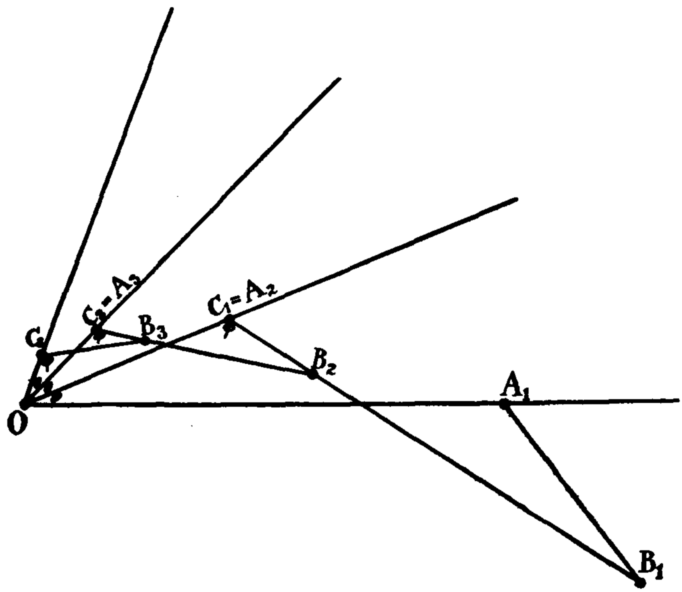

Fig. 4

which are the necessary relations determining the dimensions of the successive contraparallelograms.

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