Reciprocally related primes

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Introduction

Number theory and especially primes are an infinite source of inspirational ideas in elementary mathematics. Many of these ideas can be understood and some also explained with very basic mathematical knowledge. In our paper we explain the cycle which appears between the first six multiples of the decimal expression of $\frac{1}{7}$ and explore an inspiringly simple relation between primes, which somehow makes them gather into mysterious groups. And while exploring these simple questions, we are led to unveil some aspects of the power and limits of human mind and those of a computer.

Mysterious 7

One of the well-known observations related to number 7 is the fact that, if one takes the 6-digit cycle of the decimal expression of $\frac{1}{7}$, and compares it to the cycles of the associated multiples

$$\begin{aligned}
\tfrac{1}{7} &= 0.\overline{142857} \\
\tfrac{2}{7} &= 0.\overline{285714} \\
\tfrac{3}{7} &= 0.\overline{428571} \\
\tfrac{4}{7} &= 0.\overline{571428} \\
\tfrac{5}{7} &= 0.\overline{714285} \\
\tfrac{1}{7} &= 0.\overline{857142},
\end{aligned} \qquad (1)$$

one gets 'cyclic shifts' with the very same digits. Is number 7 unique or are there other numbers with analogous 'cyclic shifts' of their decimal expression cycles? Explanation of this phenomenon will follow shortly, but let us first observe another related 'mystery'. Consider the 6-digit number 142857, which appears as a cycle of the decimal expression of $\frac{1}{7}$. Factorising it we get

$$142857 = 3 \cdot 3 \cdot 3 \cdot 11 \cdot 13 \cdot 37.$$

Now take one of these prime factors, for example 3, write $\frac{1}{3}$ in decimal form and notice its 6 digit period $\frac{1}{3} = 0.\overline{333333}$. Take the number 333333 and factorise it $333333 = 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Doing the same for the other three different factors, we notice an interesting pattern:

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$$\begin{aligned}
\tfrac{1}{3} &= 0.\overline{333333} \text{ and } 333333 = 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \\
\tfrac{1}{7} &= 0.\overline{142857} \text{ and } 142857 = 3 \cdot 3 \cdot 3 \cdot 11 \cdot 13 \cdot 37 \\
\tfrac{1}{11} &= 0.\overline{090909} \text{ and } 090909 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 13 \cdot 37 \\
\tfrac{1}{13} &= 0.\overline{076923} \text{ and } 076923 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 37 \\
\tfrac{1}{17} &= 0.\overline{027027} \text{ and } 027027 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13.
\end{aligned} \tag{2}$$

What is so special about prime numbers 3, 7, 11, 13, 37? What makes them '*reciprocally related*' in such a nice way? Are there other 'families' of 'reciprocally related primes'? For example, are primes 23, 29, 41 reciprocally related in a similar way?

Reciprocally related primes

We notice that, multiplying all the factors in the above factorisations, we obtain

$$3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 = 999999 = 10^6 - 1.$$

Obviously

$$1 = \frac{3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37}{999999} = \frac{3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37}{10^6 - 1}$$

and since

$$\frac{1}{10^6 - 1} = \frac{1}{10^6} + \frac{1}{10^{12}} + \frac{1}{10^{18}} + \dots$$

also

$$1 = \frac{999999}{10^6 - 1} = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot \left(\frac{1}{10^6} + \frac{1}{10^{12}} + \frac{1}{10^{18}} + \dots\right). \tag{3}$$

Dividing this equation by any $p \in \{3, 3, 3, 7, 11, 13, 37\}$, for example in the cases $p = 7$ or $p = 13$, we get

$$\frac{1}{7} = \frac{3 \cdot 3 \cdot 3 \cdot 11 \cdot 13 \cdot 37}{10^6 - 1} = \frac{142857}{10^6 - 1} = 142857\left(\frac{1}{10^6} + \frac{1}{10^{12}} + \frac{1}{10^{18}} + \dots\right)$$

and

$$\frac{1}{13} = \frac{3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 37}{10^6 - 1} = \frac{76923}{10^6 - 1} = 76923\left(\frac{1}{10^6} + \frac{1}{10^{12}} + \frac{1}{10^{18}} + \dots\right)$$

respectively, which explains why these primes are 'reciprocally related' and the pattern observed in (2) becomes obvious.

Our above reasoning was based on the factorisation of $10^6 - 1$. Analogously we can think about $10^6 - 1$ for every $n \in \mathbb{N}$. Let us define:

Definition 1: For any positive integer n we define the set of reciprocally related primes $\mathcal{RRP}_n$ as the set of all prime factors of $10^n - 1$.

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In general if $10^n - 1$ factorises as $10^n - 1 = p \cdot q \cdot r$, we have

$$\overset{n \text{ digits}}{\overleftrightarrow{99\cdots99}} = 10^n - 1 = p \cdot q \cdot r$$

and

$$1 = \frac{p \cdot q \cdot r}{10^n - 1} \text{ from where}$$

$$\frac{1}{p} = q \cdot r \cdot \left(\frac{1}{10^n} + \frac{1}{10^{2n}} + \frac{1}{10^{3n}} + \ldots \right)$$

$$\frac{1}{q} = p \cdot r \cdot \left(\frac{1}{10^n} + \frac{1}{10^{2n}} + \frac{1}{10^{3n}} + \ldots \right).$$

From these observations we conclude that an analogous pattern to the one presented in (2) can be observed for an arbitrary set of reciprocally related primes $\mathcal{RRP}_n$:

Theorem 1: If $10^n - 1 = p \cdot q \cdot r$ and particularly if $p, q \in \mathcal{RRP}_n$, then the decimal form of the unit fractions $\frac{1}{p} = 0 \cdot \overline{a_1 a_2 \ldots a_n}$ and $\frac{1}{q} = 0 \cdot \overline{b_1 b_2 \ldots b_n}$, with digits $a_i, b_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, both have cycles of length n. If we consider the two integers $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_n$ we have

$$p \cdot a_1 a_2 \ldots a_n = q \cdot b_1 b_2 \ldots b_n.$$

First we write the factorisations of $10^n - 1$ for $n \in \{1, 2, 3, \ldots, 20\}$ Table 1 and some of the $\mathcal{RRP}_n$ in Table 2.

$$9 = 3 \cdot 3$$

$$99 = 3 \cdot 3 \cdot 11$$

$$999 = 3 \cdot 3 \cdot 3 \cdot 37$$

$$9\,999 = 3 \cdot 3 \cdot 11 \cdot 101$$

$$99\,999 = 3 \cdot 3 \cdot 41 \cdot 271$$

$$999\,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$$

$$9\,999\,999 = 3 \cdot 3 \cdot 239 \cdot 4649$$

$$99\,999\,999 = 3 \cdot 3 \cdot 11 \cdot 73 \cdot 101 \cdot 137$$

$$999\,999\,999 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 37 \cdot 33667$$

$$9\,999\,999\,999 = 3 \cdot 3 \cdot 11 \cdot 41 \cdot 271 \cdot 9091$$

$$99\,999\,999\,999 = 3 \cdot 3 \cdot 21649 \cdot 513239$$

$$999\,999\,999\,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 37 \cdot 101 \cdot 9901$$

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$$9\,999\,999\,999\,999 = 3 \cdot 3 \cdot 53 \cdot 79 \cdot 265371653$$
$$99\,999\,999\,999\,999 = 3 \cdot 3 \cdot 3 \cdot 11 \cdot 239 \cdot 4649 \cdot 909091$$
$$999\,999\,999\,999\,999 = 3 \cdot 3 \cdot 3 \cdot 31 \cdot 37 \cdot 41 \cdot 271 \cdot 2906161$$
$$9\,999\,999\,999\,999\,999 = 3 \cdot 3 \cdot 11 \cdot 17 \cdot 73 \cdot 101 \cdot 137 \cdot 5882353$$
$$99\,999\,999\,999\,999\,999 = 3 \cdot 3 \cdot 2071723 \cdot 5363222357$$
$$999\,999\,999\,999\,999\,999 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 52579 \cdot 33667$$
$$9\,999\,999\,999\,999\,999\,999 = 3 \cdot 3 \cdot 1111111111111111111$$
$$99\,999\,999\,999\,999\,999\,999 = 3 \cdot 3 \cdot 11 \cdot 41 \cdot 101 \cdot 271 \cdot 3541 \cdot 9091 \cdot 27961$$

TABLE 1: Factorisation of $10^n - 1$ for $n \in \{1, 2, 3, \dots, 20\}$

$$\mathcal{RRP}_1 = \{3, 3\}$$
$$\mathcal{RRP}_2 = \{3, 3, 11\}$$
$$\mathcal{RRP}_3 = \{3, 3, 3, 37\}$$
$$\mathcal{RRP}_4 = \{3, 3, 11, 101\}$$
$$\mathcal{RRP}_5 = \{3, 3, 41, 271\}$$
$$\mathcal{RRP}_6 = \{3, 3, 3, 7, 11, 13, 37\}$$
$$\mathcal{RRP}_7 = \{3, 3, 239, 4649\}$$
$$\mathcal{RRP}_8 = \{3, 3, 11, 73, 101, 137\}$$
$$\mathcal{RRP}_{12} = \{3, 3, 3, 7, 11, 13, 101, 9901\}$$
$$\mathcal{RRP}_{20} = \{3, 3, 11, 41, 101, 271, 3541, 9091, 27961\}$$

TABLE 2: Some of the sets $\mathcal{RRP}_n$

Which prime numbers do appear in $\mathcal{RRP}_n$ sets and how do they group together? Obviously, neither 2 nor 5 divide $10^n - 1$, so prime numbers 2 and 5 do not appear in any of the sets of reciprocally related primes $\mathcal{RRP}_n$. But it is worth mentioning that (3) has a very simple analogue

$$1 = \frac{2 \cdot 5}{10} \Rightarrow \frac{1}{2} = 0.5 \quad \text{and} \quad \frac{1}{5} = 0.2$$

and it therefore makes sense to say that primes 2 and 5 are 'reciprocally related'. They are the only two primes associated with finite decimal expressions $\frac{1}{2} = 0.5$ and $\frac{1}{5} = 0.2$. Because of the similar and very special status of the primes 2 and 5 it would make sense to denote the associated set as $\mathcal{RRP}_0 = \{2, 5\}$.

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What about other primes? Which primes do appear in the sets of reciprocally related primes $\mathcal{RRP}_n$? Is it possible that all other primes are members of certain sets of reciprocally related primes? It is easy to see that the answer is affirmative.

Theorem 2: If $m \in \mathbb{N}$ is any number relatively prime to 10, then m divides $10^n - 1$ for some $n \in \mathbb{N}$.

The theorem easily follows from the well known Euler-Fermat theorem (see for example [1, p. 176]), which states that if $\gcd(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$, where $\phi(m)$ is the Euler phi-function counting the positive integers not exceeding m that are co-prime to m.

Theorem 2 can also be proved directly by very elementary reasoning. Similar reasoning will later lead to the understanding of the 'mystery' presented in (1) above.

An alternative proof of Theorem 2: Let $m \in \mathbb{N}$ be any number relatively prime to 10. We know from elementary knowledge of rational numbers that $\frac{1}{m}$ is a rational number, which can be written in decimal form as

$$\frac{1}{m} = 0{\cdot}a_1a_2\ldots a_l\overline{b_1b_2\ldots b_k}$$

with $a_i, b_j \in \{0, 1, 2, \ldots, 9\}$ being standard decimal digits and where the repeating cycle of digits is over-lined.

Therefore we have

$$\frac{10^l}{m} = a_1a_2\ldots a_l.\overline{b_1b_2\ldots b_k}.$$

Defining an l-digit integer $h = a_1a_2\ldots a_l$ and a k-digit integer $R = b_1b_2\ldots b_k$, we obviously get

$$0.\overline{b_1b_2\ldots b_k} = \frac{R}{10^k} + \frac{R}{10^{2k}} + \frac{R}{10^{3k}} + \ldots = \frac{R}{10^k - 1}$$

and

$$\frac{10^l}{m} = h + \frac{R}{10^k - 1}.$$

This equation is equivalent to

$$\left(10^l - m \cdot h\right) \cdot \left(10^k - 1\right) = m \cdot R.$$

Since 10^l and m are relatively prime, so are $10^l - m \cdot h$ and m. Thus m must divide $10^k - 1$.

In the next two corollaries we explain the interesting cycle of the decimal cycles which appears in the decimal expressions of $\frac{k}{7}$, for $1 \leqslant k \leqslant 6$ presented in (1) above.

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Corollary 3: If $m \in \mathbb{N}$ is relatively prime to 10^k, then for every $1 \leqslant k < m$ the fraction $\frac{k}{m}$ is written in decimal form as

$$\frac{k}{m} = 0.\overline{b_1 b_2 \dots b_n}.$$

(In decimal expression the cycle starts immediately after the decimal point.)

Proof: By Theorem 2, m divides $10^n - 1$ for some n and we take the smallest n for which

$$m \cdot r = 10^n - 1.$$

Then

$$\frac{1}{m} = \frac{r}{10^n - 1} = r\left(\frac{1}{10^n} + \frac{1}{10^{2n}} + \frac{1}{10^{3n}} + \dots\right),$$

where for $b_i \in \{0, 1, 2, \dots, 9\}$, we have a n-digit integer $r = b_1 b_2 \dots b_n$ and

$$\frac{1}{m} = 0.\overline{b_1 b_2 \dots b_n},$$

which completes the proof for $k = 1$.

If $1 < k$, then we have

$$\frac{k}{m} = \frac{k \cdot r}{10^n - 1} = k \cdot r\left(\frac{1}{10^n} + \frac{1}{10^{2n}} + \frac{1}{10^{3n}} + \dots\right).$$

The number $k \cdot r$ has fewer than n digits, as $k \cdot r < 10^n - 1$. Therefore, the decimal expression of $\frac{k}{m}$ has the cycle of the same length as the cycle of decimal expression of $\frac{1}{m}$. Rather than number r, the number $k \cdot r$ appears in the cycle of the decimal expression of $\frac{k}{m}$.

Referring to Theorem 2, to the Euler-Fermat theorem and to the definition of Euler's phi-function $\phi(m)$, we know that if we take the smallest n for which m divides $10^n - 1$, we surely have $n < m$. Furthermore, it is only possible that $n = m - 1$ in the case when m is a prime.

Corollary 4: If p is a prime such that $n = p - 1$ is the smallest n for which p divides $10^n - 1$, then the fractions $\frac{k}{p}$ for $1 \leqslant k < p$ written in decimal form are given by cyclic transformation of the same digits (as $\frac{1}{p} - 0.\overline{b_1 b_2 \dots b_n}$).

Proof: The argument follows directly from elementary division procedure. Namely, to find the decimal expression of $\frac{1}{p}$, we first divide 10 by p to get the quotient b_1 and the associated remainder. In the next step we multiply the remainder by 10 and divide this product by p to get the next decimal digit and so on. Remainders are smaller then p and since the cycles are of length $p - 1$, we know that all numbers $1 \leqslant r \leqslant p - 1$ must appear as the remainders of these divisions. Digits of a specific cycle of the decimal form corresponding to $\frac{k}{p}$ are uniquely defined by remainders. The corresponding cycle of $\frac{k}{p}$ is therefore just a cyclic transformation of the cycle corresponding to $\frac{1}{p}$, but starting with a digit, which belongs to the remainder k in the division procedure associated with $\frac{1}{p}$.

We can now find many cases analogous to the cycle given in (1). In Table 3 we write (with the help of a computer) the smallest associated numbers of the form $10^n - 1$ in which the first twenty primes (except 2 and 5) are factors.

prime	factor of	prime	factor of
3	$10^1 - 1$	41	$10^5 - 1$
7	$10^6 - 1$	43	$10^{21} - 1$
11	$10^2 - 1$	47	$10^{46} - 1$
13	$10^6 - 1$	53	$10^{13} - 1$
17	$10^{16} - 1$	59	$10^{58} - 1$
19	$10^{18} - 1$	61	$10^{66} - 1$
23	$10^{22} - 1$	67	$10^{33} - 1$
29	$10^{28} - 1$	71	$10^{35} - 1$
31	$10^{15} - 1$	73	$10^8 - 1$
37	$10^3 - 1$	79	$10^{13} - 1$

TABLE 3: $10^n - 1$ for the first twenty primes (except 2 and 5)

Analogous cycles to the one presented in (1) therefore appear also in many other cases with much longer cycles. For example the cycle for $\frac{k}{17}$. $1 \leqslant k < 17$ has length 16; the cycle for $\frac{k}{19}$, $1 \leqslant k < 19$ has length 18; cycle for $\frac{k}{23}$, $1 \leqslant k < 23$ has length 22; the cycle for $\frac{k}{59}$, $1 \leqslant k < 59$ has length 58 and with a computer we can find many more. For example, the cycle for $\frac{k}{983}$, $1 \leqslant k < 983$ has length 982 and so on.

To illustrate the meaning of Corollary 4 we write the 16-elements cycle of the 16-digit decimal cycles belonging to numbers $\frac{k}{17}$:

$$\begin{array}{llll}
\frac{1}{17} = 0.\overline{0588235294117647} & \frac{9}{17} = 0.\overline{5294117647058823} \\
\frac{2}{17} = 0.\overline{1176470588235294} & \frac{10}{17} = 0.\overline{5882352941176470} \\
\frac{3}{17} = 0.\overline{1764705882352941} & \frac{11}{17} = 0.\overline{6470588235294117} \\
\frac{4}{17} = 0.\overline{2352941176470588} & \frac{12}{17} = 0.\overline{7058823529411764} \\
\frac{5}{17} = 0.\overline{2941176470588235} & \frac{13}{17} = 0.\overline{7647058823529411} \\
\frac{6}{17} = 0.\overline{3529411764705882} & \frac{14}{17} = 0.\overline{8235294117647058} \\
\frac{7}{17} = 0.\overline{4117647058823529} & \frac{15}{17} = 0.\overline{8823529411764705} \\
\frac{8}{17} = 0.\overline{4705882352941176} & \frac{16}{17} = 0.\overline{9411764705882352}
\end{array}$$

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Returning to Theorem 2, our main theorem, we know that any number m or particularly any prime p, which is relatively prime to 10, divides $10^n - 1$ for some $n \in \mathbb{N}$. Therefore any number relatively prime to 10 is a factor of

$$\overset{n \text{ digits}}{\overleftrightarrow{999\ldots 99}}$$

for some n and belongs to the associated set of reciprocally related primes $\mathcal{RRP}_n$.

Therefore an analogous pattern to the one we observed in (2) and which is related to $\mathcal{RRP}_6$ appears within $\mathcal{RRP}_n$ for every $n \in \mathbb{N}$.

Note also, that, from Theorem 2 it immediately follows that any number relatively prime to 10 is a factor of

$$\overset{n \text{ digits}}{\overleftrightarrow{111\ldots 11}}$$

for some n.

$\mathcal{RRP}_n$ containing 23, 29 *and* 41

Initially we observed the interesting reciprocal relation between the primes 3, 7, 11, 13, 37 and asked rather naive question whether also for example primes 23, 29, 41 might be reciprocally related in a similar way? For Euler's phi-function we have $\phi(23 \cdot 29 \cdot 41) = 22 \cdot 28 \cdot 40 = 24640$ and therefore by Theorem 2 and the Euler-Fermat theorem we know that $23, 29, 41 \in \mathcal{RRP}_{24640}$. To understand $\mathcal{RRP}_{24640}$ we would need to factorise $10^{24640} - 1$ which is certainly not an easy task. But $\mathcal{RRP}_{24640}$ is not the smallest set of reciprocally related primes containing 23, 29, 41. The following rather elementary lemma will be helpful in finding the smallest set of reciprocally related primes $\mathcal{RRP}_n$ containing 23, 29, 41.

Lemma 5: Assume $a, m, n \in \mathbb{N}$ and m and n are relatively prime. If m divides $a^M - 1$ and n divides $a^N - 1$, then $m \cdot n$ divides $a^h - 1$, where h is any multiple of M and N.

Proof: As h is a multiple of M and N we can write $M \cdot N' = M' \cdot N = h$. If we respectively insert values $x = a^M$, $k = N'$ and $x = a^N$, $k = M'$ into the simple geometric series formula

$$1 + x + x^2 + \ldots + x^{k-1} = \frac{x^k - 1}{x - 1},$$

we see that $a^M - 1$, $a^N - 1$, m and n all divide $a^h - 1$. Since m and m are relatively prime, $m \cdot n$ divides $a^h - 1$.

From Table 3 we know the smallest numbers of the form $10^n - 1$ which are the multiples of 23, 29 and 41.

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prime	factor of
23	$10^{22} - 1$
29	$10^{28} - 1$
41	$10^{5} - 1$.

If we factorise $22 = 2 \cdot 11$, $28 = 2 \cdot 2 \cdot 7$, we find the smallest multiple of 22, 28 and 5 to be $2 \cdot 2 \cdot 7 \cdot 11 \cdot 5 = 1540$. Therefore, by Lemma 5, the prime numbers 23, 29, 41 are factors of $10^{1540} - 1$. We can check with a computer, that this huge number (but indeed much smaller than $10^{24640} - 1$) is indeed the smallest multiple of 23, 29, 41 of the form $10^n - 1$. Therefore $\mathcal{RRP}_{1540}$ is the smallest of the sets $\mathcal{RRP}_n$, which contains 23, 29, 41. To understand and fully describe $\mathcal{RRP}_{1540}$, we would need to factorise $10^{1540} - 1$. How can one factorise such a big number? Could this be easily done with a computer? Before addressing this questions let us pause to think, how different could our task be if we considered a different triple of primes? By Lemma 5 we could, for example, easily obtain Table 4.

$\mathcal{RRP}_n$ containing	contains all the factors of
13, 17 and 19	$10^{144} - 1$
19, 23 and 29	$10^{2772} - 1$
23, 29 and 31	$10^{4620} - 1$
23, 29 and 41	$10^{1540} - 1$
23, 37 and 73	$10^{264} - 1$

TABLE 4

Smaller prime numbers obviously do not necessarily mean an easier task of factorisation. To get a taste of how big these numbers are, let us write down the prime factors of $10^{144} - 1$ obtained by brute computer force.

$$10^{144} - 1 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 101 \cdot 137 \cdot 3169$$
$$\cdot 8929 \cdot 9901 \cdot 52579 \cdot 98641 \cdot 333667 \cdot 5882353 \cdot 99990001$$
$$\cdot 999999000001 \cdot 3199044596370769 \cdot 9999999900000001$$
$$\cdot 111994624580356142905139433307201254339791169$$

An average modern PC, which can perform the above factorisation of $10^{144} - 1$ in a split of a second remains powerless in trying to factorise $10^{1540} - 1$. With the help of Paul Zimmermann from INRIA (*Institut National de Recherche en Sciences et Technologies du Numérique*), a co-author of the February 2020 factorisation of RSA-250 (see [2]), referring to Cunningham tables (see [3]) and some very technical procedures and with the use of a computer we were able to factorise $10^{1540} - 1$ (see[4]).

The number $10^{1540} - 1$ has altogether 74 prime factors. Of those only 3 and 11 appear twice. There are 28 prime factors with over 10 digits, 10 prime factors with over 30 digits, 6 with over 100 digits and the biggest prime factor has 241 digits.

The 'mysterious' nature of the original set $\mathcal{RRP}_6 = \{3, 3, 3, 7, 11, 13, 37\}$ is therefore fully understood and we also comprehend the nature of all other $\mathcal{RRP}_n$. By increasing n, these sets expand so very quickly that the beauty of the pattern observed in the case of $\mathcal{RRP}_6$ (2) is lost in the sheer enormity of sets like $\mathcal{RRP}_{1540}$. But observing the sets of reciprocally related primes $\mathcal{RRP}_n$ we notice new mysteries. For example some prime numbers seem much more 'social' than others. Of course it is obvious that the prime 3 appears in every $\mathcal{RRP}_n$ (with multiplicity of at least two). With a computer we explored all the primes appearing in the sets $\mathcal{RRP}_i$ for $1 \leqslant i \leqslant 80$. How to explain for example the fact that prime 5882353 appears 5 times, while there are 52203 primes before and 19556 primes after the prime 5882353, which do not show up at all in our $(1 - 80)$ sets of *reciprocally related primes*?

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References

1. J. J. Tattersall, *Elementary number theory in nine chapters* (2nd edn.) Cambridge (2005).
2. RSA numbers, *Wikipedia*, accessed April 2024 at https://en.wikipedia.org/wiki/RSA numbers
3. The Cunningham Project, accessed April 2024 at https://homes.cerias.purdue.edu/~ssw/cun/
4. Damjan Kobal, Factorisation of $10^{1540} - 1$, accessed March 2024 at https://ko.fmf.uni-lj.si/fact/Factor-10-to-1540-1.pdf

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