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Simultaneous Approximation and Interpolation on Arakelian Sets

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Abstract. We extend results of P. M. Gauthier, W. Hengartner and A. A. Nersesyan on simultaneous approximation and interpolation on Arakelian sets.

In [4], a monotonicity problem for the multipole Lempert function was solved. The proof there was based on a simultaneous interpolation and approximation result for a special Arakelian set of the unit disc. So it was natural to ask what is going on for an arbitrary Arakelian set. The final result presented here turns out to be an extension of a result due to P. M. Gauthier, W. Hengartner, and A. A. Nersesyan [2, 3].

Let us first recall the definition of an Arakelian set and the known results.

Definition 1 A relatively closed subset E of a plane domain D is called *an Arakelian set of* D, if $D^* \setminus E$ is connected and locally connected, where D^* denotes the one-point compactification of D.

For a plane domain D, let $\mathcal{O}(D)$ be the set of all holomorphic functions on D. If E is a relatively closed subset of D, we denote by $\mathcal{A}(E)$ the set of all functions continuous on E and holomorphic on E^0, where E^0 is the interior of E.

The main theorems we will need are the following ones.

Theorem 2 (Arakelian's Theorem [1]) *Let E be an Arakelian set of a plane domain D. If $f \in \mathcal{A}(E)$ and $\varepsilon > 0$, then there exists $g \in \mathcal{O}(D)$ with $|g(z) - f(z)| < \varepsilon$, $z \in E$.*

Theorem 3 (The Gauthier–Hengartner–Nersesyan Theorem [2, 3]) *Let D and E be as in Theorem 2 and let Λ be a sequence in $E \setminus E^0$ without limit points in D. Assume that for any $\lambda \in \Lambda$ a finite sequence $(\beta_\lambda^\nu)_{\nu=1}^{\nu(\lambda)}$ of complex numbers is given. Let $f \in \mathcal{A}(E)$ and $\varepsilon > 0$. Then there is a $g \in \mathcal{O}(D)$ satisfying the conclusion of Theorem 2 and in addition, $g(\lambda) = f(\lambda)$ and $g^{(\nu)}(\lambda) = \beta_\lambda^\nu$ for any $\lambda \in \Lambda$ and any $\nu = 1, \dots, \nu(\lambda)$.*

In the result presented in this note, even more specified functions g are provided. In fact, we have the following theorem.

Theorem 4 *Let D, E, Λ (Λ may be empty), β_λ^ν be as in Theorem 3, and let $b_1, \dots b_k \in E^0$. Then for given $f \in \mathcal{A}(E)$, $\varepsilon > 0$, and $m \in \mathbb{Z}_+$, there is a $g \in \mathcal{O}(D)$ satisfying the conclusions of Theorem 3, and in addition, $g^{(\nu)}(b_j) = f^{(\nu)}(b_j)$ for any $j = 1, \dots, k$ and any $\nu = 0, \dots, m$.*

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Proof Obviously, we may assume that $E \neq D$. Now we present the proof in four steps.

Step 1: For any $j = 1, \dots, k$ there exists an $s_j \in \mathcal{O}(D)$ that is bounded on E such that $s'_j(b_j) \neq 0$, $s_j(b_j) = 0$ and $s_j(b_q) \neq 0$ for every $q \neq j$.[1]

To see this, choose a point $c \in D \setminus E$. Since $E \cup \{c\}$ is an Arakelian set of D, by Theorem 2, there is an $\tilde{s} \in \mathcal{O}(D)$ with $|\tilde{s}| < 1$ on E and $|\tilde{s}(c) - 2| < 1$. Set $\hat{s}_j := \tilde{s} - \tilde{s}(b_j)$. Since $\hat{s}_j(c) \neq 0$, then $\hat{s}_j \not\equiv 0$. Now $|\hat{s}_j| < 2$ on E implies that the function s_j,

$$s_j(z) := \frac{(z - b_j)\hat{s}_j(z)}{\prod_{q=1}^{k}(z - b_q)^{\operatorname{ord}_{b_q} \hat{s}_j}}, \quad z \in D,$$

has all the required properties. (As usual, $\operatorname{ord}_\lambda p$ denotes the smallest integer $q \geq 0$ with $p^{(q)}(\lambda) \neq 0$.)

Step 2: There is a function $p \in \mathcal{O}(D)$ that is bounded on E such that $p(b_j) \neq 0$ for any $j = 1, \dots, k$ and $\operatorname{ord}_\lambda p \geq \nu(\lambda) + 1$ for any $\lambda \in \Lambda$.

Indeed, if $q = 0$ on E and $q(c) = 1$, where $c \in D \setminus E$, then it is enough to apply Theorem 3 for $E \cup \{c\}$, q, $\varepsilon = 1$, and $\beta_\lambda^\nu = 0$, $\nu = 1, \dots, \nu(\lambda) + 1$, $\lambda \in \Lambda$. Hence, we get a non–constant function $\tilde{p} \in \mathcal{O}(D)$ such that $|\tilde{p}| < 1$ on E and $\operatorname{ord}_\lambda \tilde{p} \geq \nu(\lambda) + 1$, $\lambda \in \Lambda$. What remains is to put

$$p(z) := \frac{\tilde{p}(z)}{\prod_{j=1}^{k}(z - b_j)^{\operatorname{ord}_{b_j} \tilde{p}}}, \quad z \in D.$$

Step 3: Let s_j be the function from Step 1, $j = 1, \dots, k$. For a non-negative integer ν set

$$\tilde{h}_j^\nu := \frac{p}{s_j} \prod_{q=1}^{k} s_q^{\nu+1},$$

where p is the function from Step 2. Then

$$h_j^\nu := \frac{\tilde{h}_j^\nu}{(\tilde{h}_j^\nu)^{(\nu)}(b_j)}$$

is a well-defined function on D. Put

$$M_\nu := \sup_E \sum_{j=1}^{k} |h_j^\nu|.$$

Step 4: Finally, we are going to prove Theorem 4 by induction on m. Fix $m = 0$ and let g be the function from Theorem 3 for Λ, $(\beta_\lambda^\nu)_{\nu=1}^{\nu(\lambda)}$, and $\frac{\varepsilon}{M_0+1}$. Then it is easy to check that the function

$$g_0 := g + \sum_{j=1}^{k} (f(b_j) - g(b_j)) h_j^0$$

[1] Observe that this step is obvious whenever D is biholomorphic to a bounded domain, in particular, when $\overline{D} \neq \mathbb{C}$.

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satisfies all the required properties.

Set $d := \min_{1\leq j\leq k} \operatorname{dist}(b_j, \mathbb{C}\setminus E^0)$. Assume now that Theorem 4 is true for some integer $m \geq 0$ and let g_m satisfy the conclusion of Theorem 4 for

$$\varepsilon(1 + M_{m+1}(m+1)!d^{-m-1})^{-1}.$$

By virtue of the Cauchy inequalities it follows that the function

$$g_{m+1} := g_m + \sum_{j=1}^{k}(f^{(m+1)}(b_j) - g_m^{(m+1)}(b_j))h_j^{m+1}$$

fulfills the conclusions of Theorem 4 for $m+1$. ■

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