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# On denseness of horospheres in higher rank homogeneous spaces

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*Abstract.* Let $G$ be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. Let $N$ denote a maximal horospherical subgroup of $G$, and $P = MAN$ the minimal parabolic subgroup which is the normalizer of $N$. Let $\mathcal{E}$ denote the unique $P$-minimal subset of $\Gamma\backslash G$ and let $\mathcal{E}_0$ be a $P^\circ$-minimal subset. We consider a notion of a horospherical limit point in the Furstenberg boundary $G/P$ and show that the following are equivalent for any $[g] \in \mathcal{E}_0$:
(1) $gP \in G/P$ is a horospherical limit point;
(2) $[g]NM$ is dense in $\mathcal{E}$;
(3) $[g]N$ is dense in $\mathcal{E}_0$.
The equivalence of items (1) and (2) is due to Dal'bo in the rank one case. We also show that unlike convex cocompact groups of rank one Lie groups, the $NM$-minimality of $\mathcal{E}$ does not hold in a general Anosov homogeneous space.

Key words: horospheres, horospherical limit points, limit cones, infinite volume
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## 1. *Introduction*

Let $G$ be a connected semisimple real algebraic group. Let $(X, d)$ denote the associated Riemannian symmetric space. Let $P = MAN$ be a minimal parabolic subgroup of $G$ with fixed Langlands decomposition, where $A$ is a maximal real split torus of $G$, $M$ the maximal compact subgroup of $P$ commuting with $A$, and $N$ the unipotent radical of $P$. Note that $N$ is a maximal horospherical subgroup of $G$, which is unique up to conjugations.

Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \log A$ so that $\log N$ consists of positive root subspaces, and we set $A^+ = \exp \mathfrak{a}^+$. This means that $N$ is a contracting horospherical subgroup in the sense that for any $a$ in the interior of $A^+$,

$$N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

Let $\Gamma$ be a *Zariski dense* discrete subgroup of $G$. In this paper, we are interested in the topological behavior of the action of the horospherical subgroup $N$ on $\Gamma\backslash G$ via the right translations. When $\Gamma < G$ is a cocompact lattice, every $N$-orbit is dense in $\Gamma\backslash G$, that is, the $N$-action on $\Gamma\backslash G$ is minimal. This is due to Hedlund [11] for $G = \mathrm{PSL}_2(\mathbb{R})$ and to Veech [19] in general. Dani gave a full classification of possible orbit closures of $N$-action for any lattice $\Gamma < G$ [8].

For a general discrete subgroup $\Gamma < G$, the quotient space $\Gamma\backslash G$ does not necessarily admit a dense $N$-orbit, even a dense $NM$-orbit, for instance in the case where $\Gamma$ does not have a full limit set. Let $\mathcal{F}$ denote the Furstenberg boundary $G/P$. We denote by $\Lambda = \Lambda_\Gamma$ the limit set of $\Gamma$,

$$\Lambda = \Big\{ \lim_{i\to\infty} \gamma_i(o) \in \mathcal{F} : \gamma_i \in \Gamma \Big\},$$

where $o \in X$ and the convergence is understood as in Definition 2.2. This definition is independent of the choice of $o \in X$. The limit set $\Lambda$ is known to be the unique $\Gamma$-minimal subset of $\mathcal{F}$ (see [1, 9, 15]). Thus, the set

$$\mathcal{E} = \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

is the unique $P$-minimal subset of $\Gamma\backslash G$. For a given point $[g] \in \mathcal{E}$, the topological behavior of the horospherical orbit $[g]N$ (or of $[g]NM$) is closely related to the ways in which the orbit $\Gamma(o)$ approaches $gP$ along its limit cone. The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of $\Gamma$ is defined as the smallest closed cone of $\mathfrak{a}^+$ containing the Jordan projection $\lambda(\Gamma)$. It is a convex cone with non-empty interior: $\operatorname{int} \mathcal{L} \neq \emptyset$ [1]. If rank $G = 1$, then $\mathcal{L} = \mathfrak{a}^+$. In higher ranks, the limit cone of $\Gamma$ depends more subtly on $\Gamma$.

1.1. *Horospherical limit points.* Recall that in the rank one case, a horoball in $X$ based at $\xi \in \mathcal{F}$ is a subset of the form $gN(\exp \mathfrak{a}^+)(o)$, where $g \in G$ is such that $\xi = gP$ [5]. Our generalization to higher rank of the notion of a horospherical limit point involves the limit cone of $\Gamma$. By a $\Gamma$-tight horoball based at $\xi \in \mathcal{F}$, we mean a subset of the form $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$, where $g \in G$ is such that $\xi = gP$ and $\mathcal{C}$ is a closed cone contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. For $T > 0$, we write

$$\mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o,$$

where $\mathcal{C}_T = \{u \in \mathcal{C} : \|u\| < T\}$ for a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}$.

*Definition 1.1.* We call a limit point $\xi \in \Lambda$ a horospherical limit point of $\Gamma$ if one of the following equivalent conditions holds:

- there exists a $\Gamma$-tight horoball $\mathcal{H}_\xi$ based at $\xi$ such that for any $T > 1$, $\mathcal{H}_\xi(T)$ contains some point of $\Gamma(o)$;
- there exist a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ and a sequence $\gamma_j \in \Gamma$ satisfying that $\beta_\xi(o, \gamma_j o) \in \mathcal{C}$ for all $j \geq 1$ and $\beta_\xi(o, \gamma_j o) \to \infty$ as $j \to \infty$, where $\beta$ denotes the $\mathfrak{a}$-valued Busemann map (Definition 2.3).

See Lemma 3.3 for the equivalence of the above two conditions. We denote by

$$\Lambda_h \subset \Lambda$$

the set of all horospherical limit points of $\Gamma$. The attracting fixed point $y_\gamma$ of a loxodromic element $\gamma \in \Gamma$ whose Jordan projection $\lambda(\gamma)$ belongs to $\operatorname{int} \mathcal{L}$ is always a horospherical

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limit point (Lemma 3.5). Moreover, for any $u \in \operatorname{int} \mathcal{L}$, any $u$-directional radial limit point $\xi$ (i.e. $\xi = gP$ for some $g \in G$ such that $\limsup_{t\to\infty} \Gamma g \exp(tu) \neq \emptyset$) is also a horospherical limit point (Lemma 5.3).

*Remarks 1.2*

(1) There exists a notion of horospherical limit points in the geometric boundary associated to a symmetric space, see [**10**]. When rank $G \geq 2$, this notion and the one considered here are different.

(2) Unlike the rank one case, a sequence $\gamma_i(o) \in \mathcal{H}_\xi(T_i)$, with $T_i \to \infty$, does not necessarily *converge* to $\xi$ for a $\Gamma$-tight horoball $\mathcal{H}_\xi$ based at $\xi$. It is hence plausible that a general discrete group $\Gamma$ would support a horospherical limit point outside of its limit set.

1.2. *Deneness of horospheres.* The following theorem generalizes Dal'bo's theorem [**5**] to discrete subgroups in higher rank semisimple Lie groups.

THEOREM 1.3. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $[g] \in \mathcal{E}$, the following are equivalent:*

(1) $gP \in \Lambda_h$;

(2) *$[g]NM$ is dense in $\mathcal{E}$.*

*Remarks 1.4.* Conze and Guivarc'h considered the notion of a horospherical limit point for Zariski dense discrete subgroups $\Gamma$ of $\mathrm{SL}_d(\mathbb{R})$ using the description of $\mathrm{SL}_d(\mathbb{R})/P$ as the full flag variety and the standard linear action of $\Gamma$ on $\mathbb{R}^d$ [**4**]. By duality, this notion coincides with ours and hence the special case of Theorem 1.3 for $G = \mathrm{SL}_d(\mathbb{R})$ also follows from [**4**, Theorem 4.2]. (However the claim in [**4**, Theorem 6.3] is incorrect.)

To extend Theorem 1.3 to $N$-orbits, we fix a $P^\circ$-minimal subset $\mathcal{E}_0$ of $\Gamma\backslash G$, where $P^\circ$ denotes the identity component of $P$. Clearly, $\mathcal{E}_0 \subset \mathcal{E}$. Since $P = P^\circ M$, any $P^\circ$-minimal subset is a translate of $\mathcal{E}_0$ by an element of the finite group $M^\circ\backslash M$, where $M^\circ$ is the identity component of $M$. Denote by $\mathfrak{D}_\Gamma = \{\mathcal{E}_0, \ldots, \mathcal{E}_p\}$ the finite collection of all $P^\circ$-minimal sets in $\mathcal{E}$. To understand $N$-orbit closures, it is hence sufficient to restrict to $\mathcal{E}_0$.

The following is a refinement of Theorem 1.3.

THEOREM 1.5. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $[g] \in \mathcal{E}_0$, the following are equivalent:*

(1) $gP \in \Lambda_h$;

(2) *$[g]N$ is dense in $\mathcal{E}_0$.*

*Remark 1.6.* We may consider horospherical limit points outside the context of $\Lambda$. In this case, our proofs of Theorems 1.3 and 1.5 show that if $gP \in \mathcal{F}$ is a horospherical limit point, then the closures of $[g]MN$ and $[g]N$ contain $\mathcal{E}$ and $\mathcal{E}_i$ for some $\mathcal{E}_i \in \mathfrak{D}_\Gamma$, respectively.

For $G = \mathrm{SO}^\circ(n, 1)$, $n \geq 2$, Theorem 1.5 was proved in [**16**]. When $G$ has rank one and $\Gamma < G$ is convex cocompact, every limit point is horospherical and Winter's mixing theorem [**20**] implies the $N$-minimality of $\mathcal{E}_0$.

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1.3. *Directional horospherical limit points.* We also consider the following seemingly much stronger notion.

*Definition 1.7.* For $u \in \mathfrak{a}^+$, a point $\xi \in \mathcal{F}$ is called $u$-horospherical if there exists a sequence $\gamma_j \in \Gamma$ such that $\sup_j \|\beta_\xi(o, \gamma_j o) - \mathbb{R}_+ u\| < \infty$ and $\beta_\xi(o, \gamma_j o) \to \infty$ as $j \to \infty$.

Denote by $\Lambda_h(u)$ the set of $u$-horospherical limit points. Surprisingly, it turns out that every horospherical limit point is $u$-horospherical *for all* $u \in \operatorname{int} \mathcal{L}$.

THEOREM 1.8. *For all* $u \in \operatorname{int} \mathcal{L}$, *we have*

$$\Lambda_h = \Lambda_h(u).$$

1.4. *Existence of non-dense horospheres.* A finitely generated subgroup $\Gamma < G$ is called an Anosov subgroup (with respect to $P$) if there exists $C > 0$ such that for all $\gamma \in \Gamma$, $\alpha(\mu(\gamma)) \geq C|\gamma| - C$ for all simple roots $\alpha$ of $(\mathfrak{g}, \mathfrak{a}^+)$, where $\mu(\gamma) \in \mathfrak{a}^+$ denotes the Cartan projection of $\gamma$ and $|\gamma|$ is the word length of $\gamma$ with respect to a fixed finite generating set of $\Gamma$.

For Zariski dense Anosov subgroups of $G$, almost all $NM$-orbits are dense in $\mathcal{E}$ and almost all $N$-orbits are dense in $\mathcal{E}_0$ with respect to any Patterson–Sullivan measure on $\Lambda$ [15, 14]. In particular, the set of all horospherical limit points has full Patterson–Sullivan measures.

However, as Anosov subgroups are regarded as higher rank generalizations of convex cocompact subgroups, it is a natural question whether the minimality of the $NM$-action persists in the higher rank setting. It turns out that it is not the case. Our example is based on Thurston's theorem [18, Theorem 10.7] together with the following observation on the implication of the existence of a Jordan projection of an element of $\Gamma$ lying in the boundary $\partial\mathcal{L}$ of the limit cone.

PROPOSITION 1.9. *Let* $\Gamma < G$ *be a Zariski dense discrete subgroup. For any loxodromic element* $\gamma \in \Gamma$, *we have*

$$\lambda(\gamma) \in \operatorname{int} \mathcal{L} \quad \textit{if and only if } \{y_\gamma, y_{\gamma^{-1}}\} \subset \Lambda_h,$$

*where* $y_\gamma$ *and* $y_{\gamma^{-1}}$ *denote the attracting fixed points of* $\gamma$ *and* $\gamma^{-1}$, *respectively.*

*In particular, if* $\lambda(\Gamma) \cap \partial\mathcal{L} \neq \emptyset$, *then* $\Lambda \neq \Lambda_h$ *and hence there exists a non-dense* $NM$*-orbit in* $\mathcal{E}$.

Thurston's work [18] provides many examples of Anosov subgroups satisfying that $\lambda(\Gamma) \cap \partial\mathcal{L} \neq \emptyset$. To describe them, let $\Sigma$ be a torsion-free cocompact lattice of $\mathrm{PSL}_2(\mathbb{R})$ and let $\pi : \Sigma \to \mathrm{PSL}_2(\mathbb{R})$ be a discrete faithful representation. Let $0 < d_-(\pi) \leq d_+(\pi) < \infty$ be the minimal and maximal geodesic stretching constants:

$$d_+(\pi) = \sup_{\sigma \in \Sigma - \{e\}} \frac{\ell(\pi(\sigma))}{\ell(\sigma)} \quad \text{and} \quad d_-(\pi) = \inf_{\sigma \in \Sigma - \{e\}} \frac{\ell(\pi(\sigma))}{\ell(\sigma)}, \tag{1.1}$$

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where $\ell(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\Sigma\backslash\mathbb{H}^2$ corresponding to $\sigma$ and $\ell(\pi(\sigma))$ is defined similarly.

Consider the following self-joining subgroup:

$$\Gamma_\pi := (\mathrm{id} \times \pi)(\Sigma) = \{(\sigma, \pi(\sigma)) : \sigma \in \Sigma\} < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}).$$

It is easy to see that $\Gamma$ is an Anosov subgroup of $G = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Moreover, when $\pi$ is not a conjugate by a Möbius tranformation, $\Gamma_\pi$ is Zariski dense in $G$ (cf. [12, Lemma 4.1]). Identifying $\mathfrak{a} = \mathbb{R}^2$, the Jordan projection $\lambda(\gamma_\pi)$ of $\gamma_\pi = (\sigma, \pi(\sigma)) \in \Gamma_\pi$ is given by $(\ell(\sigma), \ell(\pi(\sigma))) \in \mathbb{R}^2$. Hence, the limit cone $\mathcal{L}$ of $\Gamma_\pi$ is given by

$$\mathcal{L} := \{(v_1, v_2) \in \mathbb{R}^2_{\geq 0} : d_-(\pi)v_1 \leq v_2 \leq d_+(\pi)v_1\}.$$

Thurston [18, Theorem 10.7] showed that $d_+(\pi)$ is realized by a simple closed geodesic of $\Sigma\backslash\mathbb{H}^2$ in *most of the cases*, which hence provides infinitely many examples of $\Gamma_\pi$ which satisfy $\lambda(\Gamma_\pi) \cap \partial\mathcal{L} \neq \emptyset$. Therefore, Proposition 1.9 implies (in this case, we have $NM = N$) the following corollary.

COROLLARY 1.10. *There are infinitely many non-conjuagte Zariski dense Anosov subgroups* $\Gamma_\pi < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ *with non-dense* $NM$*-orbits in* $\mathcal{E}$.

We close the introduction by the following question (cf. [13, 17]).

*Question 1.11.* For a simple real algebraic group $G$ with rank $G \geq 2$, is every discrete subgroup $\Gamma < G$ with $\Lambda = \Lambda_h = \mathcal{F}$ necessarily a cocompact lattice in $G$?

## 2. *Preliminaries*

Let $G$ be a connected, semisimple real algebraic group. We fix, once and for all, a Cartan involution $\theta$ of the Lie algebra $\mathfrak{g}$ of $G$, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of $\theta$, respectively. We denote by $K$ the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$.

Choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of $A$ in $K$ is denoted by $M$, and we set $N$ to be the maximal contracting horospherical subgroup: for $a \in \mathrm{int}\, A^+$,

$$N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

We set $P = MAN$, which is the unique minimal parabolic subgroup of $G$, up to conjugation.

For $u \in \mathfrak{a}$, we write $a_u = \exp u \in A$. We denote by $\|\cdot\|$ the norm on $\mathfrak{g}$ induced by the Killing form. Consider the Riemannian symmetric space $X := G/K$ with the metric induced from the norm $\|\cdot\|$ on $\mathfrak{g}$ and $o = K \in X$.

Let $\mathcal{F} = G/P$ denote the Furstenberg boundary. Since $K$ acts transitively on $\mathcal{F}$ and $K \cap P = M$, we may identify $\mathcal{F} = K/M$. We denote by $\mathcal{F}^{(2)}$ the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$.

Denote by $w_0 \in K$ the unique element in the Weyl group such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$; it is the longest Weyl element. We then have $\check{P} := w_0 P w_0^{-1}$ is an opposite parabolic

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subgroup of $G$, with $\check{N}$ its unipotent radical. The map $\mathrm{i} = -\operatorname{Ad}_{w_0} : \mathfrak{a}^+ \to \mathfrak{a}^+$ is called the opposition involution.

For $g \in G$, we consider the following visual maps:

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := gw_0P \in \mathcal{F}.$$

Then $\mathcal{F}^{(2)} = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}$.

Any element $g \in G$ can be uniquely decomposed as the commuting product $g_h, g_e, g_u$, where $g_h, g_e$, and $g_u$ are hyperbolic, elliptic, and unipotent elements, respectively. The Jordan projection of $g$ is defined as the element $\lambda(g) \in \mathfrak{a}^+$ satisfying $g_h = \varphi \exp \lambda(g) \varphi^{-1}$ for some $\varphi \in G$.

An element $g \in G$ is called loxodromic if $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$; in this case, $g_u$ is necessarily trivial. For a loxodromic element $g \in G$, the point $\varphi^+ \in \mathcal{F}$ is called the attracting fixed point of $g$, which we denote by $y_g$. For any loxodromic element $g \in G$ and $\xi \in \mathcal{F}$ with $(\xi, y_{g^{-1}}) \in \mathcal{F}^{(2)}$, we have $\lim_{k\to\infty} g^k \xi = y_g$ and the convergence is uniform on compact subsets.

Note that for any loxodromic element $g \in G$,

$$\lambda(g^{-1}) = \mathrm{i}\,\lambda(g).$$

Let $\Gamma < G$ be a Zariski dense discrete subgroup of $G$. The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of $\Gamma$ is the smallest closed cone of $\mathfrak{a}^+$ containing $\lambda(\Gamma)$. It is a convex cone with non-empty interior [1].

We will use the following simple lemma.

LEMMA 2.1. *For any $v \in \lambda(\Gamma)$ and $\zeta \in \mathcal{F}$, there exists a loxodromic element $\gamma \in \Gamma$ with $\lambda(\gamma) = v$ and a neighborhood $U$ of $\zeta$ in $\mathcal{F}$ such that $\{y_\gamma\} \times U$ is a relatively compact subset of $\mathcal{F}^{(2)}$ and as $k \to \infty$,*

$$\gamma^{-k}\zeta \to y_{\gamma^{-1}} \quad \textit{uniformly on } U.$$

*Proof.* Let $\zeta \in \mathcal{F}$. Choose $\gamma_1 \in \Gamma$ such that $\lambda(\gamma_1) = v$. Since the set of all loxodromic elements of $\Gamma$ is Zariski dense in $G$ [2] and $\mathcal{F}^{(2)}$ is Zariski open in $\mathcal{F} \times \mathcal{F}$, there exists $\gamma_2 \in \Gamma$ such that $(\zeta, \gamma_2 y_{\gamma_1}) \in \mathcal{F}^{(2)}$. Let $\gamma = \gamma_2\gamma_1\gamma_2^{-1}$, so that $y_\gamma = \gamma_2 y_{\gamma_1}$. It now suffices to take any neighborhood $U$ of $\zeta$ such that $U \times \{\gamma_2 y_{\gamma_1}\}$ is a relatively compact subset of $\mathcal{F}^{(2)}$. □

2.1. *Convergence of a sequence in $X$ to $\mathcal{F}$.* By the Cartan decomposition $G = KA^+K$, for $g \in G$, we may write

$$g = \kappa_1(g) \exp(\mu(g))\kappa_2(g) \in KA^+K,$$

where $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of $g$, is uniquely determined, and $\kappa_1(g), \kappa_2(g) \in K$. If $\mu(g) \in \operatorname{int} \mathfrak{a}^+$, then $[\kappa_1(g)] \in K/M = \mathcal{F}$ is uniquely determined.

Let $\Pi$ be the set of simple roots for $(\mathfrak{g}, \mathfrak{a})$. For a sequence $g_i \to G$, we say $g_i \to \infty$ regularly if $\alpha(\mu(g_i)) \to \infty$ for all $\alpha \in \Pi$. Note that if $g_i \to \infty$ regularly, then for all sufficiently large $i$, $\mu(g_i) \in \operatorname{int} \mathfrak{a}^+$ and hence $[\kappa_1(g_i)]$ is well defined.

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*Definition 2.2.* A sequence $p_i \in X$ is said to converge to $\xi \in \mathcal{F}$ if there exists $g_i \to \infty$ regularly in $G$ with $p_i = g_i(o)$ and $\lim_{i\to\infty}[\kappa_1(g_i)] = \xi$.

2.2. *$P^\circ$-minimal subsets.* We denote by $\Lambda \subset \mathcal{F}$ the limit set of $\Gamma$, which is defined as

$$\Lambda = \{\lim \gamma_i(o) : \gamma_i \in \Gamma\}. \tag{2.1}$$

For a non-Zariski dense subgroup, $\Lambda$ may be an empty set. For $\Gamma < G$ Zariski dense, this is the unique $\Gamma$-minimal subset of $\mathcal{F}$ [1, 15].

It follows that the following set $\mathcal{E}$ is the unique $P$-minimal subset of $\Gamma\backslash G$:

$$\mathcal{E} = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}.$$

Let $P^\circ$ denote the identity component of $P$. Then $\mathcal{E}$ is a disjoint union of at most $[P : P^\circ]$-number of $P^\circ$-minimal subsets. We fix one $P^\circ$-minimal subset $\mathcal{E}_0$ once and for all. Note that any $P^\circ$-minimal subset is then of the form $\mathcal{E}_0 m$ for some $m \in M$. We set

$$\Omega := \{[g] \in \Gamma\backslash G : g^+, g^- \in \Lambda\} \quad \text{and} \quad \Omega_0 := \Omega \cap \mathcal{E}_0. \tag{2.2}$$

2.3. *Busemann map.* The Iwasawa cocycle $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$ with $\xi = [k]$ for $k \in K$, $\exp \sigma(g, \xi)$ is the $A$-component of $gk$ in the $KAN$ decomposition, that is,

$$gk \in K \exp(\sigma(g, \xi))N.$$

The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times X \times X \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(ho, go) := \sigma(h^{-1}, \xi) - \sigma(g^{-1}, \xi).$$

We note that for any $g \in G$, $\xi \in \mathcal{F}$, and $x, y, z \in X$,

$$\beta_\xi(x, y) = \beta_{g\xi}(gx, gy) \quad \text{and} \quad \beta_\xi(x, y) = \beta_\xi(x, z) + \beta_\xi(z, y). \tag{2.3}$$

In particular, $\beta_\xi(o, go) \in \mathfrak{a}$ is defined by

$$g^{-1}k_\xi \in K \exp(-\beta_\xi(o, go))N, \tag{2.4}$$

and hence $\beta_P(o, a_u o) = u$ for any $u \in \mathfrak{a}$. For $h, g \in G$, we set $\beta_\xi(h, g) := \beta_\xi(ho, go)$.

2.4. *Shadows.* For $q \in X$ and $r > 0$, we set $B(q, r) = \{x \in X : d(x, q) \le r\}$. For $p = g(o) \in X$, the shadow of the ball $B(q, r)$ viewed from $p$ is defined as

$$O_r(p, q) := \{(gk)^+ \in \mathcal{F} : k \in K,\ gk \operatorname{int} A^+ o \cap B(q, r) \neq \emptyset\}.$$

Similarly, for $\xi \in \mathcal{F}$, the shadow of the ball $B(q, r)$ as viewed from $\xi$ is

$$O_r(\xi, q) := \{h^+ \in \mathcal{F} : h \in G \text{ satisfies } h^- = \xi,\ ho \in B(q, r)\}.$$

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LEMMA 2.3. [15, Lemmas 5.6 and 5.7]

(1) *There exists $\kappa > 0$ such that for any $g \in G$ and $r > 0$,*

$$\sup_{\xi \in O_r(g(o),o)} \|\beta_\xi(g(o), o) - \mu(g^{-1})\| \le \kappa r.$$

(2) *If a sequence $p_i \in X$ converges to $\xi \in \mathcal{F}$, then for any $0 < \varepsilon < r$, we have*

$$O_{r-\varepsilon}(p_i, o) \subset O_r(\xi, o) \subset O_{r+\varepsilon}(p_i, o)$$

*for all sufficiently large $i$.*

## 3. *Horospherical limit points*

Let $\Gamma < G$ be a Zariski dense discrete subgroup. A $\Gamma$-tight horoball based at $\xi \in \mathcal{F}$ is a subset of the form $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$, where $g \in G$ is such that $\xi = gP$ and $\mathcal{C}$ is a closed cone contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. For $T > 0$, we write $\mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o$. We recall the definition from the introduction.

*Definition 3.1.* We say that $\xi \in \mathcal{F}$ is a horospherical limit point of $\Gamma$ if there exists a $\Gamma$-tight horoball $\mathcal{H}_\xi$ based at $\xi$ such that $\mathcal{H}_\xi(T) \cap \Gamma(o) \neq \emptyset$ for all $T > 1$.

In this section, we provide a mostly self-contained proof of the following theorem.

THEOREM 3.2. *Let $[g] \in \mathcal{E}$. The following are equivalent:*

(1) *$g^+ = gP \in \Lambda$ is a horospherical limit point;*

(2) *$[g]NM$ is dense in $\mathcal{E}$.*

The main external ingredient in our proof is the density of the group generated by the Jordan projection $\lambda(\Gamma)$, due to Benoist [2], that is,

$$\mathfrak{a} = \overline{\langle \lambda(\Gamma) \rangle}$$

for every Zariski dense discrete subgroup $\Gamma < G$. In fact, for every cone $\mathcal{C} \subset \mathcal{L}$ with non-empty interior, there exists a Zariski dense subgroup $\Gamma' < \Gamma$ with $\mathcal{L}_{\Gamma'} \subset \mathcal{C}$ (see [1]); therefore, we have

$$\mathfrak{a} = \overline{\langle \lambda(\Gamma) \cap \operatorname{int} \mathcal{L} \rangle}.$$

It is convenient to use a characterization of horospherical limit points in terms of the Busemann function.

LEMMA 3.3. *For $\xi \in \Lambda$, we have $\xi \in \Lambda_h$ if and only if there exists a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ and a sequence $\gamma_j \in \Gamma$ satisfying*

$$\beta_\xi(o, \gamma_j o) \to \infty \quad \text{and} \quad \beta_\xi(o, \gamma_j o) \in \mathcal{C} \quad \text{for all large } j \ge 1. \tag{3.1}$$

*Proof.* Let $\xi = gP \in \Lambda_h$ be as defined in Definition 3.1. Then there exists $\gamma_j = gpn_j a_{u_j} k_j \in \Gamma$ for some $p \in P$, $n_j \in N$, $k_j \in K$, and $u_j \to \infty$ in some closed cone $\mathcal{C}$ contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. Fix some closed cone $\mathcal{C}' \subset \operatorname{int} \mathcal{L} \cup \{0\}$ whose interior

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contains $\mathcal{C}$. Note that

$$\begin{aligned}\beta_\xi(o,\gamma_j o) &= \beta_{gP}(e,g)+\beta_{gP}(g,gpn_ja_{u_j})\\ &= \beta_P(g^{-1},e)+\beta_P(e,p)+\beta_P(e,n_j)+\beta_P(e,a_{u_j})\\ &= \beta_P(g^{-1},p)+u_j.\end{aligned}$$

Therefore, the sequence $\beta_\xi(o,\gamma_j)-u_j$ is uniformly bounded. Since $u_j\in\mathcal{C}$, $\beta_\xi(o,\gamma_j o)\in\mathcal{C}'$ for all large $j$. Therefore, equation (3.1) holds. For the other direction, let $\gamma_j$ and $\mathcal{C}$ satisfy equation (3.1) for $\xi=gP$ for $g\in G$. Since $G=gNAK$, we may write $\gamma_j=gn_ja_{u_j}k_j$ for some $n_j\in N$, $u_j\in\mathfrak{a}$ and $k_j\in K$. By a similar computation as above, the sequence $\beta_\xi(o,\gamma_j o)-u_j$ is uniformly bounded. It follows that $u_j\in\mathcal{C}'$ for all large $j$ and $u_j\to\infty$. Therefore, for any $T>1$, there exists $j>1$ such that $\gamma_j(o)\in gN\exp(\mathcal{C}'-\mathcal{C}'_T)(o)$. This proves $\xi\in\Lambda_h$. □

We note that the condition in equation (3.1) is independent of the choice of basepoint $o$. Indeed, for any $g\in G$ and $\xi\in\mathcal{F}$ and for all $\gamma\in\Gamma$, we have

$$\beta_\xi(o,\gamma o)=\beta_\xi(o,go)+\beta_\xi(go,\gamma go)+\beta_\xi(\gamma go,\gamma o),$$

and hence

$$\begin{aligned}\|\beta_\xi(o,\gamma o)-\beta_\xi(go,\gamma go)\| &= \|\beta_\xi(o,go)+\beta_\xi(\gamma go,\gamma o)\|\\ &= \|\beta_\xi(o,go)-\beta_{\gamma^{-1}\xi}(o,go)\|\\ &\le 2\cdot\max_{\eta\in\mathcal{F}}\|\beta_\eta(o,go)\|.\end{aligned}$$

Since this bound is independent of $\gamma\in\Gamma$, the condition in equation (3.1) implies that for any $p=go\in X$,

$$\beta_\xi(p,\gamma_j p)\to\infty \quad\text{and}\quad \beta_\xi(p,\gamma_j p)\in\mathcal{C} \quad\text{for all large } j. \tag{3.2}$$

Let us now consider the following seemingly stronger condition for a limit point being horospherical.

*Definition 3.4.* For $u\in\mathfrak{a}^+$, a point $\xi\in\mathcal{F}$ is called a $u$-horospherical limit point if for some $p\in X$ (and hence for any $p\in X$), there exists a constant $R>0$ and a sequence $\gamma_j\in\Gamma$ satisfying

$$\beta_\xi(p,\gamma_j p)\to\infty \quad\text{and}\quad \|\beta_\xi(p,\gamma_j p)-\mathbb{R}_+u\|<R \quad\text{for all } j.$$

We denote the set of $u$-horospherical limit points by $\Lambda_h(u)$.

By $G$-invariance of the Busemann map, the set of horospherical (respectively $u$-horospherical) limit points is $\Gamma$-invariant. Therefore, for $x=[g]\in\Gamma\backslash G$, we may say $x^+:=\Gamma gP$ horospherical (respectively $u$-horospherical) if $g^+$ is.

For $u\in\mathfrak{a}$, we call $x\in\Gamma\backslash G$ a $u$-periodic point if $xa_u=xm_0$ for some $m_0\in M$; note that $xa_{\mathbb{R}u}M_0$ is then compact. Note that for $u\in\operatorname{int}\mathfrak{a}^+$, the existence of a $u$-periodic point is equivalent to the condition that $u\in\lambda(\Gamma)$.

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LEMMA 3.5. *Let $u \in \mathfrak{a}^+$. If $x \in \Gamma\backslash G$ is u-periodic, then $x^+ \in \mathcal{F}$ is a u-horospherical limit point.*

*Proof.* Since $x$ is $u$-periodic, there exist $g \in G$ with $x = [g]$ and $\gamma \in \Gamma$ such that $\gamma = g a_u m g^{-1}$ for some $m \in M$, and $y_\gamma = g^+ \in \Lambda$. Moreover, for any $k \geq 1$,

$$\beta_{gP}(go, \gamma^k go) = \beta_P(o, a_u^k o) = ku.$$

This implies $gP$ is $u$-horospherical. □

PROPOSITION 3.6. *Let $x \in \Gamma\backslash G$. If $x^+$ is u-horospherical for some $u \in \lambda(\Gamma)$, then the closure $\overline{xN}$ contains a u-periodic point.*

*Proof.* Choose $g \in G$ so that $x = [g]$. We may assume without loss of generality that $g = k \in K$, since $kanN = kNa$, and a translate of a $u$-periodic point by an element of $A$ is again a $u$-periodic point. Since $u \in \lambda(\Gamma)$, there exists a $u$-periodic point, say, $x_0 \in \Gamma\backslash G$. It suffices to show that

$$\overline{[k]N} \cap x_0 AM \neq \emptyset \tag{3.3}$$

as every point in $x_0 AM$ is $u$-periodic.

Since $k^+$ is $u$-horospherical and using equation (2.4), there exists $R > 0$ and sequences $\gamma_j \in \Gamma$, $u_j \to \infty$ in $\mathfrak{a}^+$ and $k_j \in K$ and $n_j \in N$ satisfying $\gamma_j^{-1} k = k_j a_{-u_j} n_j$ or

$$k_j = \gamma_j^{-1} k n_j^{-1} a_{u_j}, \tag{3.4}$$

with $\|\mathbb{R}_+ u - u_j\| < R$ for all $j$. Let $\ell_j \to \infty$ be a sequence of integers satisfying

$$\|\ell_j u - u_j\| < R + \|u\| \quad \text{for all}\, j \geq 1. \tag{3.5}$$

By passing to a subsequence, we may assume without loss of generality that $\gamma_j^{-1} kP$ converges to some $\xi_0 \in \mathcal{F}$. Since $\check{N}P$ is Zariski open and $\Gamma$ is Zariski dense, we may choose $g_0 \in G$ such that $x_0 = [g_0]$ and $g_0^{-1}\xi_0 \in \check{N}P$. Let $h_0 \in \check{N}$ be such that $\xi_0 = g_0 h_0 P$. Since $g_0 \check{N} P$ is open and $\gamma_j^{-1} kP \to g_0 h_0 P$, we may assume that for all $j$, there exists $h_j \in \check{N}$ satisfying $g_0 h_j P = \gamma_j^{-1} kP = k_j P$ with $h_j \to h_0$. Let $p_j = a_{v_j} m_j \tilde{n}_j \in P = AMN$ be such that $g_0 h_j p_j = k_j$; since $h_j \to h_0$ and the product map $\check{N} \times P \to \check{N}P$ is a diffeomorphism, the sequence $p_j$, as well as $v_j \in \mathfrak{a}$, are bounded.

Therefore, by equation (3.4), we get for all $j$,

$$\begin{aligned} g_0 &= k_j p_j^{-1} h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} a_{u_j} (\tilde{n}_j^{-1} m_j^{-1} a_{-v_j}) h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} (a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) a_{u_j} m_j^{-1} a_{-v_j} h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} (a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) m_j^{-1} (a_{u_j - v_j} h_j^{-1} a_{-u_j + v_j}) a_{u_j - v_j}. \end{aligned}$$

Since $h_j^{-1} \in \check{N}$ and $v_j \in \mathfrak{a}$ are uniformly bounded and since $u_j \to \infty$ within a bounded neighborhood of the ray $\mathbb{R}_+ u \in \operatorname{int} \mathfrak{a}^+$, we have

$$\tilde{h}_j = a_{u_j - v_j} h_j^{-1} a_{-u_j + v_j} \to e \quad \text{in}\ \check{N}.$$

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By setting $n'_j = n_j^{-1}(a_{u_j}\tilde{n}_j^{-1}a_{-u_j}) \in N$, we may now write

$$g_0 = \gamma_j^{-1} k n'_j m_j^{-1} \tilde{h}_j a_{u_j - v_j}.$$

Since $x_0$ is $u$-periodic, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0 = g_0 a_u m_0 g_0^{-1}$ for some $m_0 \in M$. Hence, for all $j \geq 1$,

$$\gamma_0^{-\ell_j} = g_0 a_{-\ell_j u} m_0^{-\ell_j} g_0^{-1} = (\gamma_j^{-1} k n'_j m_j^{-1} \tilde{h}_j a_{u_j - v_j})(a_{-\ell_j u} m_0^{-\ell_j}) g_0^{-1}.$$

In other words,

$$\gamma_j^{-1} k n'_j = \gamma_0^{-\ell_j} g_0 m_0^{\ell_j} a_{-u_j + \ell_j u + v_j} \tilde{h}_j^{-1} m_j.$$

Since the sequence $-u_j + \ell_j u + v_j \in \mathfrak{a}$ is uniformly bounded by equation (3.5) and $\tilde{h}_j \to e$ in $\check{N}$, we conclude that the sequence $\Gamma k n'_j$ has an accumulation point in $\Gamma g_0 AM$. This proves equation (3.3). □

It turns out that a horospherical limit point is also $u$-horospherical for any $u \in \operatorname{int} \mathcal{L}$.

PROPOSITION 3.7. *For each $u \in \operatorname{int} \mathcal{L}$, we have $\Lambda_h = \Lambda_h(u)$.*

*Proof.* Let $\xi \in \Lambda_h$. By definition, there is a sequence $\gamma_j \in \Gamma$ satisfying $v_j := \beta_\xi(e, \gamma_j) \to \infty$ with the sequence $\|v_j\|^{-1} v_j$ converging to some point $v_0 \in \operatorname{int} \mathcal{L}$. By passing to a subsequence, we may assume that $\gamma_j^{-1}\xi$ converges to some $\xi_0 \in \mathcal{F}$.

Let $u \in \operatorname{int} \mathcal{L}$. We claim that $\xi \in \Lambda_h(u)$. We first consider the case $u \notin \mathbb{R}_+ v_0$. Let $r := \operatorname{rank} G - 1 \geq 0$. Since $\bigcup_{\gamma \in \Gamma} \mathbb{R}_+ \lambda(\gamma)$ is dense in $\mathcal{L}$, there exist $w_1, \ldots, w_r \in \lambda(\Gamma)$ such that $v_0$ belongs to the interior of the convex cone spanned by $u, w_1, \ldots, w_r$, so that

$$v_0 = c_0 u + \sum_{\ell=1}^{r} c_\ell w_\ell$$

for some positive constants $c_0, \ldots, c_\ell$.

Since $\|v_j\|^{-1} v_j \to v_0$, we may assume, by passing to a subsequence, that for each $j \geq 1$, we have

$$\|v_j\|^{-1} v_j = c_{0,j} u + \sum_{\ell=1}^{r} c_{\ell,j} w_\ell \tag{3.6}$$

for some positive $c_{\ell,j}$, $\ell = 0, \ldots, r$. Note that for each $0 \leq \ell \leq r$, $c_{\ell,j} \to c_\ell$ as $j \to \infty$.

By Lemma 2.1, we can find a loxodromic element $g_1 \in \Gamma$ and a neighborhood $U_1$ of $\xi_0$ such that $\lambda(g_1^{-1}) = w_1$, $\{y_{g_1}\} \times U_1 \subset \mathcal{F}^{(2)}$ and $g_1^{-k} U_1 \to y_{g_1^{-1}}$ uniformly. Applying Lemma 2.1 once more, we can find $g_2 \in \Gamma$ satisfying $\lambda(g_2^{-1}) = w_2$ and a neighborhood $U_2 \subset \mathcal{F}$ of $y_{g_1^{-1}}$ satisfying $\{y_{g_2}\} \times U_2 \subset \mathcal{F}^{(2)}$ and that $g_2^{-k} U_2 \to y_{g_2^{-1}}$ uniformly.

Continuing inductively, we get elements $g_1, \ldots, g_r \in \Gamma$ and open sets $U_1, \ldots, U_r \subset \mathcal{F}$ satisfying that for all $\ell = 1, \ldots, r$:

(1) $w_\ell = \lambda(g_\ell^{-1})$;

(2) $y_{g_{\ell-1}^{-1}} \in U_\ell$;

(3) $g_\ell^{-k} U_\ell \to y_{g_\ell^{-1}}$ uniformly; and

(4) $\{y_{g_\ell}\} \times U_\ell$ is a relatively compact subset of $\mathcal{F}^{(2)}$.

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We set $\xi_\ell := y_{g_\ell^{-1}}$ for each $1 \le \ell \le r$; so $U_\ell$ is a neighborhood of $\xi_{\ell-1}$ for each $1 \le \ell \le r$.

Since $\mathcal{Q}_{\eta_0} := \{\eta \in \mathcal{F} : (\eta_0, \eta) \in \mathcal{F}^{(2)}\} = \bigcup_{R>0} O_R(\eta_0, o)$ for any $\eta_0 \in \mathcal{F}$ and $U_\ell \subset \mathcal{Q}_{y_{g_\ell}}$ is a relatively compact subset of $\mathcal{F}^{(2)}$, there exists $R_\ell > 0$ such that $U_\ell \subset O_{R_\ell}(y_{g_\ell}, o)$. Since $g_\ell^k o$ converges to $y_{g_\ell}$ as $k \to +\infty$, by Lemma 2.3(2),

$$O_{R_\ell}(y_{g_\ell} o, o) \subset O_{R_\ell+1}(g_\ell^k o, o) \tag{3.7}$$

for all sufficiently large $k > 1$.

For each $1 \le \ell \le r$ and $j \ge 1$, let $k_{\ell,j}$ be the largest integer smaller than $c_{\ell,j}\|v_j\|$. As $\|v_j\| \to \infty$ and $c_{\ell,j} \to c_\ell$, we have $k_{\ell,j} \to \infty$ as $j \to \infty$. By the uniform contraction $g_\ell^{-k} U_i \to \xi_\ell$, there exists $j_0 > 1$ such that for all $j \ge j_0$,

$$\gamma_j^{-1}\xi \in U_1, \quad g_\ell^{-k_{\ell,j}} U_\ell \subseteq U_{\ell+1}, \quad \text{and} \quad U_\ell \subset O_{R_\ell+1}(g_\ell^{k_{\ell,j}} o, o) \tag{3.8}$$

for all $\ell = 1, \ldots, r$.

For each $j \ge j_0$, we now set

$$\tilde{\gamma}_j := \gamma_j g_1^{k_{1,j}} g_2^{k_{2,j}} \ldots g_r^{k_{r,j}} \in \Gamma.$$

We claim that $\beta_\xi(e, \tilde{\gamma}_j) \to \infty$ as $j \to \infty$ and that

$$\sup_{j \ge j_0} \|\beta_\xi(e, \tilde{\gamma}_j) - \mathbb{R}_+ u\| < \infty; \tag{3.9}$$

this proves that $\xi$ is $u$-horospherical.

Fix $j \ge j_0$ and for each $1 \le \ell \le r$, let $k_\ell := k_{\ell,j}$, $b_\ell := c_{\ell,j}\|v_j\|$, and set

$$h_\ell = g_1^{k_1} g_2^{k_2} \ldots g_\ell^{k_\ell},$$

and $g_0 = e$. The cocycle property of the Busemann function gives that

$$\beta_\xi(e, \tilde{\gamma}_j) = \beta_\xi(e, \gamma_j) - \sum_{\ell=1}^{r} \beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e). \tag{3.10}$$

By equation (3.8), $\gamma_j^{-1}\xi \in U_1$ and for each $1 \le \ell \le r$,

$$h_{\ell-1}^{-1}\gamma_j^{-1}\xi \in g_\ell^{-k_\ell} \ldots g_1^{-k_1} U_1 \subset U_{\ell+1} \subset O_{R_\ell+1}(g_\ell^{k_\ell} o, o).$$

Hence, by Lemma 2.3(1), there exists $\kappa \ge 1$ such that for each $1 \le \ell \le r$,

$$\|\beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e) - \mu(g_\ell^{-k_\ell})\| \le \kappa(R_\ell + 1).$$

Note that for some $C_\ell > 0$, $\|\mu(g_\ell^{-k}) - k\lambda(g_\ell^{-1})\| \le C_\ell$ for all $k \ge 1$. Since $\lambda(g_\ell^{-1}) = w_\ell$, we get

$$\|\beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e) - k_\ell w_\ell\| \le \kappa(R_\ell + 1) + C_\ell.$$

Therefore, by equation (3.10), we obtain

$$\left\|\beta_\xi(e, \tilde{\gamma}_j) - \left(v_j - \sum_{\ell=1}^{r} k_\ell w_\ell\right)\right\| \le \kappa \sum_{\ell=1}^{r} (R_\ell + C_\ell + 1).$$

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By equation (3.6), we have

$$c_{0,j}\|v_j\|u = v_j - \sum_{\ell=1}^{r} b_\ell w_\ell.$$

Since $|b_\ell - k_\ell| \le 1$ and $c_{0,j} > 0$, we deduce that for all $j \ge j_0$,

$$\begin{aligned}\|\beta_\xi(e,\tilde{\gamma}_j) - \mathbb{R}_+ u\| &\le \|\beta_\xi(e,\tilde{\gamma}_j) - c_{0,j}\|v_j\|\cdot u\| \\ &\le \left\|\beta_\xi(e,\tilde{\gamma}_j) - \left(v_j - \sum_{\ell=1}^{r} k_\ell w_\ell\right)\right\| + \sum_{\ell=1}^{r}\|k_\ell w_\ell - b_\ell w_\ell\| \\ &\le \kappa\sum_{\ell=1}^{r}(R_\ell + C_\ell + \|w_\ell\| + 1).\end{aligned}$$

This proves equation (3.9) and, consequently, $\xi$ is $u$-horospherical for any $u \notin \mathbb{R}_+ v_0$. To show that $\xi$ is $v_0$-horospherical, fix any $u \notin \mathbb{R}_+ v_0$ and $\tilde{\gamma}_j \in \Gamma$ be a sequence as in equation (3.9) associated to $u$. If we set $\tilde{v}_j = \beta_\xi(e,\tilde{\gamma}_j)$, then $\|\tilde{v}_j\|^{-1}\tilde{v}_j$ converges to a unit vector in int $\mathcal{L}$ proportional to $u$. Therefore, by repeating the same argument only now switching the roles of $v_0$ and $u$, we prove that $\xi$ is $v_0$-horospherical as well. This completes the proof. $\square$

We may now prove Theorem 3.2.

*Proof of Theorem 3.2.* Let $g \in G$ be such that $\xi = g^+ \in \Lambda$ is a horospherical limit point. Set $Y := \overline{[g]NM}$. We claim that $Y = \mathcal{E}$. By Benoist [1], the group generated by $\lambda(\Gamma) \cap$ int $\mathcal{L}$ is dense in $\mathfrak{a}$. Hence, for every $\varepsilon > 0$, there exist loxodromic elements $\gamma_1, \ldots, \gamma_q \in \Gamma$ such that

$$\lambda(\gamma_1), \ldots, \lambda(\gamma_q) \in \mathrm{Int}\mathcal{L}$$

and the group $\mathbb{Z}\lambda(\gamma_1) + \cdots + \mathbb{Z}\lambda(\gamma_q)$ is an $\varepsilon$-net in $\mathfrak{a}$, that is, its $\varepsilon$-neighborhood covers all $\mathfrak{a}$. Denote $u_i = \lambda(\gamma_i)$ for $i = 1, \ldots, q$. By Proposition 3.7, the point $\xi$ is $u_1$-horospherical. By Proposition 3.6, there exists a $u_1$-periodic point $x_1 \in \mathcal{E}$ contained in $Y$, set

$$Y_1 := \overline{x_1 NM} \subset Y.$$

By Lemma 3.5, $x_1^+$ is $u_1$-horospherical; in particular, it is a horospherical limit point. Therefore, we can inductively find a $u_i$-periodic point $x_i$ in $Y_{i-1} = \overline{x_{i-1}NM}$ for each $2 \le i \le q$. By periodicity, $x_i(\exp u_i)M = x_i M$, and hence $Y_i \exp \mathbb{Z}u_i = Y_i$ for each $1 \le i \le q$. Therefore, we obtain

$$Y \supset Y_1 \exp \mathbb{Z}u_1 \supset Y_2 \exp(\mathbb{Z}u_1 + \mathbb{Z}u_2) \supset \cdots \supset Y_q \exp\left(\sum_{i=1}^{q}\mathbb{Z}u_i\right).$$

Recalling the dependence of $Y_q$ and $\sum_{i=1}^{q}\mathbb{Z}u_i$ on $\varepsilon$, set

$$Z_\varepsilon := Y_q MN \exp\left(\sum_{i=1}^{q}\mathbb{Z}u_i\right) \subset Y.$$

Since $MN\exp(\sum_{i=1}^{q}\mathbb{Z}u_i)$ is an $\varepsilon$-net of $P$ and $\mathcal{E}$ is $P$-minimal, $Z_\varepsilon$ is a $2\varepsilon$-net of $\mathcal{E}$ for all $\varepsilon > 0$. Since $Y$ contains a $2\varepsilon$-net of $\mathcal{E}$ for all $\varepsilon > 0$ and $Y$ is closed, it follows that $Y = \mathcal{E}$.

For the other direction, suppose that $[g]NM$ is dense in $\mathcal{E}$ for $g \in G$. Choose any $u \in \operatorname{int} \mathcal{L}$ and a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ which contains $u$. Then $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$ is a $\Gamma$-tight horoball. Let $t > 1$. Since $ga_{-2tu} \in \mathcal{E}$, there exist $\gamma_i \in \Gamma$, $n_i \in N$, $m_i \in M$, and $q_i \to e$ in $G$ such that for all $i \geq 1$, $\gamma_i g n_i m_i q_i = ga_{-2tu}$. Since $d(\gamma_i^{-1} g, g n_i m_i a_{2tu}) \leq d(q_i a_{2tu}, a_{2tu}) \to 0$ as $i \to \infty$, it follows that for all sufficiently large $i \geq 1$, $\gamma_i^{-1} go \in \mathcal{H}_\xi(t)$. Hence, $g^+$ is a horospherical limit point by Definition 3.1. □

## 4. *Topological mixing and directional limit points*

There is a close connection between denseness of $N$-orbits and the topological mixing of one-parameter diagonal flows with direction in $\operatorname{int} \mathcal{L}$. This connection allows us to make use of recent topological mixing results by Chow and Sarkar [3]: recall the notation $\Omega_0$ from equation (2.2).

THEOREM 4.1. [3] *For any $u \in \operatorname{int} \mathcal{L}$, $\{a_{tu} : t \in \mathbb{R}\}$ is topologically mixing on $\Omega_0$, that is, for any open subsets $\mathcal{O}_1, \mathcal{O}_2$ of $\Gamma\backslash G$ intersecting $\Omega_0$,*

$$\mathcal{O}_1 \exp tu \cap \mathcal{O}_2 \neq \emptyset \quad \textit{for all large } |t| \gg 1.$$

The above theorem was predated by a result of Dang [6] in the case where $M$ is abelian.

### 4.1. *$N$-orbits based at directional limit points along* $\operatorname{int} \mathcal{L}$.

*Definition 4.2.* For $u \in \operatorname{int} \mathfrak{a}^+$, denote by $\Lambda_u$ the set of all $u$-directional limit points, that is, $\xi \in \Lambda_u$ if and only if $\limsup_{t\to+\infty} \Gamma g \exp(tu) \neq \emptyset$ for some (and hence any) $g \in G$ with $gP = \xi$.

It is easy to see that $\Lambda_u \subset \Lambda$ for $u \in \operatorname{int} \mathfrak{a}^+$.

PROPOSITION 4.3. *If $[g] \in \mathcal{E}_0$ satisfies $g^+ \in \Lambda_u$ for some $u \in \operatorname{int} \mathcal{L}$, then*

$$\overline{[g]N} = \mathcal{E}_0.$$

*Proof.* Since $\Omega_0 N = \mathcal{E}_0$, we may assume without loss of generality that $x = [g] \in \Omega_0$. There exist $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i u}$ converges to some $h \in G$. In particular, $x \exp(t_i u) \to [h]$. Since $x a_{t_i u} \in \Omega_0$ and $\Omega_0$ is $A$-invariant and closed, we have $[h] \in \Omega_0$. We write $\gamma_i g a_{t_i u} = h q_i$, where $q_i \to e$ in $G$. Therefore, $xN = [h] q_i N a_{-t_i u}$ for all $i \geq 1$. Let $\mathcal{O} \subset \Gamma\backslash G$ be any open subset intersecting $\Omega_0$. It suffices to show that $xN \cap \mathcal{O} \neq \emptyset$. Let $\mathcal{O}_1$ be an open subset intersecting $\Omega_0$ and $V \subset \check{P}$ be an open symmetric neighborhood of $e$ such that $\mathcal{O}_1 V \subset \mathcal{O}$.

Since $q_i \to e$ and $NV$ is an open neighborhood of $e$ in $G$, there exists an open neighborhood, say, $U$ of $e$ in $G$ and $i_0$ such that $U \subset q_i NV$ for all $i \geq i_0$. By Theorem 4.1, we can choose $i > i_0$ such that $[h]U \cap \mathcal{O}_1 a_{t_i u} \neq \emptyset$. It follows that $[h] q_i N V a_{-t_i u} \cap \mathcal{O}_1 \neq \emptyset$. Since $V \subset a_{-t_i u} V a_{t_i u}$ as $u \in \mathfrak{a}^+$, we have

$$[h] q_i N V a_{-t_i u} \cap \mathcal{O}_1 \subset [h] q_i N a_{-t_i u} V \cap \mathcal{O}_1.$$

Since $V = V^{-1}$, we get $[h] q_i N a_{-t_i u} \cap \mathcal{O}_1 V \neq \emptyset$. Therefore, $xN \cap \mathcal{O} \neq \emptyset$, as desired. □

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This immediately implies the following corollary.

COROLLARY 4.4. *If* $[g] \in \Omega_0$ *is u-periodic for some* $u \in \operatorname{int} \mathcal{L}$, *then*

$$\overline{[g]N} = \mathcal{E}_0.$$

*Proof.* Since $[g](\exp ku) = [g]m_0^k$ for any integer $k$ and $M$ is compact, we have $g^+ \in \Lambda_u$. Therefore, the claim follows from Proposition 4.3. □

We may now conclude our main theorem in its fullest form.

THEOREM 4.5. *Let* $[g] \in \mathcal{E}_0$. *The following are equivalent:*

(1) $g^+ \in \Lambda$ *is a horospherical limit point;*
(2) $[g]N$ *is dense in* $\mathcal{E}_0$*;*
(3) $[g]NM$ *is dense in* $\mathcal{E}$.

*Proof.* The implication (2) ⇒ (3) is trivial and (3) ⇒ (1) was shown in Theorem 3.2. Hence, let us prove (1) ⇒ (2).

Let $x = [g] \in \mathcal{E}_0$. Suppose that $g^+ \in \Lambda_h$. Fix any $u \in \lambda(\Gamma) \cap \operatorname{int} \mathcal{L}_\Gamma$. By Propositions 3.7 and 3.6, $xN$ contains a $u$-periodic point, say, $x_0$. Hence, by Corollary 4.4, $\overline{xN} \supset \overline{x_0 N} \supset \Omega_0 N = \mathcal{E}_0$. This proves (1) ⇒ (2). □

## 5. *Conical limit points, minimality, and Jordan projection*

A point $\xi \in \mathcal{F}$ is called a *conical* limit point of $\Gamma$ if there exists a sequence $u_j \to \infty$ in $\mathfrak{a}^+$ such that for some (and hence every) $g \in G$ with $\xi = gP$,

$$\limsup_{j\to\infty} \Gamma g a_{u_j} \neq \emptyset.$$

A conical limit point of $\Gamma$ is indeed contained in $\Lambda$. We consider the following restricted notion.

*Definition 5.1.* We call $\xi \in \mathcal{F}$ a *strongly conical* limit point of $\Gamma$ if there exists a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ and a sequence $u_j \to \infty$ in $\mathcal{C}$ such that for some (and hence every) $g \in G$ with $\xi = gP$,

$$\limsup_{j\to\infty} \Gamma g a_{u_j} \neq \emptyset.$$

*Remarks 5.2.* We mention that a conical limit point defined in [4] for $\Gamma < \mathrm{SL}_d(\mathbb{R})$ coincides with our strongly conical limit point.

LEMMA 5.3. *Any strongly conical limit point of* $\Gamma$ *is horospherical.*

*Proof.* Suppose that $\xi = gP$ is strongly conical, that is, there exist $\gamma_j \in \Gamma$ and $u_j \to \infty$ in some closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ such that $\gamma_j g a_{u_j}$ converges to some $h \in G$. Write $\gamma_j g a_{u_j} = hq_j$, where $q_j \to e$ in $G$. Let $\mathcal{C}'$ be a closed cone contained in $\operatorname{int} \mathcal{L} \cup \{0\}$ whose interior contains $\mathcal{C} \setminus \{0\}$.

Then $\gamma_j^{-1} = g a_{u_j} q_j^{-1} h^{-1}$ and

$$\beta_{gP}(e, \gamma_j^{-1}) = \beta_P(g^{-1}, a_{u_j} q_j^{-1} h^{-1}) = \beta_P(g^{-1}, q_j^{-1} h^{-1}) + \beta_P(e, a_{u_j}).$$

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Since $\beta_P(e, a_{u_j}) = u_j$ and $q_j^{-1}h^{-1}$ are uniformly bounded, the sequence

$$\beta_{gP}(e, \gamma_j^{-1}) - u_j$$

is uniformly bounded. Since $u_j \in \mathcal{C}$ and $\mathcal{C} \subset \operatorname{int} \mathcal{C}' \cup \{0\}$, it follows that

$$\beta_{gP}(e, \gamma_j^{-1}) \in \mathcal{C}'$$

for all sufficiently large $j$. This proves that $\xi \in \Lambda_h$. □

COROLLARY 5.4. *For any $g \in G$ with strongly conical $g^+ \in \mathcal{F}$, we have*

$$\overline{[g]NM} = \mathcal{E}.$$

5.1. *Directionally conical limit points.* If $v \in \operatorname{int} \mathcal{L}$, then clearly $\Lambda_v$ is contained in the horospherical limit set of $\Gamma$, and hence any $NM$-orbit based at a point of $\Lambda_v$ is dense in $\mathcal{E}$. However, we would like to show in this section that the existence of a point in $\Lambda_v$ for $v \in \partial\mathcal{L}_\Gamma$ implies the existence of a non-dense $NM$-orbit in $\mathcal{E}$.

The flow $\exp(\mathbb{R}u)$ is said to be topologically transitive on $\Omega/M = \{\Gamma g M : g^\pm \in \Lambda\}$ if for any open subsets $\mathcal{O}_1, \mathcal{O}_2$ intersecting $\Omega/M$, there exists a sequence $t_n \to +\infty$ such that $\mathcal{O}_1 \cap \mathcal{O}_2 a_{t_n u} \neq \emptyset$.

We make the following simple observation.

LEMMA 5.5. *For $g \in \Omega$, we have*

$$\overline{gNM} \supset \Omega \quad \textit{if and only if} \quad \overline{gw_0\check{N}M} \supset \Omega.$$

*Proof.* We have $\check{N} = w_0 N w_0^{-1}$. Note that $[g] \in \Omega$ if and only if $[gw_0] \in \Omega$, since $(gw_0)^\pm = g^\mp$. So $\Omega w_0 = \Omega$. Hence, $gNM$ is dense in $\Omega$ if and only if $gw_0\check{N}Mw_0^{-1}$ is dense in $\Omega$ if and only if $[g]w_0\check{N}M$ is dense in $\Omega w_0 = \Omega$. □

Since the opposition involution preserves $\mathcal{L}$ and $\lambda(g^{-1}) = \mathrm{i}\,\lambda(g)$ for any loxodromic element, it follows that $\lambda(\gamma) \in \partial\mathcal{L}$ if and only if $\lambda(\gamma^{-1}) \in \partial\mathcal{L}$.

PROPOSITION 5.6

(1) *If $\Lambda = \Lambda_h$, then* $\exp(\mathbb{R}v)$ *is topologically transitive on $\Omega/M$ for any $v \in \operatorname{int} \mathfrak{a}^+$ such that $\Lambda_v \neq \emptyset$.*

(2) *For any loxodromic element $\gamma \in \Gamma$ with $\{y_\gamma, y_{\gamma^{-1}}\} \subset \Lambda_h$, the flow* $\exp(\mathbb{R}\lambda(\gamma))$ *is topologically transitive on $\Omega/M$.*

*Proof.* Assume that $\Lambda = \Lambda_h$; so the $NM$-action on $\mathcal{E}$ is minimal. Suppose that $\Lambda_v \neq \emptyset$ for some $v \in \operatorname{int} \mathfrak{a}^+$. We claim that for any $\mathcal{O}_1, \mathcal{O}_2$ be two right $M$-invariant open subsets intersecting $\Omega$, $\mathcal{O}_1 \exp(t_i v) \cap \mathcal{O}_2 \neq \emptyset$ for some sequence $t_i \to +\infty$. Choose $x = [g] \in \Omega$ so that $g^+ \in \Lambda_v$. Then there exists $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i v}$ converges to some $g_0$. Note that $x_0 := [g_0] \in \Omega$. So write $\gamma_i g a_{t_i v} = g_0 h_i$ with $h_i \to e$. By the $NM$-minimality assumption, $xNM$ intersects every open subset of $\Omega$. Since $v \in \operatorname{int} \mathfrak{a}^+$ and hence $a_{-tv} n a_{tv} \to e$ as $t \to +\infty$, we may assume without loss of generality that $x \in \mathcal{O}_1$. Choose an open neighborhood $U$ of $e$ in $G$ so that $\mathcal{O}_1 \supset xUM$. Note that there

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exists a sequence $T_i \to \infty$ as $i \to \infty$ such that for all $i$,

$$xUMa_{t_i v} \supset xa_{t_i v}a_{-t_i v}\check{N}_\varepsilon Ma_{t_i v} \supset x_0 h_i \check{N}_{T_i},$$

where $\check{N}_R = \check{N} \cap B_R^G$ is the set of elements of $\check{N}$ of norm$\le R$. So $\mathcal{O}_1 a_{t_i v} \supset x_0 h_i \check{N}_{T_i}$.

Choose an open neighborhood $V$ of $e$ in $G$ and some open subset $\mathcal{O}_2'$ intersecting $\Omega$ so that $\mathcal{O}_2 \supset \mathcal{O}_2' V$. Since $x_0 \check{N} M$ is dense in $\Omega$, $x_0 n \in \mathcal{O}_2'$ for some $n \in \check{N}$. Hence, $x_0 h_i n = x_0 n(n^{-1} h_i n) \in \mathcal{O}_2' V \subset \mathcal{O}_2$ for all $i$ large enough so that $n^{-1} h_i n \in V$. Therefore, for all $i$ such that $n \in \check{N}_{T_i}$, we get

$$x_0 h_i n \in \mathcal{O}_1 a_{t_i v} \cap \mathcal{O}_2 \ne \emptyset.$$

This proves the first claim.

Now suppose that $\gamma \in \Gamma$ is a loxodromic element with $y_\gamma, y_{\gamma^{-1}} \in \Lambda_h$. Write $\gamma = g m a_v g^{-1}$ for some $g \in G$ and $m \in M$. Since $y_\gamma = g^+$ and $y_{\gamma^{-1}} = g w_0^+$, we have each $[g]NM$ and $[g]w_0 NM$ contains $\Omega$ in its closure. Now in the notation of the proof of the first claim, note that $x_0 = [g_0] \in [g]M$ since $[g]\exp(\mathbb{R}v)M$ is closed. Therefore, each $\overline{x_0 NM}$ and $\overline{x_0 \check{N} M}$ contains $\Omega$. Based on this, the same argument as above shows the topological transitivity of $\exp \mathbb{R}v$, which finishes the proof since $v = \lambda(\gamma)$. □

Since $\mathcal{L}$ is invariant under the opposition involution i and $\lambda(\gamma) = \mathrm{i}\,\lambda(\gamma^{-1})$ for any loxodromic element $\gamma \in \Gamma$, the Jordan projection $\lambda(\gamma)$ belongs to $\partial\mathcal{L}$ if and only if the Jordan projection $\lambda(\gamma^{-1})$ belongs to $\partial\mathcal{L}$. Together with the result of Dang and Gloriuex [7, Proposition 4.7], which says that $\exp(\mathbb{R}u)$ is not topologically transitive on $\Omega/M$ for any $u \in \partial\mathcal{L} \cap \operatorname{int}\mathfrak{a}^+$, Proposition 5.6 implies the following corollary.

COROLLARY 5.7

(1) *If* $\Lambda_v \ne \emptyset$ *for some* $v \in \partial\mathcal{L} \cap \operatorname{int}\mathfrak{a}^+$, *then*

$$\Lambda \ne \Lambda_h.$$

(2) *For any loxodromic element* $\gamma \in \Gamma$, *we have* $\lambda(\gamma) \in \partial\mathcal{L}$ *if and only if*

$$\{y_\gamma, y_{\gamma^{-1}}\} \not\subset \Lambda_h.$$

*Hence, if* $\Lambda = \Lambda_h$, *then* $\lambda(\Gamma) \subset \operatorname{int}\mathcal{L}$.

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