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POSITIVE DERIVATIONS ON PARTIALLY ORDERED STRONGLY REGULAR RINGS

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Abstract

The author presents a proof that a partially ordered strongly regular ring S which has the additional property that the square of each member of S is greater than or equal to zero cannot have nontrivial positive derivations.

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1. Introduction

In recent years efforts have been made to study positive derivations on certain classes of partially ordered rings, see Colville [1], and also on partially ordered linear algebras, see Dai [2]. This note is concerned exclusively with the investigation of positive derivations on partially ordered strongly regular rings. It will be shown that a p.o. ring $(S, +, \cdot, \leqslant)$ which is strongly regular and has the additional property that $x^2 \geqslant 0$ for each $x \in S$ cannot have nontrivial positive derivations. Also, an example is included to complement the given theorem.

Recall that a ring $(S, +, \cdot)$ is strongly regular if for each $x \in S$ there exists a $y \in S$ such that $x^2 y = x$. Terminology and background material on p.o. rings needed for this article may be found in Fuchs [3].

DEFINITION. The statement that δ is a positive derivation on a p.o. ring $(S, +, \cdot, \leqslant)$ means that δ is a map from S into S such that (1) $\delta(x+y) = \delta(x) + \delta(y)$ for each $x, y \in S$, (2) $\delta(xy) = x\delta(y) + \delta(x)y$ for each $x, y \in S$, and (3) $\delta(x) \geqslant 0$ for each $x \in S$, with $x \geqslant 0$.

2. Main results

Before proving the main theorem, we need the following lemma.

LEMMA. *Suppose $(S, +, \cdot, \leqslant)$ is a partially ordered strongly regular ring such that $x^2 \geqslant 0$ for each $x \in S$. If δ is a positive derivation defined on S and $x \in S$, with $x \geqslant 0$, then $\delta(x) = 0$.*

PROOF. Let $x \in S$ with $x \geqslant 0$. Since the ring is strongly regular, there exists a $y \in S$ such that $x^2y = x$. Using the fact that $xyx = x$, it follows that $\delta(x) = \delta(xyx) = \delta(xy)x + (xy)\delta(x)$. Thus, $x\delta(x) = x\delta(xy)x + x\delta(x)$ and this implies that $x\delta(xy)x = 0$. Hence $[x\delta(xy)]^2 = 0$ and consequently $x\delta(xy) = 0$, since $(S, +, \cdot, \leqslant)$ contains no nonzero nilpotent elements. In a similar manner, $\delta(xy)x = 0$ and thus $\delta(x) = xy\delta(x)$. Recalling that it can be shown that y will commute with x in a strongly regular ring, we have that $\delta(xy) = \delta(yx) = y\delta(x) + \delta(y)x$ and this implies, with $x\delta(xy) = 0$, that $0 = xy\delta(x) + x\delta(y)x$. Hence $xy\delta(x) = -x\delta(y)x$. Therefore $\delta(x) = -x\delta(y)x$. From $0 \leqslant xy^2$, we have $0 \leqslant \delta(xy^2) = \delta(xy)y + (xy)\delta(y)$. Multiplying on the left by x, gives $0 \leqslant x\delta(xy)y + x\delta(y) = 0 + x\delta(y) = x\delta(y)$. Next, multiplying on the right by x, we obtain $0 \leqslant x\delta(y)x = -\delta(x)$. Hence $\delta(x) \leqslant 0$. Thus $\delta(x) = 0$, since it is also true that $\delta(x) \geqslant 0$.

THEOREM. *Suppose $(S, +, \cdot, \leqslant)$ is a partially ordered strongly regular ring such that $x^2 \geqslant 0$ for each $x \in S$. If δ is a positive derivation defined on S, then $\delta(x) = 0$ for each $x \in S$.*

PROOF. Suppose $x_1 \in S$ such that $\delta(x_1) \neq 0$. Consider x_1^2. From the preceding lemma, $\delta(x_1^2) = 0$, since $x_1^2 \geqslant 0$. Appealing again to strong regularity, there exists a $y_1 \in S$ such that $x_1^2 y_1 = x_1$. Thus, $\delta(x_1) = \delta(x_1^2)y_1 + x_1^2\delta(y_1) = x_1^2\delta(y_1)$. Multiplying the preceding equation by y_1 on the left and using the fact that $y_1x_1^2 = x_1$ is also true, we obtain $y_1\delta(x_1) = x_1\delta(y_1)$. Consequently, $\delta(x_1) = x_1[x_1\delta(y_1)] = x_1y_1\delta(x_1) = y_1x_1\delta(x_1)$. Hence $x_1\delta(x_1) \neq 0$ since $\delta(x_1) \neq 0$. Thus $[x_1\delta(x_1)]^2 \neq 0$, since the ring contains no nonzero nilpotent elements. Next, $0 = \delta(x_1^2) = x_1\delta(x_1) + \delta(x_1)x_1$ implies that $x_1\delta(x_1) = -\delta(x_1)x_1$. Multiplying the preceding equation on the left by x_1 and on the right by $\delta(x_1)$, gives $0 \leqslant x_1^2[\delta(x_1)]^2 = -[x_1\delta(x_1)]^2$.

Hence, by the hypothesis on the ring, $[x_1\delta(x_1)]^2 = 0$ and this contradicts the earlier observation that $[x_1\delta(x_1)]^2 \neq 0$. Therefore $\delta(x_1) = 0$. Consequently $\delta(x) = 0$ for each $x \in S$.

3. An example

We conclude this note with an example of a partially ordered strongly regular ring $(S, +, \cdot, \leqslant)$ such that (1) the positive cone $P \neq \{0\}$, (2) contains at least one element $x \in S$ such that $x^2 \ngeqslant 0$, and (3) has associated with it a positive derivation δ such that $\delta(x) \not\equiv 0$.

Consider the following construction. Let $(S, +, \cdot)$ denote the ring of all formal truncated Laurent series $\sum_{i=n}^{\infty} a_i x^i$, where a_i is a real number and n is an arbitrary integer, with ordinary addition and multiplication of series as the two operations defined on S. Let the positive cone P, for the partial ordering on S, consist of all series of the form $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \geqslant 0$ for $0 \leqslant i < \infty$. Note that the sum and product of two members of P is again a member of P and $P \neq \{0\}$. This ring is a field and thus $(S, +, \cdot, \leqslant)$ is a partially ordered strongly regular ring. Now define a mapping δ from S into S by $\delta[\sum_{i=n}^{\infty} a_i x^i] = \sum_{i=n}^{\infty} i a_i x^{i-1}$. It is easily seen that δ is a positive derivation and is obviously nontrivial. Last, we observe that the series $g(x) = 1 - x$ has the property that $[g(x)]^2 \ngeqslant 0$.

References

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