Theorems connecting different classes of Self-Reciprocal Functions

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1. The question, "How, from a given function which is self-reciprocal for a transform of a particular order, can we construct other functions which are self-reciprocal for transforms of different orders?" was first raised by Hardy and Titchmarsh[1] who gave some rules for constructing such functions. Following their method, I have shown, in a recent paper,[2] that there are certain general theorems of the following type:—

If $f(x)$ *is its own* J_μ *transform,* $g(x)$ *is its own* J_ν *transform.* In this note I add a few more such theorems, the interest lying mainly in the results themselves and not in a rigorous proof thereof; and hence only the formal procedure is given here.

For his constant guidance in my work I wish to express my thanks to Prof. E. C. Titchmarsh at whose suggestion I started the investigation.

2. Following Hardy and Titchmarsh I will say that a function is R_ν if it is self-reciprocal for J_ν transforms, and it is $-R_\nu$ if it is skew-reciprocal for J_ν transforms. Also, for $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ I will write R_s and R_c respectively.

I will make use of the following result of Hardy and Titchmarsh.[3]

A necessary and sufficient condition that a function $f(x)$ should be R_μ is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s\right)\psi(s)\,x^{-s}\,ds, \tag{2.1}$$

where $0 < c < 1$, and

$$\psi(s) = \psi(1-s). \tag{2.2}$$

[1] Hardy and Titchmarsh IV.

[2] Mehrotra V.

[3] Hardy and Titchmarsh III, Theorem 8.

3. Theorem I. If $f(x)$ is R_1, the function

$$g(x) = x^{-\frac{1}{2}\nu} \int_0^\infty y^{-\frac{1}{2}\nu} H_{\frac{1}{2}(\nu-1)}(xy) f(y)\, dy,$$

where $H_\nu(x)$ is Struve's function of order[1] ν, is R_ν.

This theorem can be proved on the lines of a similar theorem of Hardy and Titchmarsh[2], or it can be derived from a more general theorem proved in my paper referred to above[3], with the help of a formula given by Watson[4].

4. Theorem II. If $f(x)$ is R_μ, the function

$$g(x) = \frac{1}{x} \int_0^x Q\left(\log \frac{x}{y}\right) f(y)\, dy$$

is R_ν, provided that

$$\left. \begin{aligned} Q(x) &= \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma\left(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s\right) \Gamma\left(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s\right) \chi(s)\, e^{xs}\, ds \qquad (x>0) \\ &= 0 \qquad (x<0), \end{aligned} \right\} \tag{4.1}$$

where k is any positive number, and $\chi(s)$ satisfies (2.2).

By (2.1) we have

$$g(x) = \frac{1}{2\pi i x} \int_0^x Q\left(\log \frac{x}{y}\right) dy \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\, \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\mu + \tfrac{1}{2}s\right) \psi(s)\, y^{-s}\, ds,$$

where $\psi(s) = \psi(1-s)$. Hence

$$\begin{aligned} g(x) &= \frac{1}{2\pi i x} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\, \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\mu + \tfrac{1}{2}s\right) \psi(s)\, ds \int_0^x Q\left(\log \frac{x}{y}\right) y^{-s}\, dy \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\, \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\mu + \tfrac{1}{2}s\right) \psi(s)\, x^{-s}\, ds \int_0^\infty e^{(s-1)u} Q(u)\, du. \end{aligned}$$

Now, using a form of Mellin's Inversion Formula[5], from (4.1) we have

$$\int_0^\infty e^{-xs} Q(x)\, dx = \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\mu + \tfrac{1}{2}s\right) \Gamma\left(\tfrac{3}{4} + \tfrac{1}{2}\nu - \tfrac{1}{2}s\right) \chi(s),$$

where $\chi(s) = \chi(1-s)$. Changing s into $1-s$, we get

$$\int_0^\infty e^{(s-1)x} Q(x)\, dx = \Gamma\left(\tfrac{3}{4} + \tfrac{1}{2}\mu - \tfrac{1}{2}s\right) \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}s\right) \chi(s).$$

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[1] Watson VII, § 10.4 (2).

[2] Hardy and Titchmarsh IV, § 2.

[3] Mehrotra V, § 8.

[4] Watson VII, § 13.24 (2).

[5] See Bateman I, Hardy II and Pincherle VI.

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Hence

$$g(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\chi(s)\psi(s)x^{-s}ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\psi_1(s)\,x^{-s}\,ds,$$

where $\psi_1(s)=\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\,\chi(s)\,\psi(s)$.

As $\psi_1(s)$ satisfies the equation

$$\psi_1(s)=\psi_1(1-s),$$

it follows from (2.1) that $g(x)$ is R_ν.

5. If, in Theorem II, we put $\mu=\nu$, we get a corollary. If $f(x)$ is R_ν, the function

$$g(x)=\frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right)f(y)\,dy$$

is R_ν provided that

$$Q(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{xs}\,\lambda(s)\,ds \qquad (x>0)$$
$$=0 \qquad (x<0),$$

where $\lambda(s)=\lambda(1-s)$.

6. Theorem III. If $f(x)$ is R_μ, the function

$$g(x)=\int_0^{1/x} Q\left(\log\frac{1}{xy}\right)f(y)\,dy$$

is R_ν, provided that

$$\left.\begin{aligned} Q(x)&=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{s}\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\,\chi(s)\,e^{xs}\,ds \quad (x>0)\\ &=0 \qquad (x<0),\end{aligned}\right\} \tag{6.1}$$

where $\chi(s)$ satisfies (2.2).

The proof of this theorem is similiar to that of Theorem II.

The symmetry of the integral in (6.1) shows that if $f(x)$ is R_ν, $g(x)$ is R_μ.

7. For the particular case $\mu=\mp\tfrac{1}{2}$, $\nu=\pm\tfrac{1}{2}$, the above theorem takes the simpler form:

If $f(x)$ is R_c (or R_s), the function

$$g(x)=\int_0^{1/x} Q\left(\log\frac{1}{xy}\right)f(y)\,dy$$

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is R_s (or R_c) provided that

$$Q(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(s)\,\chi(s)\, e^{xs}\, ds \qquad (x > 0)$$
$$= 0 \qquad (x < 0),$$

where $\chi(s)$ satisfies (2.2).

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