RATIONAL POLYGONS

T. K. SHENG

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1. Introduction

A polygon is said to be rational if all its sides and diagonals have rational lengths. I. J. Schoenberg has posed the interesting problem, "Can any polygon be approximated as closely as we like by a rational polygon?" Besicovitch [2] proved that right-angled triangles and parallelograms can be approximated in Schoenberg's sense, the proofs were improved by Daykin [5]. Mordell [7] proved that any quadrilateral can be approximated by a rational quadrilateral. By adapting Mordell's proof, Almering [1] generalised Mordell's result by showing that, if A, B, C are three distinct points with the distances AB, BC, CA all rational, then the set of points P for which PA, PB, PC are rational is everywhere dense in the plane that contains ABC. Daykin [4] extended the results of Besicovitch [3] and Mordell [7] by adding the requirement that the approximating quadrilaterals have rational area. He also proved that any hexagon with an axis of symmetry through two corners can be approximated by a rational hexagon with rational area and an axis of symmetry through two corners.

In this paper we prove

THEOREM 1. *Let CB be a diameter of the unit circle $\mathscr{C}$ with centre 0, and let D be a point on CB (produced if necessary). Then given $\varepsilon > 0$, there exists on CB, a point A within ε of D and a finite set S_ε of points on $\mathscr{C}$ such that*

(i) *Given any point P on $\mathscr{C}$, there exist a point P_i of S_ε within ε of P,*
(ii) *The set of points S_ε is symmetric about CB,*
(iii) *The polygon formed by the points of S_ε and the points 0, A, B, C is a rational polygon with rational area.*

It is easy to see that Daykin's extension of Mordell's result on general quadrilaterals is a special case of Theorem 1.

The key to our proof of Theorem 1 lies in finding rational solutions of a diophantine equation, namely

(1.1) $$y^2 = x^4+2mx^2+1 \text{ where } m \text{ is rational, } m > 1.$$

This equation is derived from the geometric properties of the polygon described in (iii) of Theorem 1. We give the derivation in Lemma 2. Now the proofs given by Almering, Daykin and Mordell in their papers cited above depend on showing that there is an everywhere dense set of rational points on some cubic curve. To prove theorem 1, however, we do not need a dense set of rational points of (1.1), but just a finite number of rational points.

2. Derivation of (1.1)

We start by defining Θ to be the set of all real numbers θ such that $\sin\theta$ and $\cos\theta$ are both rational. Besicovitch's result on right angled triangles [3] implies that Θ is dense in the set of all real numbers. Also it is easy to prove [8] by elementary trigonometry that

(i) $\theta \in \Theta$ iff either $\tan\frac{1}{2}\theta$ is rational or θ is a multiple of π, and
(ii) any integral linear combination of elements of Θ belongs to Θ.

We now prove

LEMMA 1. *Let CB be a diameter of the unit circle $\mathscr{C}$ with centre O. Let P_2, Q be points of $\mathscr{C}$ and put $\phi = \frac{1}{2}\angle BOP_2$ and $\psi = \frac{1}{2}\angle BOQ$. Further let A be the point of intersection of P_2Q and CB produced. Then distances OA, P_2A, QA are all rational iff $\psi, \phi \in \Theta$.*

PROOF. We note that no matter where A lies on CB the angles $\angle OAP_2$ and $\angle AQO$ and $\angle AP_2O$ are all expressible in the form $\pm\frac{1}{2}\pi\pm\phi\pm\psi$ and the result follows by applying (i), (ii) and the sine rule.

LEMMA 2. *Let $\mathscr{C}$ be the unit circle centre O. Also let A be any point other than O with $l = OA$ rational and put*

$$m = 2(l+1)^2(l-1)^{-2}-1. \tag{2.1}$$

For any point P on $\mathscr{C}$ put

$$z = AP \text{ and } x = \tan(\tfrac{1}{4}\angle AOP) \text{ and } y = z(1+x^2)/(l-1). \tag{2.2}$$

Then (x, y) is a rational solution of (1.1) iff both $z = AP$ is rational and $\angle AOP \in \Theta$.

PROOF. Let $\xi = \frac{1}{2}\angle AOP$ then

$$\sin\xi = 2x/(1+x^2) \text{ and } \cos\xi = (1-x^2)/(1+x^2). \tag{2.3}$$

Also by the cosine rule

$$z^2 = l^2+1-2l\cos 2\xi = (l-1)^2+4l\sin^2\xi. \tag{2.4}$$

Upon substitution from (2.3) and multiplication by $(1+x^2)^2/(l-1)^2$, equation (2.4) becomes

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$$z^2(1+x^2)^2/(l-1)^2 = (1+x^2)^2+16lx^2/(l-1)^2,$$

which is equation (1.1) with m defined by (2.1) and y defined by (2.2). The lemma now follows from (2.2) and the properties (i), (ii) of Θ.

3. On rational solutions of (1.1)

LEMMA 3. *Let* (α_1, β_1), (α_2, β_2) *be two rational points on the curve*

(3.1) $$y^2 = x^4+2mx^2+1 \text{ where } m \text{ is rational } m > 1,$$

satisfying

(3.2) $$\delta \leqq \alpha_2 < \alpha_1 \leqq 1,$$

(3.3) $$0 < \alpha_1-\alpha_2 < \delta,$$

(3.4) $$\beta_1 > 1, \beta_2 > 1,$$

where δ *is a positive number satisfying*

(3.5) $$(2m+2)\delta < 1.$$

Then there exists a further rational point (α_3, β_3) *on the curve satisfying*

(3.6) $$0 < \alpha_3 < \alpha_2, \quad \beta_3 > 1,$$

(3.7) $$1-(2m+2)(\alpha_1-\alpha_2) < (\alpha_2-\alpha_3)/(\alpha_1-\alpha_2) < 1.$$

PROOF. First we note that if (α, β) is a rational point of (3.1), then trivially so are $(\pm\alpha, \pm\beta)$ and $|\beta| \geqq 1$. Hence $(\alpha_2, -\beta_2)$ is a rational point of (3.1) and we consider the parabola (cf. [6], p. 642)

(3.8) $$y = ax^2+bx+c$$

which passes through (α_1, β_1) and touches (3.1) at $(\alpha_2, -\beta_2)$. For this parabola we have

(3.9) $$\begin{cases} a\alpha_1^2+b\alpha_1+c = \beta_1, \\ a\alpha_2^2+b\alpha_2+c = -\beta_2, \\ 2a\alpha_2+b \qquad = -\beta_2', \end{cases}$$

where

(3.10) $$\beta_2' = (2\alpha_2^3+2m\alpha_2)/\beta_2$$

and β_2' is the gradient of (3.1) at (α_2, β_2). We will make use of the fact that β_2' is rational and positive. Now since the equations (3.9) have rational coefficients, their solution a, b, c is rational. In particular

$$a = (\beta_1+\beta_2)(\alpha_1-\alpha_2)^{-2}+\beta_2'(\alpha_1-\alpha_2)^{-1}$$

and

$$c = \beta_1+\beta_2'+a\alpha_1[\alpha_2-(\alpha_1-\alpha_2)].$$

Hence $a > 1$ because $\beta_i > 1$, $\beta_2' > 0$ by (3.10) and $0 < \alpha_1 - \alpha_2 < 1$ by (3.2). Also $c > 1$ because $\beta_1 > 1$, $\beta_2' > 0$, $\alpha_1 > 0$ by (3.2) and $\alpha_1 - \alpha_2 < \delta \leqq \alpha_2$ by (3.2), (3.3). Now the equation

$$(ax^2+bx+c)^2 = x^4+2mx^2+1$$

gives the x-coordinates of the four points of intersection of (3.1) and (3.8). It is the fourth point of intersection which provides the point (α_3, β_3) of this lemma. Three of these x-coordinates are α_1, α_2, α_2, hence for the fourth α_3, we have

$$\alpha_1\alpha_2^2\alpha_3 = (c^2-1)/(a^2-1)$$

and so α_3 is rational. Moreover because α_1, $\alpha_2 > 0$ and $a, c > 1$ we have

$$\alpha_3 > 0,$$

which is the first of conditions (3.6). Now since the parabola (3.8) is a continuous curve from $(0, c)$ to $(\alpha_2, -\beta_2)$ and (3.1) is a continuous curve from $(0, 1)$ to (α_2, β_2), it is obvious that

$$\alpha_3 < \alpha_2$$

and

$$\beta_3 = a\alpha_3^2+b\alpha_3+c > 1. \tag{3.11}$$

Thus (3.6) holds.

We finally establish (3.7). Eliminating a, b, c from (3.9) and (3.11), we obtain

$$\begin{vmatrix} \alpha_1^2 & \alpha_1 & 1 & \beta_1 \\ \alpha_2^2 & \alpha_2 & 1 & -\beta_2 \\ \alpha_3^2 & \alpha_3 & 1 & \beta_3 \\ 2\alpha_2 & 1 & 0 & -\beta_2' \end{vmatrix} = 0;$$

whence, by elementary transformations of the determinant,

$$\lambda_2^2/\lambda_1^2 = (\beta_3+\beta_2-\lambda_2\beta_2')/(\beta_1+\beta_2+\lambda_1\beta_2') \tag{3.12}$$

where

$$\lambda_1 = \alpha_1-\alpha_2, \quad \lambda_2 = \alpha_2-\alpha_3.$$

Since $0 < \alpha_3 < \alpha_2$ and the upper branch of the curve (3.1) is increasing for $x > 0$, we have $\beta_3 < \beta_1$. We showed already that $\beta_2' > 0$, and it follows that

$$\lambda_2/\lambda_1 < 1. \tag{3.13}$$

Now

$$\beta_1^2 = (\alpha+\lambda_1)^4+2m(\alpha+\lambda_1)^2+1$$
$$\beta_2^2 = \alpha^4+2m\alpha^2+1,$$

(writing α for α_2 for convenience). Hence

$$\beta_1^2-\beta_2^2 = \lambda_1[2\alpha+\lambda_1)][(\alpha+\lambda_1)^2+\alpha^2+2m],$$

i.e.

$$\beta_1-\beta_2 = \lambda_1[2\alpha+\lambda_1][(\alpha+\lambda_1)^2+\alpha^2+2m]/(\beta_1+\beta_2).$$

Since

$$\lambda_1 < \varepsilon \leqq \alpha = \alpha_1-\lambda_1 \leqq 1-\lambda_1,$$

we have

$$2\alpha+\lambda < 2, \quad \alpha+\lambda < 1,$$

and so

$$\beta_1-\beta_2 < \lambda_1(2)(2+2m)/2 = 2(1+m)\lambda_1.$$

Similarly, we may show that

$$\beta_3 > \beta_2-2(1+m)\lambda_2,$$

whence

$$\beta_3 > \beta_2-2(1+m)\lambda_1.$$

Then from (3.12), we obtain

$$\lambda_2^2/\lambda_1^2 > [2\beta_2-2(1+m)\lambda_1-\lambda_2\beta_2']/2\beta_2+2(1+m)\lambda_1+\lambda_1\beta_2'].$$

Also from (3.10), since $0 < \alpha < 1, \beta > 1$, we have

$$0 < \beta_2' < 2(1+m)$$

and therefore

$$\lambda_2^2/\lambda_1^2 > [\beta_2-2(1+m)\lambda_1]/[\beta_2+2(1+m)\lambda_1].$$

Now by hypothesis $(2m+2)\lambda_1 < (2m+2)\varepsilon < 1 < \beta_2$, and so

$$\begin{aligned}\lambda_2^2/\lambda_1^2 &> [1-2(1+m)\lambda_1\beta_2^{-1}][1+2(1+m)\lambda_1\beta_2^{-1}]^{-1}\\ &> [1-2(1+m)\lambda_1\beta_2^{-1}]^2\\ \lambda_2/\lambda_1 &> 1-2(1+m)\lambda_1\beta_2^{-1} > 1-2(1+m)\lambda_1;\end{aligned}$$

and Lemma 3 is proved.

LEMMA 4. *Suppose that the hypotheses of Lemma 3 are satisfied. Then there exists a sequence of rational points* (α_i, β_i) *for* $i = 1, 2, \cdots, N$ *on the curve* (1) *satisfying*

$$(\alpha_N, \beta_N) = (0, 1),$$

$$0 < \alpha_i-\alpha_{i+1} < \delta \text{ for } i = 1, 2, \cdots, N-1.$$

PROOF. Lemma 3 establishes the existence of (α_3, β_3) constructed from the given points (α_1, β_1), (α_2, β_2). Now consider the pair of points (α_2, β_2) and (α_3, β_3). The inequalities (3.6), together with $\alpha_2-\alpha_3 < \alpha_1-\alpha_2$ from (3.7) show that

$$\alpha_3 < \alpha_2 \leqq 1 \text{ and } 0 \leqq \alpha_2-\alpha_3 < \delta$$

and by definition $\beta_2 > 1, \beta_3 > 1$. Hence, provided that $\alpha_3 \geqq \delta$, from (α_2, β_2)

and $(\alpha_3 \beta_3)$ we can, by using lemma 3 obtain yet another point (α_4, β_4). In this way we can obtain a sequence of points (α_i, β_i) for $i = 1, 2, 3, \cdots$ which will be infinite unless for some i we have $\alpha_{i-1} < \delta$. Lemma 4 now follows if, for some N, we obtain

$$\alpha_{N-1} < \delta. \tag{3.14}$$

We now show that (3.14) does in fact hold if N is sufficiently large. For suppose not; then, the construction yields an infinite sequence of points (α_1, β_i) with

$$\alpha_i \geqq \delta \quad (i = 1, 2, 3, \cdots).$$

Let

$$\lambda_i = \alpha_i - \alpha_{i+1} \ (> 0),$$

so that

$$\sum_1^n \lambda_i = \alpha_1 - \alpha_{n+1} < \alpha_1 - \delta \text{ for all } n \geqq 1,$$

and so the series $\sum_1^\infty \lambda_i$ is convergent. However, by Lemma 3,

$$\lambda_i > \lambda_{i+1} > \lambda_i\{1-(2m+2)\lambda_i\} > 0. \tag{3.16}$$

Putting $\mu_i = (2m+2)\lambda_i$, we obtain from (3.16),

$$\mu_i > \mu_{i+1} > \mu_i(1-\mu_i) > 0. \tag{3.17}$$

Obviously $\sum_1^\infty \mu_i$ and $\sum_1^\infty \lambda_i$ converge or diverge together. We now prove that for all $i \geqq i_0$,

$$\mu > c/i \tag{3.18}$$

for some constant c. If

$$\mu_i > \tfrac{1}{2}/i \text{ for all } i \geqq i_0,$$

there is nothing to prove. Suppose

$$\mu_i = c'/i, \text{ where } c' < \tfrac{1}{2} \text{ for some } i > i_0 > 2,$$

then by (3.17) we have

$$\mu_{i+1} > (c'/i)\{1-(c'/i)\} = \{c'/(i+1)\}\{(i^2+i-ic'-c')/i^2\} > c'/(i+1).$$

Hence it follows easily by induction that (3.18) holds. Lemma 4 then follows since $\sum \lambda_i$ is divergent.

4. Proof of Theorem 1

Let the unit circle $\mathscr{C}$, the points O, B, C, D and $\varepsilon > 0$ of Theorem 1 be given. We recall that the set Θ is dense in the set of all real numbers. Hence given $\lambda > 0$ we can choose $\phi \in \Theta$ such that if P_2 is the point on $\mathscr{C}$ with

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$\angle P_2OB = 2\phi$, then the distance CP_2 is $< \lambda$. Also we can then choose $\psi \in \Theta$ such that, if Q is the point on $\mathscr{C}$ with $\angle QOB = 2\psi$, and if A is the point of intersection of the lines CB and P_2Q produced, then A is within ε of D. Without loss of generality, we may assume that A is on the same side of O as B. By Lemma 1 the distances $OA = l$ and P_2A are both rational. Then by lemma 2 the point $P = P_2$ yields, by the definitions of x, y in (2.2), a rational point (α_2, β_2) of (1.1) with m defined by (2.1). Since $(\pm\alpha_2, \pm\beta_2)$ are also points on (1.1) we may assume $\alpha_2 > 0$, $\beta_2 > 1$. In fact, $\alpha_2 = \tan\frac{1}{2}\phi$. Also the point C on $\mathscr{C}$ yields the rational point $(\alpha_1, \beta_1) = (1, 2(l+1)/(l-1))$ of (1.1). Now since $\alpha_2 \to \alpha_1 = 1$ as $\phi \to \frac{1}{2}\pi$, we can choose λ sufficiently small such that $\lambda \leqq 2\varepsilon$ and there is a δ, $0 < \delta < \frac{1}{2}\varepsilon$ satisfying (3.5), (3.2) and (3.3). Hence we obtain the sequence of rational points (α_i, β_i) for $i = 1, 2, \cdots, N$ defined in Lemma 4.

Now it follows from Lemma 2 that, to each rational point (α_i, β_i), there corresponds the point P_i on $\mathscr{C}$ such that if $\theta_i = \frac{1}{2}\angle AOP_i$ then

$$\alpha_i = \tan\tfrac{1}{2}\theta_i,$$

and AP_i is rational. We write P_i^* for the point symmetric with P_i about the line CB. Then we assert that if S_ε is the set of all $2(N-1)$ points

$$P_i,\ P_i^* \text{ for } i = 1, 2, \cdots, N,$$

then S_ε has properties (i), (ii), (iii) of theorem 1. Trivially S_ε has the symmetry property (ii). So we now show S_ε has property (i).

We show that the distance between successive points P_i, P_{i+1} of S_ε is $< 2\varepsilon$. This chord distance is less than the arc distance $2\theta_i - 2\theta_{i+1}$, and since $0 < \theta_i < \frac{1}{2}\pi$,

$$\begin{aligned} 2\theta_i - 2\theta_{i+1} < 4\tan\tfrac{1}{2}(\theta_i - \theta_{i+1}) &< 4(\tan\tfrac{1}{2}\theta_i - \tan\tfrac{1}{2}\theta_{i+1}) \\ &= 4(\alpha_i - \alpha_{i+1}) < 4\delta < 2\varepsilon, \end{aligned}$$

the last inequality but one coming from Lemma 4. This proves (i). Now it follows from the property (ii) of Θ that all the angles in the polygon described in (iii) of theorem 1 are in Θ and hence this polygon is rational with rational area.

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University of Malaya
Kuala Lumpur

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