Glasgow Math. J. **46** (2004) 417–429. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504001909. Printed in the United Kingdom

NON-ABELIAN HOMOLOGY OF LIE ALGEBRAS

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(Received 12 September, 2003; accepted 18 November 2003)

Abstract. Non-abelian homology of Lie algebras with coefficients in Lie algebras is constructed and studied, generalising the classical Chevalley-Eilenberg homology of Lie algebras. The relationship between cyclic homology and Milnor cyclic homology of non-commutative associative algebras is established in terms of the long exact non-abelian homology sequence of Lie algebras. Some explicit formulae for the second and the third non-abelian homology of Lie algebras are obtained.

2000 *Mathematics Subject Classification.* 17B40, 17B56, 18G10, 18G50.

0. Introduction. The non-abelian homology of groups with coefficients in groups was constructed and investigated in [**16**, **17**], using the non-abelian tensor product of groups of Brown and Loday [**4**, **5**] and its non-abelian left derived functors. It generalises the classical Eilenberg-MacLane homology of groups and extends Guin's low dimensional non-abelian homology of groups with coefficients in crossed modules [**9**], having an interesting application to the algebraic K-theory of non-commutative local rings [**9**, **17**].

The purpose of this paper is to set up a similar non-abelian homology theory for Lie algebras and is mainly dedicated to state and prove several desirable properties of this homology theory.

In [**8**] Ellis introduced and studied the non-abelian tensor product of Lie algebras which is a Lie structural and purely algebraic analogue of the non-abelian tensor product of groups of Brown and Loday [**4**, **5**], arising in applications to homotopy theory of a generalised Van Kampen theorem.

Applying this tensor product of Lie algebras, Guin defined the low-dimensional non-abelian homology of Lie algebras with coefficients in crossed modules [**10**].

We construct a non-abelian homology $H_*(M, N)$ of a Lie algebra M with coefficients in a Lie algebra N as the non-abelian left derived functors of the tensor product of Lie algebras, generalising the classical Chevalley-Eilenberg homology of Lie algebras and extending Guin's non-abelian homology of Lie algebras [**10**]. We give an application of our long exact homology sequence to cyclic homology of associative algebras, correcting the result of [**10**]. In fact, for a unital associative (non-commutative) algebra A we obtain a long exact non-abelian homology sequence

$$\cdots \to H_2(A, V(A), [A, A]) \to H_2(A, V(A)) \to H_2(A, [A, A]) \to H_1(A, V(A), [A, A])$$
$$\to H_1(A, V(A)) \to H_1(A, [A, A]) \to HC_1(A) \to HC_1^M(A)$$
$$\to [A, A]/[A, [A, A]] \to 0.$$

In a forthcoming paper we will give a version of non-abelian cohomology theory of Lie algebras following [**10**] and using ideas from [**13**, **14**, **15**]. We also hope to return to detailed analysis of the non-abelian homology of Lie algebras of matrices in connection with cyclic homology and de Rham cohomology following ideas of Loday and Quillen [**19**].

NOTATIONS AND CONVENTIONS. We denote by Λ a unital commutative ring unless otherwise stated. We shall use the term Lie algebra to mean a Lie algebra over Λ. $[\ ,\ ]$ and $|\ |$ denote the Lie bracket and the coset of the quotient Lie algebra respectively. We denote the category of Lie algebras over Λ by $\mathcal{L}ie$.

1. The non-abelian tensor product. Let P and M be two Lie algebras. By an action of P on M we mean a Λ-bilinear map $P \times M \to M$, $(p, m) \mapsto {}^p m$ satisfying the following conditions:

$${}^{[p,p']}m = {}^p({}^{p'}m) - {}^{p'}({}^p m), \qquad {}^p[m, m'] = [{}^p m, m'] + [m, {}^p m'],$$

for all $m, m' \in M$ and $p, p' \in P$. For example, if P is a subalgebra of some Lie algebra Q, and if M is an ideal in Q, then Lie multiplication in Q yields an action of P on M.

Now we give the definition of the tensor product of Lie algebras due to Ellis [**8**] (see also [**6**], [**10**]). Let M and N be two Lie algebras acting on each other. The tensor product $M \otimes N$ of the Lie algebras M and N is the Lie algebra generated by the symbols $m \otimes n$, $m \in M$, $n \in N$, and subject to the following relations:

(i) $\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$,
(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$,
$m \otimes (n + n') = m \otimes n + m \otimes n'$,
(iii) $[m, m'] \otimes n = m \otimes ({}^{m'}n) - m' \otimes ({}^m n)$,
$m \otimes [n, n'] = ({}^{n'}m) \otimes n - ({}^n m) \otimes n'$,
(iv) $[(m \otimes n), (m' \otimes n')] = -({}^n m) \otimes ({}^{m'}n')$

for all $\lambda \in \Lambda$, $m, m' \in M$, $n, n' \in N$.

Suppose that $\phi : M \to A$, $\psi : N \to B$ are Lie homomorphisms, A, B act on each other, and ϕ, ψ preserve the actions in the following sense:

$$\phi({}^n m) = {}^{\psi(n)}\phi(m), \quad \psi({}^m n) = {}^{\phi(m)}\psi(n), \quad m \in M, \quad n \in N.$$

Then, by [**8**], there is a unique homomorphism $\phi \otimes \psi : M \otimes N \to A \otimes B$ such that $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$ for all $m \in M$, $n \in N$. Furthermore, if ϕ, ψ are onto, so also is $\phi \otimes \psi$.

The tensor product of Lie algebras is symmetric in the sense of the isomorphism $M \otimes N \to N \otimes M$ given by $m \otimes n \mapsto -n \otimes m$ [**8**].

https://doi.org/10.1017/S0017089504001909 Published online by Cambridge University Press

A precrossed P-module (M, μ) is a Lie homomorphism $\mu : M \to P$ together with an action of P on M satisfying the following condition:

$$\mu({}^{p}m) = [p, \mu(m)].$$

If in addition the precrossed module (M, μ) satisfies the Peiffer identity:

$${}^{\mu(m)}m' = [m, m'],$$

then it is said to be a crossed P-module. Note that for a crossed module (M, μ) the image of μ is necessarily an ideal in P and the kernel of μ is a P-invariant ideal in the center of M. Moreover the action of P on $\operatorname{Ker}\mu$ induces an action of $P/\operatorname{Im}\mu$ on $\operatorname{Ker}\mu$, making $\operatorname{Ker}\mu$ a $P/\operatorname{Im}\mu$-module.

In [**8**] the results on the tensor product $M \otimes N$ are obtained assuming the actions of M and N on each other compatible, i.e.

$$^{({}^{n}m)}n' = [n', {}^{m}n] \quad \text{and} \quad ^{({}^{m}n)}m' = [m', {}^{n}m], \tag{1}$$

for all $m, m' \in M$ and $n, n' \in N$. This is the case, for example, if (M, μ) and (N, ν) are crossed P-modules, M and N act on each other via the action of P. These compatibility conditions are not assumed to hold except in the following proposition.

Proposition 1. *Let M and N be Lie algebras acting on each other such that the compatibility conditions* (1) *hold. Then there is a natural isomorphism of Lie algebras*

$$M \otimes N \cong (M \otimes_{\Lambda} N)/D(M, N),$$

where $D(M, N)$ is the Λ-submodule of $M \otimes_{\Lambda} N$ generated by the elements

$$\begin{aligned}
&[m, m'] \otimes n - m \otimes ({}^{m'}n) + m' \otimes ({}^{m}n),\\
&m \otimes [n, n'] - ({}^{n'}m) \otimes n + ({}^{n}m) \otimes n',\\
&({}^{n}m) \otimes ({}^{m}n),\\
&({}^{n}m) \otimes ({}^{m'}n') + ({}^{n'}m') \otimes ({}^{m}n),\\
&[{}^{n}m, {}^{n'}m'] \otimes ({}^{m''}n'') + [{}^{n'}m', {}^{n''}m''] \otimes ({}^{m}n) + [{}^{n''}m'', {}^{n}m] \otimes ({}^{m'}n'),
\end{aligned}$$

for all $m,\ m',\ m'' \in M$ and $n,\ n',\ n'' \in N$.

Proof. Let us introduce the Λ-module $(M \otimes_{\Lambda} N)/D(M, N)$ with a Lie structure defined by the following formula

$$[m \otimes n, m' \otimes n'] = -({}^{n}m) \otimes ({}^{m'}n').$$

To show that this multiplication could be extended from generators to any elements of $(M \otimes_{\Lambda} N)/D(M, N)$ one has to check its compatibility with the defining relations of $(M \otimes_{\Lambda} N)/D(M, N)$, which is routine and will be omitted. Now it is easy to see the required isomorphism of Lie algebras. □

The interesting properties of the tensor product of Lie algebras, in particular its compatibility with the direct limits and the right exactness, will be given.

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PROPOSITION 2. *Let* $\{M_\alpha, \phi_\alpha^\beta, \alpha \le \beta\}$ *be a direct system of Lie algebras. Let* N *be a Lie algebra and let for every* α *the Lie algebras* M_α, N *act on each other and the homomorphisms* ϕ_α^β *preserve the actions. Then there is a natural isomorphism of Lie algebras*

$$\Big(\varinjlim_\alpha\{M_\alpha\}\Big) \otimes N \cong \varinjlim_\alpha\{M_\alpha \otimes N\}.$$

Proof. We only define the actions of $\varinjlim_\alpha\{M_\alpha\}$ and N on each other by the following way:

$${}^{|m_\alpha|}n = {}^{m_\alpha}n \quad \text{and} \quad {}^{n}|m_\alpha| = |{}^{n}m_\alpha|$$

for all $m_\alpha \in M_\alpha$, $n \in N$, and the natural isomorphism of Lie algebras

$$f : \Big(\varinjlim_\alpha\{M_\alpha\}\Big) \otimes N \longrightarrow \varinjlim_\alpha\{M_\alpha \otimes N\} \quad \text{by} \quad f(|m_\alpha| \otimes n) = |m_\alpha \otimes n|.$$

The details of the proof are straightforward. □

PROPOSITION 3. *Suppose* $0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$ *is a short exact sequence of Lie algebras,* N *is an arbitrary Lie algebra acting on* M', M *and* M''*; the Lie algebras* M', M, M'' *act on* N *and* ϕ, ψ *preserve these actions. Then there is an exact sequence of Lie algebras*

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0.$$

Proof. This is similar to the proof of Proposition 9 **[8]** since it does not use compatibility conditions (1). □

2. Construction of non-abelian homology. Let M be a Λ-module, N a Lie algebra and $\alpha : M \to \mathrm{Der}(N)$ a Λ-homomorphism, where $\mathrm{Der}(N)$ is the Lie algebra of derivations of N. Let us denote by $\mathcal{F}(M)$ the free Lie algebra on the Λ-module M, i.e. the Lie algebra $\mathcal{A}(M)/\mathcal{B}(M)$, where $\mathcal{A}(M) = \sum_{0<k} \mathcal{A}_k(M)$ with $\mathcal{A}_1(M) = M$, $\mathcal{A}_k(M) = \sum_{0<i<k} \mathcal{A}_i(M) \otimes_\Lambda \mathcal{A}_{k-i}(M)$ is the free (non-associative) algebra on M and $\mathcal{B}(M)$ is the two-sided ideal of $\mathcal{A}(M)$ generated by the elements

$$xx \quad \text{and} \quad x(yz) + y(zx) + z(xy) \quad \text{for all} \quad x, y, z \in \mathcal{A}(M).$$

Then there exists a unique Lie homomorphism $\kappa : \mathcal{F}(M) \to \mathrm{Der}(N)$ such that $\kappa i = \alpha$, where $i : M \to \mathcal{F}(M)$ is the natural homomorphism of Λ-modules This means that there is an action of the Lie algebra $\mathcal{F}(M)$ on the Lie algebra N.

Now if in addition M is an N-module, then the module action of N on M yields an N-module structure on $\mathcal{A}_k(M)$: if $x \otimes y \in \mathcal{A}_i(M) \otimes_\Lambda \mathcal{A}_{k-i}(M)$ and $n \in N$ then, inductively, we define

$$n(x \otimes y) = nx \otimes y + x \otimes ny,$$

and this extends linearly to an action of n on an arbitrary element of $\mathcal{A}_k(M)$. The action of N on $\mathcal{A}_k(M)$ extends linearly to an action of N on $\mathcal{A}(M)$, making $\mathcal{A}(M)$

into an N-module. Since $\mathcal{B}(M)$ is N-invariant, the action of N on $\mathcal{A}(M)$ induces a Lie action of N on $\mathcal{F}(M)$.

Let $\mathfrak{A}_N$ denote, for a fixed Lie algebra N, the category whose objects are all Lie algebras M together with an action of M on N by derivations of N and an action of N on M by derivations of M. Morphisms in the category $\mathfrak{A}_N$ are all Lie homomorphisms $\alpha : M \to M'$ preserving the actions, namely $\alpha({}^n m) = {}^n\alpha(m)$ and ${}^m n = {}^{\alpha(m)}n$ for all $m \in M, n \in N$.

Let $\mathcal{F} : \mathfrak{A}_N \to \mathfrak{A}_N$ be the endofunctor defined as follows: for an object M of $\mathfrak{A}_N$, let $\mathcal{F}(M)$ denote the free Lie algebra on the underlying Λ-module M with the above-mentioned actions of N on $\mathcal{F}(M)$ and $\mathcal{F}(M)$ on N; for a morphism $\alpha : M \to M'$ of $\mathfrak{A}_N$, let $\mathcal{F}(\alpha)$ be the canonical Lie homomorphism from $\mathcal{F}(M)$ to $\mathcal{F}(M')$ induced by α.

Let $\tau : \mathcal{F} \to 1_{\mathfrak{A}_N}$ be the obvious natural transformation and let $\delta : \mathcal{F} \to \mathcal{F}^2$ be the natural transformation induced for every $M \in \operatorname{ob}\mathfrak{A}_N$ by the natural inclusion of Λ-modules $M \to \mathcal{F}(M)$. We obtain the cotriple $\mathbb{F} = (\mathcal{F},\ \tau,\ \delta)$. Then for any object M there is an augmented simplicial object in the category $\mathfrak{A}_N$

$$\cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \mathcal{F}^{k+1}(M) \underset{d_k^k}{\overset{d_0^k}{\begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}}} \cdots \underset{d_2^2}{\overset{d_0^2}{\begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}}} \mathcal{F}^2(M) \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} \mathcal{F}^1(M) \overset{\tau_M}{\to} M,$$

$\mathcal{F}^1(M) = \mathcal{F}(M),\ \mathcal{F}^{k+1}(M) = \mathcal{F}(\mathcal{F}^k(M)),\ d_i^k = \mathcal{F}^i(\tau_{\mathcal{F}^{k-i}(M)}),\ s_i^k = \mathcal{F}^i(\delta_{\mathcal{F}^{k-i}(M)}),\ k \geq 1$, called the cotriple resolution of M and denoted by $(\mathcal{F}^*(M), \tau_M, M)$.

Let $\mathcal{T} : \mathfrak{A}_N \to \mathcal{L}ie$ be a covariant functor. Applying $\mathcal{T}$ dimension-wise to $\mathcal{F}^*(M)$ yields the simplicial Lie algebra $\mathcal{T}\mathcal{F}^*(M)$. Define the *k-th derived functor* $\mathcal{L}_k^{\mathbb{F}}\mathcal{T} : \mathfrak{A}_N \to \mathcal{L}ie$, $k \geq 0$, of the functor $\mathcal{T}$ relative to the cotriple $\mathbb{F}$ as the k-th homotopy of $\mathcal{T}\mathcal{F}^*(M)$. Note that $\mathcal{L}_k^{\mathbb{F}}\mathcal{T}(M)$, $k \geq 1$, is an abelian Lie algebra and will be thought as a Λ-module.

The non-abelian tensor product of Lie algebras defines a covariant functor $- \otimes N$ from the category $\mathfrak{A}_N$ to the category $\mathcal{L}ie$. Consider the derived functors $\mathcal{L}_k^{\mathbb{F}}(- \otimes N)$, $k \geq 0$, of the functor $- \otimes N$ relative to the cotriple $\mathbb{F}$.

PROPOSITION 4. *Let M be a Lie algebra and N a module over the Lie algebra M, then there are natural isomorphisms*

$$\mathcal{L}_k^{\mathbb{F}}(- \otimes N)(M) \cong H_{k+1}(M, N), \qquad k \geq 1,$$
$$\operatorname{Ker}\nu \cong H_1(M, N), \qquad \operatorname{Coker}\nu \cong H_0(M, N),$$

where N is thought as an abelian Lie algebra acting trivially on M, $\nu : M \otimes N \to N$ is a Lie homomorphism given by $\nu(m \otimes n) = {}^m n$, $m \in M$, $n \in N$.

Proof. Let $\mathcal{L}ie_M$ denote the category of Lie algebras over M and $\operatorname{Diff}_M : \mathcal{L}ie_M \to U(M) - mod$ (category of $U(M)$-modules) a functor given by

$$\operatorname{Diff}_M(W) = I(W) \otimes_{U(W)} U(M),$$

where $U(M)$ and $U(W)$ are the universal enveloping algebras of M and W respectively and $I(W)$ is the augmentation ideal. By Proposition 13 **[6]** $\mathcal{L}_*^{\mathbb{F}}(- \otimes N)(M)$ are isomorphic to the values of the cotriple derived functors of the functor $\operatorname{Diff}_M(-) \otimes_{U(M)} N : \mathcal{L}ie_M \to \Lambda - \text{mod}$ (category of Λ-modules) for the object 1_M of the category $\mathcal{L}ie_M$

https://doi.org/10.1017/S0017089504001909 Published online by Cambridge University Press

which give the classical Chevalley-Eilenberg homology $H_*(M, N)$ of Lie algebras with the usual dimension shift, similarly to the cases of group (co)homology and Hochschild (co)homology described as cotriple (co)homology [**2**, **3**]. □

Using this proposition we make the following definition.

DEFINITION 5. Let M and N be Lie algebras acting on each other. Define the non-abelian homology of M with coefficients in N by setting

$$H_k(M, N) = \mathcal{L}^{\mathbb{F}}_{k-1}(- \otimes N)(M), \qquad k \geq 2,$$
$$H_1(M, N) = \operatorname{Ker} \nu, \qquad H_0(M, N) = \operatorname{Coker} \nu,$$

where $\nu : M \otimes N \to N/H$, $\nu(m \otimes n) = |^m n|$, and H is the ideal of the Lie algebra N generated by the elements ${}^{({}^n m)}n' - [n', {}^m n]$ for all $m \in M$, $n, n' \in N$.

REMARK. (a) It is clear that $H_k(M, N)$, $k \geq 2$, are only Λ-modules, when $H_1(M, N)$ and $H_0(M, N)$ are Lie algebras. If the actions of M and N satisfy the compatibility conditions (1), then $H_1(M, N)$ is also an abelian Lie algebra.

(b) Let N be a crossed M-module, then $H_0(M, N)$ and $H_1(M, N)$ coincides with zero and first non-abelian homology Λ-modules of the Lie algebra M with coefficients in the crossed M-module N introduced by Guin [**10**].

One could define another non-abelian homology theory of Lie algebras using the derived functors of the non-abelian tensor product relative to the cotriple over sets which coincides with our theory for Lie algebras being free Λ-modules.

3. Some properties of non-abelian homology. First several long exact non-abelian homology sequences with respect to both variables will be given.

THEOREM 6. *Let $\alpha : N \to N'$ be a surjective Lie homomorphism, M an arbitrary Lie algebra acting on N and N' which act on M and α preserve the actions. Then there is a long exact sequence of non-abelian homology*

$$\cdots \to H_3(M, N') \xrightarrow{\delta_3} H_2(M, N, N') \xrightarrow{j_2} H_2(M, N) \xrightarrow{i_2} H_2(M, N') \xrightarrow{\delta_2} H_1(M, N, N')$$
$$\xrightarrow{j_1} H_1(M, N) \xrightarrow{i_1} H_1(M, N') \xrightarrow{\delta_1} H_0(M, N, N') \xrightarrow{j_0} H_0(M, N) \xrightarrow{i_0} H_0(M, N') \to 0, \tag{2}$$

where

$$H_k(M, N, N') = \pi_{k-1}(\operatorname{Ker}(1_{\mathcal{F}^*(M)} \otimes \alpha)), \quad k \geq 2,$$
$$H_1(M, N, N') = \frac{\{\operatorname{Ker}\left(1_{\mathcal{F}^1(M)} \otimes \alpha\right) \cap \left(d_0^0 \otimes 1_N\right)^{-1}(\operatorname{Ker}(1_M \otimes \alpha) \cap \operatorname{Ker} \nu)\}}{\left(d_1^1 \otimes 1_N\right)\left(\operatorname{Ker}\left(1_{\mathcal{F}^2(M)} \otimes \alpha\right) \cap \operatorname{Ker}\left(d_0^1 \otimes 1_N\right)\right)},$$
$$H_0(M, N, N') = \operatorname{Ker} \widetilde{\alpha} / \nu(\operatorname{Ker}(1_M \otimes \alpha)),$$

$(\mathcal{F}^*(M), d_0^0, M)$ is the $\mathbb{F}$ cotriple resolution of the object M of the category $\mathfrak{A}_N$ and $\widetilde{\alpha} : N/H \to N'/H'$ is the homomorphism induced by α.

Proof. The following commutative diagram of Lie algebras with exact columns

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} & \operatorname{Ker}(1_{\mathcal{F}^2(M)} \otimes \alpha) & \rightrightarrows & \operatorname{Ker}(1_{\mathcal{F}^1(M)} \otimes \alpha) & \to & \operatorname{Ker}(1_M \otimes \alpha) & \to & \operatorname{Ker} \widetilde{\alpha} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} & \mathcal{F}^2(M) \otimes N & \overset{d_0^1 \otimes 1_N}{\underset{d_1^1 \otimes 1_N}{\rightrightarrows}} & \mathcal{F}^1(M) \otimes N & \overset{d_0^0 \otimes 1_N}{\to} & M \otimes N & \overset{\nu}{\to} & N/H, \\
 & 1_{\mathcal{F}^2(M)} \otimes \alpha \downarrow & & \downarrow 1_{\mathcal{F}^1(M)} \otimes \alpha & & \downarrow 1_M \otimes \alpha & & \downarrow \widetilde{\alpha} \\
\cdots \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} & \mathcal{F}^2(M) \otimes N' & \overset{d_0^1 \otimes 1_{N'}}{\underset{d_1^1 \otimes 1_{N'}}{\rightrightarrows}} & \mathcal{F}^1(M) \otimes N' & \overset{d_0^0 \otimes 1_{N'}}{\to} & M \otimes N' & \overset{\nu'}{\to} & N'/H' \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 & & 0
\end{array}
$$

immediately induces the exactness of the sequence

$$\cdots \longrightarrow H_3(M, N') \overset{\delta_3}{\longrightarrow} H_2(M, N, N') \overset{j_2}{\longrightarrow} H_2(M, N) \overset{i_2}{\longrightarrow} H_2(M, N').$$

Applying the "snake lemma" to the last two columns of this diagram one has the following exact sequence

$$H_1(M, N) \overset{i_1}{\longrightarrow} H_1(M, N') \overset{\delta_1}{\longrightarrow} H_0(M, N, N') \overset{j_0}{\longrightarrow} H_0(M, N) \overset{i_0}{\longrightarrow} H_0(M, N') \longrightarrow 0.$$

We define the homomorphisms j_1 and δ_2 by

$$j_1(|x|) = \big(d_0^0 \otimes 1_N\big)(x)$$

for $x \in \{\operatorname{Ker}(1_{\mathcal{F}^1(M)} \otimes \alpha) \cap (d_0^0 \otimes 1_N)^{-1}(\operatorname{Ker}(1_M \otimes \alpha) \cap \operatorname{Ker} \nu)\}$ and

$$\delta_2(|y|) = \big|\big(d_1^1 \otimes 1_N\big)(y') - \big(d_0^1 \otimes 1_N\big)(y')\big|$$

for $y \in \operatorname{Ker}(d_1^1 \otimes 1_{N'}) \cap \operatorname{Ker}(d_0^1 \otimes 1_{N'})$, where $y' \in \mathcal{F}^2(M) \otimes N$ such that $(1_{\mathcal{F}^2(M)} \otimes \alpha)(y') = y$. It is easy to check that j_1 and δ_2 are well defined and that the sequence (2) is exact in terms $H_2(M, N')$, $H_1(M, N, N')$ and $H_1(M, N)$ by virtue of Proposition 3. □

REMARK. (a) If the actions of M and N satisfy the compatibility conditions (1) (in this case M and N' act on each other compatibly), then $H_0(M, N, N') = H_0(M, N'')$, where $N'' = \operatorname{Ker} \alpha$.

(b) Let $0 \to (N'', 0) \to (N, \mu) \to (N', \nu) \to 0$ be an exact sequence of crossed M-modules. Thanks to the result in **[10]** there is a six-term exact non-abelian homology sequence

$$\begin{aligned} &H_1(M, N'') \to H_1(M, N) \to H_1(M, N') \to H_0(M, N'') \to H_0(M, N) \\ &\quad \to H_0(M, N') \to 0. \end{aligned} \tag{3}$$

The first five terms of the sequence (3) coincide with the first five terms of the sequence (2) and there is a natural homomorphism of Λ-modules $H_1(M, N'') \to H_1(M, N, N')$.

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Let

$$\mathcal{D} = \begin{array}{ccc} & & M_2 \\ & & \downarrow \alpha_2 \\ M_1 & \xrightarrow{\alpha_1} & M \end{array} \tag{4}$$

be a diagram in the category $\mathfrak{A}_N$ with surjective α_1. Let $L_*(\mathcal{D}, N)$ be the pullback of the induced diagram

$$\begin{array}{ccc} & & \mathcal{F}^*(M_2)\otimes N \\ & & \downarrow \mathcal{F}^*(\alpha_2)\otimes 1_N \\ \mathcal{F}^*(M_1)\otimes N & \xrightarrow{\mathcal{F}^*(\alpha_1)\otimes 1_N} & \mathcal{F}^*(M)\otimes N. \end{array}$$

Define $H_k(\mathcal{D}, N) = \pi_{k-1}L_*(\mathcal{D}, N)$, $k \geq 2$.

THEOREM 7 (MAYER-VIETORIS SEQUENCE). *For any diagram* (4) *there is a long exact sequence of* Λ*-modules*

$$\begin{aligned} \cdots \to H_{k+1}(M, N) \to H_k(\mathcal{D}, N) \to H_k(M_1, N) \oplus H_k(M_2, N) \to H_k(M, N) \\ \cdots \to H_2(\mathcal{D}, N) \to H_2(M_1, N) \oplus H_2(M_2, N) \to H_2(M, N) \to \pi_0 L_*(\mathcal{D}, N) \\ \to \pi_0(\mathcal{F}^*(M_1)\otimes N) \oplus \pi_0(\mathcal{F}^*(M_2)\otimes N) \to \pi_0(\mathcal{F}^*(M)\otimes N) \to 0. \end{aligned} \tag{5}$$

Proof. There is a commutative diagram of simplicial Lie algebras with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_* & \xrightarrow{\sigma_*} & L_*(\mathcal{D}, N) & \xrightarrow{p_*} & \mathcal{F}^*(M_2)\otimes N & \longrightarrow & 0 \\ & & \| & & \downarrow q_* & & \downarrow \mathcal{F}^*(\alpha_2)\otimes 1_N & & \\ 0 & \longrightarrow & \mathcal{I}_* & \longrightarrow & \mathcal{F}^*(M_1)\otimes N & \xrightarrow{\mathcal{F}^*(\alpha_1)\otimes 1_N} & \mathcal{F}^*(M)\otimes N & \longrightarrow & 0, \end{array}$$

where $\mathcal{I}_* = \mathrm{Ker}(\mathcal{F}^*(\alpha_1)\otimes 1_N)$. Hence one has the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} \cdots \to \pi_1(\mathcal{F}^*(M_2)\otimes N) & \to & \pi_0(\mathcal{I}_*) & \to & \pi_0 L_*(\mathcal{D}, N) & \to & \pi_0(\mathcal{F}^*(M_2)\otimes N) & \to & 0 \\ \downarrow & & \| & & \downarrow & & \downarrow & & \\ \cdots \to \pi_1(\mathcal{F}^*(M)\otimes N) & \xrightarrow{\delta_1} & \pi_0(\mathcal{I}_*) & \to & \pi_0(\mathcal{F}^*(M_1)\otimes N) & \to & \pi_0(\mathcal{F}^*(M)\otimes N) & \to & 0. \end{array} \tag{6}$$

The connecting homomorphism $\pi_k(\mathcal{F}^*(M)\otimes N) \to \pi_{k-1}L_*(\mathcal{D}, N)$, $k \geq 1$, is the composite map $\pi_{k-1}(\sigma_*)\delta_k$. The homomorphism $\pi_k(L_*(\mathcal{D}, N)) \to \pi_k(\mathcal{F}^*(M_1)\otimes N) \oplus \pi_k(\mathcal{F}^*(M_2)\otimes N)$, $k \geq 0$, is induced by $\pi_k(q_*)$ and $\pi_k(p_*)$. The homomorphism $\pi_k(\mathcal{F}^*(M_1)\otimes N) \oplus \pi_k(\mathcal{F}^*(M_2)\otimes N) \to \pi_k(\mathcal{F}^*(M)\otimes N)$, $k \geq 0$, is given by $\pi_k(\mathcal{F}^*(\alpha_1)\otimes 1_N) - \pi_k(\mathcal{F}^*(\alpha_2)\otimes 1_N)$. To get the exactness of the sequence (5) it remains to use the diagram (6). □

COROLLARY 8. *There is a long exact sequence of the non-abelian homology of Lie algebras with respect to the first variable.*

Proof. This follows by applying Theorem 7 for $M_2 = 0$. □

Let us consider $H_1(-, N)$ as a functor from the category $\mathfrak{A}_N$ to the category $\mathcal{L}ie$ of Lie algebras and its derived functors $\mathcal{L}_k^{\mathbb{F}}(H_1(-, N))$ relative to the cotriple $\mathbb{F}$.

https://doi.org/10.1017/S0017089504001909 Published online by Cambridge University Press

THEOREM 9. *There is a natural isomorphism*

$$H_k(-, N) \cong \mathcal{L}^{\mathbb{F}}_{k-1}(H_1(-, N)), \quad k \geq 1.$$

Proof. This follows from the long exact homotopy sequence of the following short exact sequence of simplicial Lie algebras

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \vdots & & H_1(\mathcal{F}^3(M), N) & \overrightarrow{\overrightarrow{\rightarrow}} & H_1(\mathcal{F}^2(M), N) & \rightrightarrows & H_1(\mathcal{F}^1(M), N) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \vdots & & \mathcal{F}^3(M) \otimes N & \overrightarrow{\overrightarrow{\rightarrow}} & \mathcal{F}^2(M) \otimes N & \rightrightarrows & \mathcal{F}^1(M) \otimes N, \\
 & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu \\
\cdots \vdots & & \operatorname{Im} \nu & \overrightarrow{\overrightarrow{\rightarrow}} & \operatorname{Im} \nu & \rightrightarrows & \operatorname{Im} \nu \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where the bottom simplicial Lie algebra is a constant simplicial Lie algebra and $\nu : M \otimes N \to N/H$ is a homomorphism given in Definition 5. □

PROPOSITION 10. *Let* $\{M_\alpha,\ \phi^\beta_\alpha,\ \alpha \leq \beta\}$ *and* $\{N_\alpha,\ \psi^\beta_\alpha,\ \alpha \leq \beta\}$ *be direct systems of Lie algebras. Let* M *and* N *be Lie algebras and for every* α *the Lie algebras* M_α*,* N *and* M*,* N_α *act on each other and the homomorphisms* ϕ^β_α*,* ψ^β_α *preserve the actions. Then there are natural isomorphisms*

$$H_k\Big(M, \varinjlim_\alpha \{N_\alpha\}\Big) \cong \varinjlim_\alpha \{H_k(M, N_\alpha)\}, \quad k \geq 0,$$

$$H_k\Big(\varinjlim_\alpha \{M_\alpha\}, N\Big) \cong \varinjlim_\alpha \{H_k(M_\alpha, N\}), \quad k \geq 0.$$

Proof. This is straightforward. □

We end this section with explicit descriptions of the second and the third non-abelian homology of Lie algebras.

Let M and N be Lie algebras acting on each other. Let F be an object of the projective class $\mathbb{P}$ induced by the cotriple $\mathbb{F}$ and $F \xrightarrow{\varepsilon} M$ be a $\mathbb{P}$-epimorphism in the category $\mathfrak{A}_N$. Let us consider the *augmented Čech resolution* $(\check{C}(\varepsilon)_*, \varepsilon, M)$ of the object M in the category $\mathfrak{A}_N$, where

$$\check{C}(\varepsilon)_k = \underbrace{F \underset{M}{\times} \cdots \underset{M}{\times} F}_{(k+1)-\text{times}}, \quad k \geq 0,$$

$$d^k_i(x_0, \ldots, x_k) = (x_0, \ldots, \hat{x}_i, \ldots, x_k), \quad k \geq 1, \quad 0 \leq i \leq k,$$

$$s^k_i(x_0, \ldots, x_k) = (x_0, \ldots, x_i, x_i, \ldots, x_k), \quad k \geq 0, \quad 0 \leq i \leq k.$$

Applying the functor $- \otimes N$ dimension-wise to the Čech resolution of M, yields the augmented simplicial Lie algebra $(\check{C}(\varepsilon)_* \otimes N, \varepsilon \otimes 1_N, M \otimes N)$.

THEOREM 11. (i) *There is an isomorphism of* Λ*-modules*

$$H_2(M,N) \cong \{\operatorname{Ker}(d_0^1 \otimes 1_N) \cap \operatorname{Ker}(d_1^1 \otimes 1_N)\}/[\operatorname{Ker}(d_0^1 \otimes 1_N), \operatorname{Ker}(d_1^1 \otimes 1_N)];$$

(ii) *there is an epimorphism of* Λ*-modules*

$$H_3(M,N) \longrightarrow \bigcap_{i=0}^{2} \operatorname{Ker}(d_i^2 \otimes 1_N)/\sum_{I,J}[K_I, K_J],$$

where $\emptyset \neq I, J \subset \{0,1,2\}$ *such that* $I \cup J = \{0,1,2\}$, $K_I = \bigcap_{i\in I} \operatorname{Ker}(d_i^2 \otimes 1_N)$, *and* $K_J = \bigcap_{j\in J} \operatorname{Ker}(d_j^2 \otimes 1_N)$.

Proof. We have an isomorphism

$$H_2(M,N) = \mathcal{L}_1^{\mathbb{F}}(-\otimes N)(M) \cong \pi_1(\check{C}(\varepsilon)_* \otimes N), \tag{7}$$

and an epimorphism

$$H_3(M,N) = \mathcal{L}_2^{\mathbb{F}}(-\otimes N)(M) \longrightarrow \pi_2(\check{C}(\varepsilon)_* \otimes N), \tag{8}$$

(see e.g. [**12**], Theorem 2.39 (ii)).

The Lie algebra $\check{C}(\varepsilon)_2 \otimes N$ coincides with its ideal generated by the degenerate elements. In fact, for any $(x,y,z)\otimes n \in \check{C}(\varepsilon)_2 \otimes N$ there is an equality

$$\begin{aligned}(x,y,z)\otimes n &= (x,x,x)\otimes n + (0, y-x, y-x)\otimes n + (0,0,z-y)\otimes n \\ &= (s_0^1 \otimes 1_N)((x,x)\otimes n) + (s_1^1 \otimes 1_N)((0, x-y)\otimes n) \\ &\quad + (s_0^1 \otimes 1_N)((0, z-y)\otimes n).\end{aligned}$$

It is easy to verify the similar fact for $\check{C}(\varepsilon)_3 \otimes N$. Then by [**1**], Theorem 1

$$\operatorname{Im}\partial_2 = [\operatorname{Ker}(d_0^1 \otimes 1_N), \operatorname{Ker}(d_1^1 \otimes 1_N)],$$
$$\operatorname{Im}\partial_3 = \sum_{I,J}[K_I, K_J],$$

where ∂_2 and ∂_3 are differentials of the Moore complex of $\check{C}(\varepsilon)_* \otimes N$. Hence the assertion follows from (7) and (8). □

4. Application to cyclic homology. The results of [**11**, **19**, **20**] (see also [**18**]) make one think of the cyclic homology HC_* and the Milnor cyclic homology HC_*^M as additive version of the algebraic K-theory and the Milnor K-theory respectively. It is well known [**18**] that the Milnor cyclic homology groups $HC_*^M(A)$ coincide with $\Omega^*_{A|\Lambda}/d\Omega^{*-1}_{A|\Lambda}$ for commutative algebra A, where $\Omega^*_\Lambda A$ are the Kähler differentials forms of A.

Using the non-abelian group homology the relation of algebraic K-functor K_2 and Milnor K-functor K_2^M is established for non-commutative local rings [**9**, **17**]. To this end we give an additive version of this result. In particular, the relation of the first cyclic homlogy HC_1 and the first Milnor cyclic homology HC_1^M of unital associative algebras is expressed in terms of a long exact non-abelian homology sequence of Lie algebras which corrects and extends the six-term exact sequence of Theorem 5.7 [**10**].

Now we introduce the definition of the first Milnor cyclic homology by generators analogously to Dennis-Stein generators [**7**].

Definition 12. Let A be a unital associative Λ-algebra. The first Milnor cyclic homology $HC_1^M(A)$ of A is the quotient of $A \otimes_\Lambda A$ by the relations

$$\begin{aligned} a \otimes b + b \otimes a &= 0, \\ ab \otimes c - a \otimes bc + ca \otimes b &= 0, \\ a \otimes bc - a \otimes cb &= 0 \end{aligned}$$

for all $a,\ b,\ c \in A$.

Our definition of $HC_1^M(A)$ coincides with the definition in the sense of [**18**] when Λ is a field of characteristic not equal to 2.

It is well known that the first cyclic homology $HC_1(A)$ of a unital associative Λ-algebra A is the kernel of the homomorphism of Λ-modules

$$\begin{aligned} A \otimes_\Lambda A/J(A) &\longrightarrow [A, A], \\ a \otimes b &\mapsto ab - ba, \end{aligned}$$

where $[A, A]$ is the additive commutator submodule of A and $J(A)$ is the submodule of $A \otimes_\Lambda A$ generated by the elements

$$a \otimes b + b \otimes a, \quad ab \otimes c - a \otimes bc + ca \otimes b,$$

for all $a, b, c \in A$.

It is clear that $HC_1^M(A)$ coincides with $HC_1(A)$ when A is commutative.

Given a unital associative (non-commutative) Λ-algebra A, consider A as the Lie algebra with the usual induced Lie structure $[a, b] = ab - ba,\ a,\ b \in A$. Denote by $V(A)$ the quotient Lie algebra of the non-abelian tensor square $A \otimes A$ by the ideal generated by the elements

$$\begin{aligned} &a \otimes b + b \otimes a, \\ &ab \otimes c - a \otimes bc + ca \otimes b, \end{aligned}$$

for all $a,\ b,\ c \in A$. We compile the results of [**10**] on the Lie algebra $V(A)$ into the following proposition.

Proposition 13. *Let A be a unital associative Λ-algebra.*

(*i*) *There is an action of the Lie algebra A on the Lie algebra $V(A)$ defined by the formula*

$$^{a'}(a \otimes b) = [a', a] \otimes b + a \otimes [a', b]$$

and a homomorphism $\mu : V(A) \to A$ given by $a \otimes b \mapsto [a, b]$ has the structure of crossed A-module;

(*ii*) *There is a natural isomorphism of Λ-modules*

$$V(A) \cong A \otimes_\Lambda A/J(A);$$

(*iii*) *A acts trivially on $HC_1(A)$;*

(*iv*) *There is a short exact sequence of crossed A-modules of Lie algebras*

$$0 \to HC_1(A) \to V(A) \to [A, A] \to 0.$$

Proof. The proof of (i) and (iii) is given in [**10**]. To prove (ii) one can show that $J(A) \supseteq D(A, A)$ and then examine similar arguments as in Proposition 1. The proof of (iv) is straightforward from (i), (ii) and (iii). □

We have the following theorem.

THEOREM 14. *Let A be a unital associative* (*non-commutative*) Λ-*algebra. Then there is an exact sequence of* Λ-*modules*

$$\begin{aligned}\cdots &\to H_2(A, V(A), [A, A]) \to H_2(A, V(A)) \to H_2(A, [A, A]) \to H_1(A, V(A), [A, A])\\ &\to H_1(A, V(A)) \to H_1(A, [A, A]) \to HC_1(A) \to HC_1^M(A)\\ &\to [A, A]/[A, [A, A]] \to 0,\end{aligned}$$

Proof. Proposition 13, Theorem 6 and its Remark yield the following long exact sequence of Λ-modules

$$\begin{aligned}\cdots &\to H_2(A, V(A), [A, A]) \to H_2(A, V(A)) \to H_2(A, [A, A]) \to H_1(A, V(A), [A, A])\\ &\to H_1(A, V(A)) \to H_1(A, [A, A]) \to H_0(A, HC_1(A)) \to H_0(A, V(A))\\ &\to H_0(A, [A, A]) \to 0.\end{aligned}$$

It is easy to see that

$$H_0(A, HC_1(A)) = HC_1(A), \quad H_0(A, [A, A]) = [A, A]/[A, [A, A]].$$

Since $H_0(A, V(A)) = \mathrm{Coker}\, \nu$, where $\nu : A \otimes V(A) \to V(A)$ is the Lie homomorphism given by $\nu(a \otimes (b \otimes c)) = {}^a(b \otimes c)$. Calculations in the Lie algebra $V(A)$ [**10**, Lemma 5.4] say that

$${}^a(b \otimes c) = a \otimes [b, c] = a \otimes bc - a \otimes cb, \quad a,\ b,\ c \in A.$$

Now one easily deduces that there is a natural isomorphism of Λ-modules

$$H_0(A, V(A)) \cong HC_1^M(A).$$ □

ACKNOWLEDGEMENTS. The authors would like to thank the referee for useful comments and suggestions. The first and the second authors would like to thank the University of Santiago de Compostela for its hospitality during the work on this paper. The authors were partially supported by NATO PST.CLG.979167. The first and the second authors were also supported by INTAS grant No 00 566.

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