SECTION I.

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§ 1. Centroid.

*The medians of a triangle are concurrent.**

Figure 1.

Let the medians BB′, CC′ cut each other at G; join AG, and let it cut BC at A′.

Produce AA′ to L, making GL equal to GA, and join BL, CL.

Because	C′G bisects AB and AL,
therefore	C′G is parallel to BL.
Similarly	B′G „ „ „ CL;
therefore	BLCG is a parallelogram;
therefore	A′ is the mid point of BC.

This theorem may be proved in many other ways.

Def.—The point G is called sometimes the *centre of gravity* † of the triangle ABC; sometimes the *centre of mean distances* ‡ of the points A, B, C; and more frequently now the *centroid* § of the triangle ABC.

The simplest construction for obtaining G by means of the ruler and the compasses is the following ‖ :—

With B as centre and AC as radius describe a circle; with C as centre and AB as radius describe a second circle cutting the former below the base at D. Join DC and produce it to meet the second circle at E.

AD and BE intersect at the centroid G.

(1) $A'G = \frac{1}{2}AG = \frac{1}{3}AA'$.

Hence the centroid of a triangle may be found by drawing any median and trisecting it; and if two (or a series of) triangles have the same vertex and the same median drawn from that vertex, they have the same centroid.

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* Archimedes, *De planorum æquilibriis*, I. 13, 14.

† Archimedes.

‡ Carnot, *Géométrie de Position*, p. 315 (1803), and Lhuilier, *Élémens d'Analyse*, p. X. (1809).

§ This expression was suggested by T. S. Davies in 1843 in the *Mathematician* I. 58. It had been used by Dr Hey in 1814 to designate another point.

‖ Mr E. Lemoine in the Report (second part) of the 21st session of the *Association Française pour l'avancement des sciences*, p. 77 (1892).

(2) Triangle GBC = GCA = GAB = $\frac{1}{3}$ABC.

(3) The sides of triangle A′B′C′ are respectively parallel to those of ABC; hence these triangles are directly similar.

Also, since the lines AA′, BB′, CC′ joining corresponding vertices are concurrent at G, triangles ABC, A′B′C′ are homothetic, and G is their homothetic centre.

DEF.—Triangles such as the fundamental triangle ABC, and that formed by joining the feet of its medians have in recent years received the following names:—

A′B′C′ is the *complementary* triangle of ABC.
ABC „ „ *anticomplementary* „ „ A′B′C′.

These names are applied also to corresponding points * in such triangles. Thus if P be any point in or outside of triangle A B C, and P′ be the corresponding point in or outside of triangle A′B′C′,

P′ is the *complementary* point of P,
P „ „ *anticomplementary* „ „ P′.

(4) If $A_1B_1C_1$ be the triangle formed by drawing through A, B, C parallels to the opposite sides of triangle ABC,

A B C is the complementary triangle of $A_1B_1C_1$,
$A_1B_1C_1$ „ „ anticomplementary „ „ A B C.

FIGURE 2.

(5) The fundamental triangle ABC is directly similar to the triangles cut off from it by the sides of its complementary triangle, AC′B′, C′BA′, B′A′C.

(6) The centroid of the fundamental triangle is the centroid of the complementary triangle; the centroid of the complementary triangle is the centroid of its complementary triangle; and so on.

(7) All straight lines parallel to the base of a triangle and terminated by the other sides are bisected by the median to the base.

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* See Mr Emile Vigarié's articles *Sur les Points Complémentaires* in *Mathesis*, VII., 5-12, 57-62, 84-9, 105-110 (1887).

Hence, if EF, GH, KL ... be parallel to BC, the points E, G, K ... being on AC, and F, H, L ... on AB, the intersections of

BE, CF; BG, CH; BK, CL; FG, EH; FK, EL ...... will all lie on the median * from A.

(8) If two triangles have the same base, the straight line which joins their vertices is parallel to and three times as long as the straight line which joins their centroids.

(9) If G be any point in the plane of ABC, and G_a, G_b, G_c be the centroids of triangles GBC, GCA, GAB, triangle $G_aG_bG_c$ is directly similar † to triangle ABC.

(10) If P be any point on the circumcircle of ABC, the centroids of the four triangles PBC, PCA, PAB, ABC are concyclic. ‡

For if the centroids of these triangles be denoted by D, E, F, G respectively, the quadrilateral DEFG has its sides DE, EF, FG, GD respectively parallel to BA, CB, PC, AP, and one-third as long.

Mr Griffiths states § that if the circle on which the four centroids lie be called the centroid-circle of the quadrangle ABCP, it may be shown that the centroid-circles of the five quadrangles that can be formed from five concyclic points will also have their centres on the circumference of another circle of one-third the radius of the first.

Townsend gives the following generalisation ‖ of (10):

If A, B, C, D, E, F, etc., be the position of any number (n) of equal masses distributed in space, G that of their centre of gravity, and A′, B′, C′, D′, E′, F′, etc., those of the centres of gravity of their n different groups of $(n-1)$; then always the two systems of n points A, B, C, D, E, F, etc., and A′, B′, C′, D′, E′, F′, etc., are similar, oppositely placed with respect to each other, have G for their centre of similitude, and $(n-1):1$ for their ratio of similitude.

The truth of this is evident, for the several lines AA′, BB′, CC′, DD′, EE′, FF′, etc., all connect through G, and are then divided internally in the common ratio of $(n-1):1$.

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* Jacobi, *De Triangulorum rectilineorum proprietatibus*, pp. 5-6 (1825).

† Professor R. E. Allardice.

‡ Mr J. Griffiths, in *Mathematical Questions from the Educational Times*, V. 92 (1866).

§ *Notes on the Geometry of the Plane Triangle*, p. 65 (1867).

‖ *Mathematical Questions from the Educational Times*, V. 92 (1866).

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DEF.—If the vertex A of a triangle ABC be joined to any point D in the base, the fourth harmonic ray to AB, AD, AC is found by dividing BC externally at D′ in the ratio BD : CD, and joining AD′.

When the point D is the mid point of BC, namely A′, the fourth harmonic ray to AB, AA′, AC is the line through A parallel to BC, and it may be denoted by AA_∞.

Similarly, the line through B parallel to CA will be the fourth harmonic ray to BC, BB′, BA, and may be denoted by BB_∞; the line through C parallel to AB will be the fourth harmonic ray to CA, CC′, CB, and may be denoted by CC_∞.

If therefore AA′, BB′, CC′ be called the *internal medians* of triangle ABC, then AA_∞, BB_∞, CC_∞ may be called the *external medians*.

(11) The six medians, internal and external, of a triangle meet three and three in four points, which are the centroid and the points anticomplementary to the vertices of the triangle, namely G, A_1, B_1, C_1.

FIGURE 2.

DEF.—The points A_1, B_1, C_1, G form a *tetrastigm* (a system of four points, no three of which are collinear), and the three pairs of opposite connectors,

$$A_1G,\ B_1C_1\ ;\ B_1G,\ C_1A_1\ ;\ C_1G,\ A_1B_1$$

meet in A ; B ; C,

which are the *centres* of the tetrastigm, and ABC is the *central triangle* of the tetrastigm.

If ABCG be the tetrastigm, the points A′, B′, C′ are its centres, and A′B′C′ its central triangle.

(12) *If in the internal median AA′ of triangle ABC any point M be taken, and MP, MQ be drawn perpendicular to AC, AB, then MP, MQ are inversely proportional to AC, AB.*

FIGURE 3.

Join MB, MC.

Then AMB = AMC;

therefore AB · MQ = AC · MP;

therefore AB : AC = MP : MQ.

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(13) *If in the external median AA_∞ of triangle ABC any point M′ be taken, and M′P′, M′Q′ be drawn perpendicular to AC, AB, then M′P′, M′Q′ are inversely proportional to AC, AB.*

FIGURE 4.

Join M′B, M′C.

Then $$AM'B = AM'C;$$
therefore $$AB \cdot M'Q' = AC \cdot M'P';$$
therefore $$AB : AC = M'P' : M'Q'.$$

(14) If from G the centroid of ABC there be drawn p_1, p_2, p_3 perpendicular to BC, CA, AB, then

$$BC : CA : AB = \frac{1}{p_1} : \frac{1}{p_2} : \frac{1}{p_3}.$$

(15) If from G the centroid of ABC there be drawn p_1', p_2', p_3' perpendicular to B_1C_1, C_1A_1, A_1B_1, then

$$BC : CA : AB = \frac{1}{p_1'} : \frac{1}{p_2'} : \frac{1}{p_3'}.$$

(16) *If the vertex A of the triangle ABC falls on the base BC, the centroid G of the three collinear points A, B, C is found by the construction indicated in* (1):

Bisect BC in A′, and divide AA′ at G so that

$$AG : A'G = 2 : 1.$$

In this case the sum of the distances from the centroid of the points on one side of it is equal to the distance from the centroid of the point on the opposite side.

FIGURE 5.

For
$$\begin{aligned} AG + BG &= (AA' - GA') + (BA' - GA'), \\ &= AA' + BA' - 2GA', \\ &= AA' + CA' - GA, \\ &= CA - GA, \\ &= CG. \end{aligned}$$

(17) If ABC be a triangle, G its centroid, and A′, B′, C′, G′ the projections of A, B, C, G, on any straight line XY, then G′ is the centroid of the three collinear points A′, B′, C′.

(18) LEMMA.*

If a straight line BC be divided internally at M so that

$$BM : CM = n : m$$

and if from B, M, C perpendiculars BB′, MM′, CC′, be drawn to any straight line XY, then

$$(m+n)MM' = mBB' \pm nCC'$$

the upper sign being taken when B and C are on the same † *side of XY, and the lower when they are on opposite sides of XY.*

FIGURE 6.

Join BC′ meeting MM′ in N.

The triangles BB′C, NM′C are similar;

therefore $\quad BB' : NM' = BC' : NC'$

$$= BC : MC$$

$$= m+n : m ;$$

therefore $\quad m\mathrm{BB}' = (m+n)\mathrm{NM}'.$

The triangles BCC′, BMN are similar;

therefore $\quad CC' : MN = BC : BM,$

$$= m+n : n ;$$

therefore $\quad n\mathrm{CC}' = (m+n)\mathrm{MN}.$

Hence $\quad m\mathrm{BB}' + n\mathrm{CC}' = (m+n)\mathrm{NM}' + (m+n)\mathrm{MN}$

$$= (m+n)\mathrm{MM}'.$$

(19) *The distance of the centroid of a triangle from any straight line is an arithmetic mean between the distances of the vertices from the same straight line.*

FIGURE 7.

Let AM be the median from A, and G the centroid.

Take A′, B′, C′, G′, M′ the projections of A, B, C, G, M on any straight line XY.

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* Lhuilier, *Élémens d'Analyse*, pp. 1-2 (1809).

† The figure and demonstration refer only to this case. The other case and the consideration of what happens when BC is divided externally are left to the reader.

Because $BM : CM = 1 : 1$,
therefore $BB' + CC' = 2MM'$.
Because $AG : MG = 2 : 1$,
therefore $AA' + 2MM' = 3GG'$;
therefore $AA' + BB' + CC' = 3GG'$.

The figure and demonstration refer only to the case when A, B, C are all on the same side of XY. If A and B be on the same side of XY, and C on the opposite side, the result will be

$$AA' + BB' - CC' = 3GG'.$$

When XY passes through the centroid G, the sum of the distances from XY of the vertices on one side of it is equal to the distance from XY of the vertex on the opposite side.

- For a very full account of the properties of the centre of mean distances see the preliminary dissertation in Lhuilier's *Élémens d'Analyse* (1809), and Townsend's *Modern Geometry of the Point, Line, and Circle*, I. 117-143 (1863).

(20) *The sum of the squares of the distances of the vertices of a triangle from any point is equal to the sum of the squares of their distances from the centroid increased by three times the square of the distance between the point and the centroid.**

FIGURE 8.

Let G be the centroid of ABC, and P any other point. Join PG, and on it draw perpendiculars from A, B, C.

Then
$$AP^2 = AG^2 + PG^2 - 2PG \cdot A'G,$$
$$BP^2 = BG^2 + PG^2 + 2PG \cdot B'G,$$
$$CP^2 = CG^2 + PG^2 - 2PG \cdot C'G;$$
therefore
$$AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2 - 2PG(A'G - B'G - C'G),$$
$$= AG^2 + BG^2 + CG^2 + 3PG^2,$$
since
$$A'G - B'G - C'G = 0.$$

(21) If a circle be described with G as centre, and any radius, and any two points P, Q be taken on its circumference †

$$AP^2 + BP^2 + CP^2 = AQ^2 + BQ^2 + CQ^2.$$

* This is a particular case of a more general theorem proved in Robert Simson's *Apollonii Pergaei Locorum Planorum Libri II.*, pp. 179-180 (1749).

† C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 7 (1825).

(22) *That point the sum of the squares of whose distances from the vertices of a triangle is a minimum is the centroid of the triangle.**

If the triangle ABC is fixed, G is a fixed point, and AG, BG, CG fixed distances. Hence for any variable point P, $AP^2+BP^2+CP^2$ always exceeds the constant quantity $AG^2+BG^2+CG^2$ by $3PG^2$. The nearer therefore P approaches to G, the nearer does $AP^2+BP^2+CP^2$ approach this constant quantity.

(23) *That point inside a triangle which has the product of its distances from the three sides a maximum is the centroid of the triangle.*†

Let G be any point inside ABC, and GR, GS, GT its distances from BC, CA, AB.

Then $GR \times GS \times GT$ is a maximum,
when $GR\cdot\frac{1}{2}BC \times GS\cdot\frac{1}{2}CA \times GT\cdot\frac{1}{2}AB$ is a maximum,
that is, when $GBC \times GCA \times GAB$ is a maximum,
that is, when these three triangles are equal,
that is, when G is the centroid.

(24) *If three straight lines drawn from the vertices of a triangle are concurrent, the three straight lines drawn parallel to them from the mid points of the opposite sides are also concurrent; and the straight line joining the two points of concurrency passes through the centroid of the triangle and is there trisected.* ‡

The triangles ABC, A′B′C′ are similar and oppositely situated, G is their homothetic centre, and 2 : 1 is the ratio of similitude.

Hence if AD, BE, CF be concurrent at O, the corresponding straight lines A′D′, B′E′, C′F′ will pass through the corresponding point O′; OO′ the straight line joining two corresponding points, will pass through the homothetic centre G; and OG : O′G = 2 : 1.

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* J. F. de Tuschis a Fagnano in *Nova Acta Eruditorum, anni* 1775, p. 290. The article referred to is entitled: *Problemata quaedam ad methodum maximorum et minimorum spectantia*, and the volume in which it occurs was published at Leipzig in 1779.

† H. Watson in the *Ladies' Diary* for 1756.

‡ Frégier in Gergonne's *Annales*, VII. 170 (1816-7).

(25) *If ABC be a triangle, O any point whatever, and A_1, B_1, C_1 symmetrical to O with respect to the mid points of BC, CA, AB, then* *

(a) *AA_1, BB_1, CC_1 are concurrent at a point P.*

(b) *The straight line OP turns round a fixed point G when the point O moves in any manner whatever.*

(c) *The point G divides OP in a constant ratio.*

FIGURE 9.

Let A′, B′, C′ be the mid points of BC, CA, AB.

(*a*) Then A_1B_1 is parallel to A′B′ and equal to 2A′B′;
therefore it is equal and parallel to AB, but oppositely directed.
Similarly B_1C_1 and C_1A_1 are equal and parallel to BC and CA, but oppositely directed.
The three pairs of parallels BC and B_1C_1, CA and C_1A_1, AB and A_1B_1 form therefore three parallelograms, whose diagonals AA_1, BB_1, CC_1 cut each other at P the mid point of each of them.

(*b*) In triangle OAA_1 the lines OP, AA′ are medians;
therefore OP cuts AA′ at G such that AG = 2A′G.
But AA′ is a median of triangle ABC;
therefore G is the centroid of ABC, and consequently a fixed point.

(*c*) OP is divided at G so that OG = 2GP.

(26) The sum of the squares on the sides of the complementary triangle is one-fourth of the sum of the squares on the sides of the fundamental triangle.

(27) If in a triangle its complementary triangle be inscribed, and in the complementary triangle its complementary triangle be inscribed, and so on, the limit of the sum of the squares on the sides of all the triangles so formed is one-third of the sum of the squares on the sides of the fundamental triangle.†

(28) If in a triangle its complementary triangle be inscribed, and so on, the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the vertices of

* Mr Maurice d'Ocagne in the *Nouvelles Annales*, 3rd Series, I. 239 (1882); proof on p. 430.

† Leybourn's *Mathematical Repository*, new series, V. 111 (1820).

all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the vertices of the fundamental triangle.*

(29) *If $A_1B_1C_1$ be the complementary triangle of ABC, $A_2B_2C_2$ the complementary triangle of $A_1B_1C_1$, and so on ; and if P be any point in the plane of the triangle, then*†

$$PA_n^2 + PB_n^2 + PC_n^2 = 3PG^2 + \frac{1}{3 \cdot 4^n}(BC^2 + CA^2 + AB^2).$$

FIGURE 10.

Join P with A, B, C, A′, G, and with D the mid point of AG.

$$\text{Then} \qquad AB^2 + AC^2 = 2A_1A^2 + 2A_1B^2;$$
$$= 18A_1G^2 + 2A_1B^2;$$

$$\text{therefore} \quad BC^2 + CA^2 + AB^2 = 18A_1G^2 + 6A_1B^2.$$

$$\text{Again,} \qquad PG^2 + PA^2 = 2\ (GD^2 + PD^2),$$
$$PB^2 + PC^2 = 2\ (A_1B^2 + PA_1^2),$$
$$2\ (PA_1^2 + PD^2) = 4\ (A_1G^2 + PG^2);$$

therefore by addition, and subtraction of what is common,

$$PA^2 + PB^2 + PC^2 = 3PG^2 + 6A_1G^2 + 2A_1B^2$$
$$= 3PG^2 + \tfrac{1}{3}(BC^2 + CA^2 + AB^2).$$

$$\text{Similarly } PA_1^2 + PB_1^2 + PC_1^2 = 3PG^2 + \tfrac{1}{3}(B_1C_1^2 + C_1A_1^2 + A_1B_1^2)$$
$$= 3PG^2 + \frac{1}{3 \cdot 4}(BC^2 + CA^2 + AB^2)$$

.... = ...

$$\text{Hence} \quad PA_n^2 + PB_n^2 + PC_n^2 = 3PG^2 + \frac{1}{3 \cdot 4^n}(BC^2 + CA^2 + AB^2)$$

(30) *If the sides of triangle ABC be divided at A_1, B_1, C_1, so that $BA_1 : BC = CB_1 : CA = AC_1 : AB = m$, then*

$$A_1B_1C_1 = ABC\{1 - 3m(1 - m)\}.$$

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* Mr E. Conolly in *Mathematical Questions from the Educational Times*, IV. 76 (1865).

† Mr Stephen Watson, in *Mathematical Questions from the Educational Times*, XX. 109-112 (1873), where four solutions are given. The solution in the text is Mr Watson's.

FIGURE 11.

For $A_1B_1C_1 = ABC - AB_1C_1 - BC_1A_1 - CA_1B_1.$

Now $\dfrac{AB_1C_1}{ABC} = \dfrac{AC_1}{AB} \cdot \dfrac{AB_1}{AC} = m(1-m);$

therefore $AB_1C_1 = ABC \cdot m(1-m).$

Similarly $BC_1A_1 = ABC \cdot m(1-m),$

and $CA_1B_1 = ABC \cdot m(1-m);$

therefore $A_1B_1C_1 = ABC\{1 - 3m(1-m)\}.$

(31) Let there be a series of triangles

$$A_1B_1C_1,\ A_2B_2C_2,\ \ldots\ A_nB_nC_n$$

such that each is derived from the preceding in the same way as $A_1B_1C_1$ was derived from ABC; and let them be denoted by $\Delta_1, \Delta_2, \ldots \Delta_n$.

Then the formula

$$\Delta_1 = \Delta\ \{1 - 3m(1-m)\}$$

may be applied to triangle $A_2B_2C_2$;

therefore $\Delta_2 = \Delta_1\{1 - 3m(1-m)\}$

$= \Delta\ \{1 - 3m(1-m)\}^2.$

Similarly $\Delta_3 = \Delta\ \{1 - 3m(1-m)\}^3,$

and $\Delta_n = \Delta\ \{1 - 3m(1-m)\}^n.$

Hence $\Delta, \Delta_1, \Delta_2 \ldots$ form a decreasing geometrical progression, whose sum to infinity is equal to $\dfrac{\Delta}{3m(1-m)}.$

(32) This sum is a minimum when the product $m(1-m)$ is a maximum, that is, when $m = \frac{1}{2}$. Hence the minimum sum is $\frac{4}{3}\Delta$, and each triangle is then the complementary triangle of its predecessor.

When m is 0 or 1, the sum becomes infinite. This arises from the fact that then $\Delta_1, \Delta_2, \ldots$ coincide with Δ.

When m varies from 0 to 1, the sum diminishes from infinity to its maximum $\frac{4}{3}\Delta$, and then increases to infinity.

(33) *The centroid G of ABC is the centroid of* $A_1B_1C_1$, $A_2B_2C_2$ ....

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FIGURE 11.

Through B_1, C_1 draw parallels to AB, AC; these parallels will intersect on BC at a point D such that

$$BD : DC = AB_1 : B_1C = BC_1 : C_1A.$$

Hence $BD = CA_1$, and AD, B_1C_1, the diagonals of the parallelogram AB_1DC_1, bisect each other at E. Now if A′ be the mid point of BC, it will also be the mid point of DA_1;

therefore AA′, A_1E, two medians of triangle ADA_1, intersect at a point G such that

$$AG = 2A'G \text{ and } A_1G = 2EG.$$

Hence since AA′, A_1E are medians of ABC, $A_1B_1C_1$ these two triangles have the same centroid G.

What has been proved with regard to ABC, $A_1B_1C_1$ will hold equally with regard to $A_1B_1C_1$, $A_2B_2C_2$; and so on. Therefore the whole series of triangles have the same centroid.*

The last property may also be proved thus †:—

FIGURE 12.

Bisect BC and A_1B_1 at A′ and F;
join AA′, C_1F cutting each other at G;
and draw B_1D parallel to AB.

Then $BA_1 : CA_1 = CB_1 : AB_1$,
$= CD : BD$;
therefore $BA_1 = CD$;
therefore A′ is the mid point of A_1D;
therefore A′F is parallel to DB_1, and equal to $\frac{1}{2}DB_1$.

Again $B_1D : AB = CB_1 : CA$,
$= AC_1 : AB$;
therefore $B_1D = AC_1$;
therefore A′F is half of AC_1, and it is parallel to it;
therefore $AG = 2A'G$ and $C_1G = 2FG$;
therefore G is the centroid of both ABC and $A_1B_1C_1$.

* The theorem that $A_1B_1C_1$ has the same centroid as ABC will be found in Pappus's *Mathematical Collection*, VIII. 2. Chasles has some remarks on the theorem in his *Aperçu historique*, 2nd ed., p. 44.

† This mode of proof was communicated to me by Mr A. J. Pressland. Compare also Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 48-9 (1890).

Relations which Exist Between a Triangle and the Triangle whose Sides are the Medians of the Former.*

(34) *If ABC be any triangle, another triangle can always be constructed whose sides are equal to the medians of ABC.*

FIGURE 13.

Let AA′, BB′, CC′ be the medians of ABC.

Through A′ draw A′L parallel to BB′, and produce it so that A″L = A′L; join A″B′, A′C′.

Because A′ is the mid point of BC, and A′L is parallel to BB′, therefore L is the mid point of B′C.
Hence B′A′CA″ is a parallelogram, as well as B′BA′A″,
and A′A″ = BB′.

Since	A B′	is equal and parallel to C′A′
and	B′A″	,, ,, ,, ,, ,, A′C ;
therefore	A″A	,, ,, ,, ,, ,, C C′ ;

that is AA′A″ is the triangle required.

(35) *The sides of AA′A″ are parallel to the medians of ABC*
and ,, ,, ,, ABC ,, ,, ,, ,, ,, ,, AA′A″.

The first part of the theorem has been already proved.

Since AB′ : B′L = 2 : 1
therefore B′ is the centroid of AA′A″,
Now the median B′A″ is parallel to BC,
,, ,, B′A coincides with CA,
and ,, ,, B′A′ is parallel to AB.

(36) A″, B′, C′ are collinear.

(37) *If ABC be a triangle whose centroid is G, DEF the triangle whose sides are the medians of ABC, that is EF = AA′, FD = BB′, DE = CC′, then*

$$\angle D = GBC + GCB, \quad \angle E = GCA + GAC, \quad \angle F = GAB + GBA.$$

* In connection with this subject, the following authorities may be consulted :

Gergonne's *Annales*, II. 93 (1811).

Supplemente zu G. S. Klügel's Wörterbüche der reinen Mathematik, Vol. I. Art. "Dreieck" (J. A. Grunert), p. 706 (1833).

Nouvelles Annales, III. 457-460 (1844).

Battaglini's *Giornale di Matematiche*, I. 126-7 (1863).

Grunert's *Archiv*, XLI. 112-4 (1864).

FIGURE 13.

The angles which are equal have been marked with the same number; and the triangle DEF corresponds to the triangle A″AA′.

(38) *If* $ABC,\ A_1B_1C_1,\ A_2B_2C_2\ \ldots\ldots$
$DEF,\ D_1E_1F_1,\ D_2E_2F_2\ \ldots\ldots$

be two sets of triangles such that the sides of

D E F are equal to the medians of A B C
$A_1B_1C_1$ „ „ „ „ „ „ *D E F*
$D_1E_1F_1$ „ „ „ „ „ „ $A_1B_1C_1$,

and so on;
the triangles $ABC,\ A_1B_1C_1\ \ldots$ *will be similar to each other* *
and $DEF,\ D_1E_1F_1\ \ldots$ „ „ „ „ „ „

FIGURE 14.

The proof of the theorem will appear from the figure † if it be observed that

Triangles	correspond to	
D E F	A″A A′	(4, 5; 6, 1; 2, 3),
$A_1B_1C_1$	L A‴A	(1, 2; 3, 4; 5, 6),
$D_1E_1F_1$	A M A$^{\text{iv}}$	(4, 5; 6, 1; 2, 3),
$A_2B_2C_2$	A$^{\text{v}}$A N	(1, 2; 3, 4; 5, 6),
$D_2E_2F_2$	P A$^{\text{vi}}$A	(4, 5; 6, 1; 2, 3).

The theorem may be proved also as follows: ‡

If m_1, m_2 m_3, be the three medians of ABC, then

$$m_1^2 = \tfrac{1}{2}(b^2 + c^2 - \tfrac{1}{2}a^2),\quad m_2^2 = \tfrac{1}{2}(c^2 + a^2 - \tfrac{1}{2}b^2),\quad m_3^2 = \tfrac{1}{2}(a^2 + b^2 - \tfrac{1}{2}c^2).$$

Make a triangle whose sides are m_1, m_2, m_3,
and let its medians be a_1, b_1, c_1; then

$$a_1^2 = \tfrac{1}{2}(m_2^2 + m_3^2 - \tfrac{1}{2}m_1^2), b_1^2 = \tfrac{1}{2}(m_3^2 + m_1^2 - \tfrac{1}{2}m_2^2), c_1^2 = \tfrac{1}{2}(m_1^2 + m_2^2 - \tfrac{1}{2}m_3^2)$$
$$= \tfrac{9}{16}a^2 \qquad , \quad = \tfrac{9}{16}b^2 \qquad , \quad = \tfrac{9}{16}c^2;$$

therefore $a_1 = \tfrac{3}{4}a,\ b_1 = \tfrac{3}{4}b,\ c_1 = \tfrac{3}{4}c,$

and $a_1 : b_1 : c_1 = a : b : c.$

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* Gergonne's *Annales*, II. 93 (1811).

† The figure has been taken from Grunert's article "Dreieck" previously referred to.

‡ Grunert's *Archiv*, XLI. 112-4 (1864).

(39) If

$$\Delta,\ \Delta_2,\ \Delta_4\ \ldots\ldots$$
$$\Delta_1,\ \Delta_3,\ \Delta_5\ \ldots\ldots$$

denote the two sets of triangles in (38), the sides of

Δ	are	$a,$	$b,$	c
Δ_2	,,	$\frac{3}{4}a,$	$\frac{3}{4}b,$	$\frac{3}{4}c$
Δ_4	,,	$(\frac{3}{4})^2a,$	$(\frac{3}{4})^2b,$	$(\frac{3}{4})^2c$
Δ_{2n}	,,	$(\frac{3}{4})^na,$	$(\frac{3}{4})^nb,$	$(\frac{3}{4})^nc$
Δ_1	,,	$m_1,$	$m_2,$	m_3
Δ_3	,,	$\frac{3}{4}m_1,$	$\frac{3}{4}m_2,$	$\frac{3}{4}m_3$
Δ_5	,,	$(\frac{3}{4})^2m_1,$	$(\frac{3}{4})^2m_2,$	$(\frac{3}{4})^2m_3$
Δ_{2n+1}	,,	$(\frac{3}{4})^nm_1,$	$(\frac{3}{4})^nm_2,$	$(\frac{3}{4})^nm_3$

(40) *The triangles* $\Delta,\ \Delta_1,\ \Delta_2,\ \Delta_3\ \ldots$ *form a geometrical progression* whose common ratio is* $\frac{3}{4}$.

FIGURE 13.

Since $\qquad AL = \frac{3}{4}AC$

therefore $\qquad AA'L = \frac{3}{4}AA'C$;

therefore $\qquad AA'A'' = \frac{3}{4}ABC$; and so on.

(41) $\qquad \Delta + \Delta_1 + \Delta_2 + \ldots ad\ infinitum = 4\Delta.$

(42) If $p,\ p_2,\ p_4\ \ldots$ be the perimeters of $\Delta,\ \Delta_2,\ \Delta_4 \ldots$
$p + p_2 + p_4 + \ldots ad\ infinitum = 4p.$

(43) If $p_1,\ p_3,\ p_5\ \ldots$ be the perimeters of $\Delta_1,\ \Delta_3,\ \Delta_5 \ldots$
$p_1 + p_3 + p_5 + \ldots ad\ infinitum = 4p_1.$

(44) $\qquad \Delta + \Delta_2 + \Delta_4 + \ldots ad\ infinitum = \frac{16}{7}\Delta.$

(45) $\qquad \Delta_1 + \Delta_3 + \Delta_5 + \ldots ad\ infinitum = \frac{12}{7}\Delta.$

(46) If G be the centroid of ABC and another triangle $A_0B_0C_0$ be formed with sides respectively equal to $\sqrt{3}\,GA$, $\sqrt{3}\,GB$, $\sqrt{3}\,GC$, then ABC may be derived from $A_0B_0C_0$ in the same way as the latter was derived from the former, that is, the relation between the triangles is a conjugate one.†

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* Gergonne's *Annales*, II. 93 (1811).

† Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, pp. 150-1 (1888).

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(47) The areas of ABC, $A_0B_0C_0$ are equal.*

These two theorems follow without much difficulty from what precedes.

(48) *If through the centroid G of a triangle ABC a straight line be drawn cutting BC, CA, AB in D, E, F and the points E, F be on the same side of G then*†

$$\frac{1}{GE}+\frac{1}{GF}=\frac{1}{GD}.$$

FIGURE 15.

Through A draw AN parallel to BC meeting DEF in K, and through G draw LMN parallel to AB meeting BC, CA, AN in L, M, N.

Then $\quad LG = MG$, and $CM = 2AM$.

But since triangles CML, AMN are similar,

therefore $\quad ML = 2MN$;

therefore $\quad GM = MN$.

Hence AG, AM, AN, AF form a harmonic pencil;

and they are cut by the transversal GEKF;

therefore G, E, K, F form a harmonic range;

therefore
$$\frac{1}{GE}+\frac{1}{GF}=\frac{2}{GK}$$
$$=\frac{1}{GD}$$

since $\quad GK = 2GD$.

* Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, p. 151 (1888).

† This property, proved in the manner given, will be found in Maclaurin's *Algebra* (1748) in the Appendix, *De Linearum Geometricarum Proprietatibus generalibus Tractatus*, §98 or p. 57. A proof by Dr E. v. Hunyady of Pesth, by means of transversals, will be found in Schlömilch's *Zeitschrift*, VII. 268-9 (1862).

Formulæ connected with the Medians.

The medians in terms of the sides.

$$\left.\begin{aligned} 4m_1^2 &= -a^2+2b^2+2c^2 \\ 4m_2^2 &= 2a^2-b^2+2c^2 \\ 4m_3^2 &= 2a^2+2b^2-c^2 \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} 4m_1^2 &= -a^2+(b+c)^2+(b-c)^2 \\ 4m_2^2 &= -b^2+(c+a)^2+(c-a)^2 \\ 4m_3^2 &= -c^2+(a+b)^2+(a-b)^2 \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} (2m_1+b-c)(2m_1-b+c) &= (a+b+c)(-a+b+c) \\ (2m_2+c-a)(2m_2-c+a) &= (a+b+c)(a-b+c) \\ (2m_3+a-b)(2m_3-a+b) &= (a+b+c)(a+b-c) \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} (b+c+2m_1)(b+c-2m_1) &= (a+b-c)(a-b+c) \\ (c+a+2m_2)(c+a-2m_2) &= (-a+b+c)(a+b-c) \\ (a+b+2m_3)(a+b-2m_3) &= (a-b+c)(-a+b+c) \end{aligned}\right\} \quad (4)$$

$$4(m_1^2+m_2^2+m_3^2) = 3(a^2+b^2+c^2) \quad (5)$$

$$3(\mathrm{AG}^2+\mathrm{BG}^2+\mathrm{CG}^2) = a^2+b^2+c^2 \quad (6)$$

$$12(\mathrm{A'G}^2+\mathrm{B'G}^2+\mathrm{C'G}^2) = a^2+b^2+c^2 \quad (7)$$

$$m_1\cdot\mathrm{AG}+m_2\cdot\mathrm{BG}+m_3\cdot\mathrm{CG} = \tfrac{1}{2}(a^2+b^2+c^2) \quad (8)$$

$$m_1\cdot\mathrm{A'G}+m_2\cdot\mathrm{B'G}+m_3\cdot\mathrm{C'G} = \tfrac{1}{4}(a^2+b^2+c^2) \quad (9)$$

$$16(m_1^4+m_2^4+m_3^4) = 9(a^4+b^4+c^4) \quad (10)$$

$$16(m_2^2m_3^2+m_3^2m_1^2+m_1^2m_2^2) = 9(b^2c^2+c^2a^2+a^2b^2) \quad (11)$$

The sides in terms of the medians.

$$\left.\begin{aligned} 9a^2 &= -4m_1^2+8m_2^2+8m_3^2 \\ 9b^2 &= 8m_1^2-4m_2^2+8m_3^2 \\ 9c^2 &= 8m_1^2+8m_2^2-4m_3^2 \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} \tfrac{9}{4}a^2 &= -m_1^2+(m_2+m_3)^2+(m_2-m_3)^2 \\ \tfrac{9}{4}b^2 &= -m_2^2+(m_3+m_1)^2+(m_3-m_1)^2 \\ \tfrac{9}{4}c^2 &= -m_3^2+(m_1+m_2)^2+(m_1-m_2)^2 \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} (\tfrac{3}{2}a+m_2-m_3)(\tfrac{3}{2}a-m_2+m_3) &= (m_1+m_2+m_3)(-m_1+m_2+m_3) \\ (\tfrac{3}{2}b+m_3-m_1)(\tfrac{3}{2}b-m_3+m_1) &= (m_1+m_2+m_3)(m_1-m_2+m_3) \\ (\tfrac{3}{2}c+m_1-m_2)(\tfrac{3}{2}c-m_1+m_2) &= (m_1+m_2+m_3)(m_1+m_2-m_3) \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} (m_2+m_3+\tfrac{3}{2}a)(m_2+m_3-\tfrac{3}{2}a) &= (m_1+m_2+m_3)(m_1-m_2+m_3) \\ (m_3+m_1+\tfrac{3}{2}b)(m_3+m_1-\tfrac{3}{2}b) &= (m_1+m_2+m_3)(m_1+m_2-m_3) \\ (m_1+m_2+\tfrac{3}{2}c)(m_1+m_2-\tfrac{3}{2}c) &= (-m_1-m_2+m_3)(-m_1+m_2+m_3) \end{aligned}\right\} \quad (15)$$

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$$\left.\begin{aligned}3(b^2 \sim c^2) &= 4(m_3^2 \sim m_2^2)\\ 3(c^2 \sim a^2) &= 4(m_1^2 \sim m_3^2)\\ 3(a^2 \sim b^2) &= 4(m_2^2 \sim m_1^2)\end{aligned}\right\} \qquad (16)$$

If
$$\begin{aligned}2s &= a+b+c\\ 2s' &= b+c+2m_1\\ 2s'' &= c+a+2m_2\\ 2s''' &= a+b+2m_3\end{aligned}$$

then (3) and (4) become

$$\left.\begin{aligned}(s'-b)(s'-c) &= s(s-a) & s'(s'-2m_1) &= (s-b)(s-c)\\ (s''-c)(s''-a) &= s(s-b) & s''(s''-2m_2) &= (s-c)(s-a)\\ (s'''-a)(s'''-b) &= s(s-c) & s'''(s'''-2m_3) &= (s-a)(s-b)\end{aligned}\right\} \qquad (17)$$

$$\left.\begin{aligned}\triangle^2 &= s'(s'-b)(s'-c)(s'-2m_1)\\ &= s''(s''-c)(s''-a)(s''-2m_2)\\ &= s'''(s'''-a)(s'''-b)(s'''-2m_3)\end{aligned}\right\} \qquad (18)$$

For each
$$= s(s-a)(s-b)(s-c)$$

If
$$\begin{aligned}2m &= m_1+m_2+m_3\\ 2n_1 &= m_2+m_3+\tfrac{3}{2}a\\ 2n_2 &= m_3+m_1+\tfrac{3}{2}b\\ 2n_3 &= m_1+m_2+\ \ c\end{aligned}$$

then (14) and (15) become

$$\left.\begin{aligned}(n_1-m_2)(n_1-m_3) &= m(m-m_1) & n_1(n_1-\tfrac{3}{2}a) &= (m-m_2)(m-m_3)\\ (n_2-m_3)(n_2-m_1) &= m(m-m_3) & n_2(n_2-\tfrac{3}{2}b) &= (m-m_3)(m-m_1)\\ (n_3-m_1)(n_3-m_2) &= m(m-m_3) & n_3(n_3-\tfrac{3}{2}c) &= (m-m_1)(m-m_2)\end{aligned}\right\} \qquad (19)$$

Area of triangle in terms of its medians.

FIGURE 1.

Because $\quad ABC = 2ABA' = 6GBA' = 3GBL;$

and $\quad GL = \tfrac{2}{3}m_1,\ GB = \tfrac{2}{3}m_2,\ BL = \tfrac{2}{3}m_3;$

therefore $\quad \triangle^2 = 9(GBL)^2$

$$= 9\left\{\frac{m_1+m_2+m_3}{3}\cdot\frac{-m_1+m_2+m_3}{3}\cdot\frac{m_1-m_2+m_3}{3}\cdot\frac{m_1+m_2-m_3}{3}\right\}$$

$$= \tfrac{1}{9}\{(m_1+m_2+m_3)(-m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)\}.$$

Let $\quad 2m = m_1+m_2+m_3$

then
$$\triangle^2 = \tfrac{16}{9}\{m(m-m_1)(m-m_2)(m-m_3)\} \qquad (20)$$

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If

$$2m' = -m_1 + m_2 + m_3$$
$$2m'' = m_1 - m_2 + m_3$$
$$2m''' = m_1 + m_2 - m_3$$

then

$$\Delta = \tfrac{4}{3}\sqrt{mm'm''m'''} \tag{21}$$

$$\left.\begin{aligned}\Delta &= \tfrac{4}{3}\sqrt{n_1(n_1 - m_2)(n_1 - m_3)(n_1 - \tfrac{3}{2}a)}\\ &= \tfrac{4}{3}\sqrt{n_2(n_2 - m_3)(n_2 - m_1)(n_2 - \tfrac{3}{2}b)}\\ &= \tfrac{4}{3}\sqrt{n_3(n_3 - m_1)(n_3 - m_2)(n_3 - \tfrac{3}{2}c)}\end{aligned}\right\} \tag{22}$$

This is deduced from (20) by means of (19).

If R, S, T be the projections of G on the sides

$$\left.\begin{aligned}\mathrm{BR} &= \frac{3a^2 - b^2 + c^2}{6a} & \mathrm{CR} &= \frac{3a^2 + b^2 - c^2}{6a}\\ \mathrm{CS} &= \frac{a^2 + 3b^2 - c^2}{6b} & \mathrm{AS} &= \frac{-a^2 + 3b^2 + c^2}{6b}\\ \mathrm{AT} &= \frac{-a^2 + b^2 + 3c^2}{6c} & \mathrm{BT} &= \frac{a^2 - b^2 + 3c^2}{6c}\end{aligned}\right\} \tag{23}$$

$$\mathrm{ST} = \frac{4m_1\Delta}{3bc} \qquad \mathrm{TR} = \frac{4m_2\Delta}{3ca} \qquad \mathrm{RS} = \frac{4m_3\Delta}{3ab} \tag{24}$$

$$\mathrm{ST} : \mathrm{TR} : \mathrm{RS} = am_1 : bm_2 : cm_3 \tag{25}$$

Of the preceding formulæ, (8) and (9) are given by C. F. A. Jacobi, *De Triangulorum Rectilineorum Proprietatibus*, p. 7 (1825); (10) and (11) occur in Hind's *Trigonometry*, 4th ed., p. 244 (1841); (12) in Thomas Simpson's *Select Exercises*, Part II., Problem xxii. (1752); (2)-(4), (13)-(19), (21), (22) are due to Thomas Weddle. See *Lady's and Gentleman's Diary* for 1848, pp. 74-75. I have changed the notation adopted by Weddle.

On the authority of Férussac's *Bulletin des Sciences Mathématiques*, xii. 297 (1829), formula (20) should be assigned to Professor Desgranges.