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# ON THE LOWER CENTRAL FACTORS OF FREE CENTRE-BY-METABELIAN GROUPS

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**Abstract**

We describe the structure of the lower central factors of free centre-by-metabelian groups.

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## 1. Introduction

The lower central factors of free polynilpotent groups are torsion-free (Šmelkin [7], Ward [8]), but the $c$th lower central factor $\Phi_c(G_n)$ of a free centre-by-metabelian group $G_n$ of rank $n \geqslant 2$ may have 2-torsion for $c \geqslant 5$. This was first shown by Ridley [6] for $n = 2$, $c$ even, and later for $n \geqslant 5$, $c$ odd, by Hurley (unpublished). For $c \leqslant 4$, $\Phi_c(G_n)$ is the same as the corresponding factor of a free group of rank $n$.

In this paper, we give an explicit basis for $\Phi_c(G_n)$. Gupta and Levin [2] observed that the structure of the torsion group $T_{n,c}$ of $\Phi_c(G_n)$ varies drastically according to whether $c$ is even or odd, but in either case $T_{n,c}$ is an elementary abelian 2-group.

## 2. Notation

Our commutator notation is $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$, $[g_1,\ldots,g_{n+1}] = [[g_1,\ldots,g_n], g_{n+1}]$ and $[g_1, g_2; g_3, g_4] = [[g_1, g_2], [g_3, g_4]]$ for group elements, with analogous





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notation for the ring commutator $(r_1, r_2) = r_1r_2 - r_2r_1$. further, $[g_1, kg_2] = [g_1, g_2, \ldots, g_2]$, $k$ repeats of $g_2$, ($[g_1, 0g_2] = g_1$). For $n \geqslant 1$, $\gamma_n(G)$ is the $n$th term of the lower central series of $G$, $G' = \gamma_2(G)$, $G'' = \gamma_2(G')$. All other notation follows Hanna Neumann [5].

## 3. Preliminaries

Much of the technical work required in this paper has been carried out in [2]. As in [2], the main tool in our investigation is the following power series representation (cf. [3]). Let $\mathbf{P}_n = \mathbf{Z}[[y_1, \ldots, y_n]]$, $n \geqslant 2$, be the free associative power series ring over the integers and let $C$ be the ideal of $\mathbf{P}_n$ generated by all $y_i(y_j, y_k)y_m$ for all $i, j, k, m$. Set $R_n = \mathbf{P}_n/C$ and denote the generators of $R_n$ by $x_1, \ldots, x_n$, where $x_i = y_i + C$. As shown in [3], the group of units of $R_n$ is a centre-by-metabelian group, that is, satisfies the law $[g_1, g_2; g_3, g_4; g_5] = 1$. Thus if $G_n$ is the free centre-by-metabelian group of rank $n$ with generators $f_1, \ldots, f_n$, then the map $\Theta: f_i \to 1 + x_i$, $1 \leqslant i \leqslant n$, can be extended to a representation of $G_n$ in $R_n$. C. K. Gupta [1] has shown that the kernel of $\Theta$ is an elementary abelian 2-group of rank $\binom{n}{4}$ contained in the centre of $G_n$ for $n \geqslant 4$, but for $n = 2, 3$ the representation is faithful.

For any $w \in G_n$, $w\Theta$ is a power series of the form $1 + \Sigma_{i=1}^{\infty}\langle w\Theta\rangle_i$, where $\langle w\Theta\rangle_i$ denotes the component of terms of total degree $i$. As in the corresponding representation of the free group of rank $n$ (see [4], chapter 5) the elements $w \in \gamma_c(G_n)$ are characterized by $\langle w\Theta\rangle_i = 0$, $i \leqslant c - 1$. Thus, an element $w \in \gamma_c(G_n)$ will be a relator for $\Phi_c(G_n)$, the $c$th lower central factor of $G_n$, only if $\langle w\Theta\rangle_c = 0$. In particular, if $w$ is a relator for $\Phi_c(G_n)$ then $w \in \gamma_c(G_n) \cap G_n''$ and $\langle w\Theta\rangle_c = 0$.

The simple basic commutators of the form $[f_i, f_j, f_k, \ldots, f_m]$ of weight $c$ with $i > j \leqslant k \leqslant \cdots \leqslant m$ form a basis for $\Phi_c(G_n/G_n'')$ (see [5], page 106) and is a part of the basis for $\Phi_c(G_n)$. Thus if we write $\Phi_c(G_n) = F_{n,c} \times T_{n,c}$, where $F_{n,c}$ is free abelian and $T_{n,c}$ is the torsion group of $\Phi_c(G_n)$, our problem reduces to extending the simple basic commutators to a basis for $F_{n,c}$ and to finding a basis for $T_{n,c}$.

## 4. The structure of $\Phi_c(G_n)$, $c$ odd, $c \geqslant 5$

For $c \geqslant 5$, let $\omega_c^*$ and $\omega_c^{**}$ be defined as

$$\begin{aligned}\omega_c^* = {} & [f_1, f_2; f_3, f_4, f_5, \ldots, f_c] \\ & [f_2, f_3; f_1, f_4, f_5, \ldots, f_c] \\ & [f_3, f_1; f_2, f_4, f_5, \ldots, f_c],\end{aligned}$$

and

$$\omega_c^{**} = [f_1, f_2; f_4, f_5, f_3, f_6, \ldots, f_c] \\ [f_2, f_3; f_4, f_5, f_1, f_6, \ldots, f_c] \\ [f_3, f_1; f_4, f_5, f_2, f_6, \ldots, f_c].$$

Then we have the following lemma.

LEMMA 1. *For c odd,* $c \geqslant 5$,
(i) $\omega_c^*(G_n) \nleqslant \gamma_{c+1}(G_n)$ *for* $n \geqslant c$;
(ii) $\omega_c^{*2}(G_n) \leqslant \gamma_{c+1}(G_n)$ *for all n*;
(iii) $\omega_c^*(G_n) \leqslant \gamma_{c+1}(G_n)$ *for* $n < c$;
(iv) $\omega_c^*$ *is unaltered, modulo* $\gamma_{c+1}(G_n)$, *by an arbitrary permutation of* $\{f_1, \ldots, f_c\}$;
(v) $\omega_c^{**}(G_n) \leqslant \gamma_{c+1}(G_n)$ *for all n and all c even or odd.*

PROOF. The proofs of (i) to (iv) follow from Lemma 3.8 of [2]. For the proof of (v) we simply observe that, modulo $\gamma_{c+1}(G_n)$,

$$\omega_c^{**} \equiv [f_1, f_2, f_3; f_4, f_5, \ldots, f_c]^{-1} \\ [f_2, f_3, f_1; f_4, f_5, \ldots, f_c]^{-1} \\ [f_3, f_1, f_2; f_4, f_5, \ldots, f_c]^{-1},$$

which is trivial by the Jacobi congruence.

We can now give the structure of $\Phi_c(G_n)$ for $c$ odd, $c \geqslant 5$.

THEOREM 1. *For c odd,* $c \geqslant 5$, *let* $\Phi_c(G_n) = F_{n,c} \times T_{n,c}$ *where* $F_{n,c}$ *is free abelian and* $T_{n,c}$ *is the torsion subgroup. Then*
(a) *a basis for* $T_{n,c}$ *consists of the* $\binom{n}{c}$ *elements* $\omega_c^*(k(1), \ldots, k(c))$ *given by*

$$\omega_c^*(k(1), \ldots, k(c)) = [f_{k(1)}, f_{k(2)}; f_{k(3)}, f_{k(4)}, f_{k(5)}, \ldots, f_{k(c)}] \\ [f_{k(2)}, f_{k(3)}; f_{k(1)}, f_{k(4)}, f_{k(5)}, \ldots, f_{k(c)}] \\ [f_{k(3)}, f_{k(1)}; f_{k(2)}, f_{k(4)}, f_{k(5)}, \ldots, f_{k(c)}],$$

*with* $1 \leqslant k(1) \leqslant \cdots \leqslant k(c) \leqslant n$;
(b) *a basis for* $F_{n,c}$ *consists of all simple basic commutators of weight c with entries from the set* $\{f_1, \ldots, f_n\}$ *together with all commutators*

$$[f_i, f_j; f_{k(1)}, f_{k(2)}, f_{k(3)}, \ldots, f_{k(c-2)}]$$

*with* $i > j$; $i \geqslant k(1)$; $k(1) > k(2) \leqslant j \leqslant k(3) \leqslant \cdots \leqslant k(c-2)$.

PROOF. The simple basic commutators of weight $c$ with entries from the set $\{f_1,\ldots,f_n\}$ form a basic of $\Phi_c(G_n/G_n'')$ and hence constitute a part of the basis for $F_{n,c}$. Let $\omega \in \gamma_c(G_n) \cap G_n''$ be a relator for $\Phi_c(G_n)$. Then $\langle \omega\Theta \rangle_c = 0$ and it follows by Lemma 4.1(iv) of [2] that if $\omega \notin \gamma_{c+1}(G_n)$ then $\omega$ lies in the fully invariant closure of $\omega_c^*$. The proof of (a) now follows immediately by Lemma 1((i)–(iv)). For the remainder of the proof of (b) we first observe, using Lemma 3.1((i) and (iii)) of [2], that $\omega$ can be written, modulo $\gamma_{c+1}(G_n)$, as a product of commutators of weight $c$ of the form

$$z = [f_i, f_j; f_{k(1)}, f_{(2)}, f_{k(3)}, \ldots, f_{k(c-2)}]$$

with $i > j \geqslant k(2)$; $k(1) > k(2) \leqslant k(3) \leqslant \cdots \leqslant k(c-2)$. It remains to show that modulo $\omega_c^*(G_n)$ we can further assume that $i \geqslant i(1)$ and $j \leqslant k(3)$ in $z$. Let $m = \max\{i, j, k(1), k(3)\}$.

*Type* 1. ($z$ with $k(3) = m$). Let $\{i, j, k(1)\} = \{a, b, c\}$ with $a \leqslant b \leqslant c$. The modulo $\omega_c^*(G_n)$, $z$ can be expressed in terms of commutators $[f_c, f_a; f_b, f_{k(1)}, f_m, \ldots]$ and $[f_c, f_b; f_a, f_{k(1)}, f_m, \ldots]$ each of which is of the required form.

*Type* 2. ($z$ with $i = m$). Let $\{j, k(1), k(3)\} = \{a, b, c\}$. Then $z$ is of the form $[f_m, f_a; f_b, f_{k(1)}, f_c, \ldots]$. If $a > c$ then modulo $\omega_c^{**}(G_n)$, $z$ can be expressed as a product of Type 1 and the required commutator $[f_m, f_c; f_b, f_{k(1)}, f_a, \ldots]$.

*Type* 3. ($z$ with $k(1) = m$). Here modulo $\omega_c^*(G_n)$, $z$ can be expressed as a product of Type 1 and Type 2 commutators. This completes the proof of Theorem 1.

## 5. The structure of $\Phi_c(G_n)$, $c$ even, $c \geqslant 6$

We begin by recalling some results from [2]. For the remainder of the paper, we assume $c$ even, $c \geqslant 6$.

LEMMA 2. *Let* $z = [f_i, f_j; f_i, f_j, p_1 f_1, \ldots, p_n f_n]$ *be an element of* $\gamma_c(G_n)$, *where* $c = 4 + p_1 + \cdots + p_n$, $p_i \geqslant 0$.

(i) *In* $R_n$, $z\Theta$ *is obtained by expanding*

$$z\Theta = 1 + (x_i, x_j)x_1^{p_1} \cdots x_n^{p_n}(1 + x_i)^{-1}(1 + x_j)^{-1} \cdot \{(1 + x_1)^{-p_1} \cdots (1 + x_n)^{-p_n} - 1\}(x_i, x_j).$$

(ii) $$\langle z\Theta \rangle_{c+1} = -\sum_{m=1}^{n} p_m(x_i, x_j)x_1^{p_1} \cdots x_m^{p_{m+1}} \cdots x_n^{p_n}(x_i, x_j),$$

(iii) $\langle z^2\Theta \rangle_{c+1} = 2\langle z\Theta \rangle_{c+1}$.

(iv) *If* $w \in \gamma_c(G_n) \cap G_n''$ *with* $\langle w\theta \rangle_c = 0$, *then* $w$ *is a product of commutators of the form* $z$.

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(v) *Let $w$ be a product of commutators of the form $z$ with weight at least* 1 *in each* $f_i$. *If* $\langle w\Theta\rangle_c = 0$ *and* $\langle w\Theta\rangle_{c+1} \equiv 0 \pmod 2$, *then* $w \in \gamma_{c+1}(G_n)$.

(vi) *Let* $\alpha_{ii}\langle w\Theta\rangle_c$ *denote the component of* $\langle w\Theta\rangle_c$ *of those terms which begin and end with* $x_i$. *If* $w \in \gamma_{c+1}(G_n) \cap G_n''$ *then* $\alpha_{ii}\langle w\Theta\rangle_{c+1} \equiv 0(2)$.

The proof of (i) follows from Lemma 3.4 of [2]; (ii), (iii) are easy consequence of (i); (iv) follows from Lemma 4.1(i) of [2]; (v) follows from Lemma 4.4 of [2] and (vi) follows from Corollary 3.5 of [3].

LEMMA 3. *If $G$ is a centre-by-metabelian group, then for all* $d \in \gamma_m(G)$, $g_i \in G$, $k \geqslant 1$, $m \geqslant 2$,

$$[d; d, g_1, \ldots, g_{2k}]^2 \equiv \prod_{i=1}^{2k} [d; d, g_1, \ldots, g_i, g_i, \ldots, g_{2k}]$$

*modulo* $\gamma_{2m+2k+2}(G)$.

PROOF. By Lemma 3.1 of [2], for any $d_1$, $d_2 \in G'$, $g \in G$, $[d_1; d_2, g][d_1, g; d_2] = [d_1, g; d_2, g]^{-1}$. Further, for any $g_i \in G$, $[d_1; d_2, g_1, g_2] = [d_1; d_2, g_2, g_1]$. Thus,

$$\begin{aligned}[d; d, g_1, \ldots, g_{2k}]&\\ &= [d, g_1; d, g_2, \ldots, g_{2k}]^{-1}[d, g_1; d, g_1, g_2, \ldots, g_{2k}]^{-1}\\ &\equiv [d, g_1; d, g_2, \ldots, g_{2k}]^{-1}[d; d, g_1, g_1, g_2, \ldots, g_{2k}]\end{aligned}$$

$$(\text{modulo } \gamma_{2m+2k+2}(G))$$

Similarly,

$$\begin{aligned}[d, g_1; d, g_2, \ldots, g_{2k}]^{-1}&\\ &\equiv [d, g_1, g_2; d, g_3, \ldots, g_{2k}][d, g_1, g_2; d, g_2, g_3, \ldots, g_{2k}]\\ &\equiv [d, g_1, g_2; d, g_3, \ldots, g_{2k}][d; d, g_1, g_2, g_2, g_3, \ldots, g_{2k}].\end{aligned}$$

Repeated applications of this step yield

$$\begin{aligned}[d; d, g_1, \ldots, g_{2k}]&\\ &\equiv [d, g_1, \ldots, g_{2k}; d] \prod_{i=1}^{2k} [d; d, g_1, \ldots, g_i, g_i, \ldots, g_{2k}];\end{aligned}$$

or equivalently,

$$[d; d, g_1, \ldots, g_{2k}]^2 \equiv \prod_{i=1}^{2k} [d; d, g_1, \ldots, g_i, g_i, \ldots, g_{2k}].$$

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Lemmas 3 and 2(ii) yield the following as a corollary.

LEMMA 4. *Let $z$ be as in Lemma 2. Then $\langle z^2\Theta\rangle_c = 0$ and*

$$\langle z^2\Theta\rangle_{c+1} = \sum_{m=1}^{n} p_m\big(x_i, x_j; x_i, x_j, p_1x_1, \ldots, (p_m+1)x_m, \ldots, p_nx_n\big).$$

*More generally, if $w = z_1\cdots z_t$ with each $z_i$ as above, then*

$$\langle w^2\Theta\rangle_{c+1} = \big\langle z_1^2\Theta\big\rangle_{c+1} + \cdots + \big\langle z_t^2\Theta\big\rangle_{c+1}.$$

We now establish a useful criterion for identifying relations in $\Phi_c(G_n)$, $c$ even;

LEMMA 5. *Let $s(w) = \langle w^2\Theta\rangle_{c+1}$. An element $w \in \gamma_c(G_n)$ is a relator of $\Phi_c(G_n)$ if and only if $s(w) = \langle v^2\Theta\rangle_{c+1}$ for some $v \in \gamma_{c+1}(G_n)$.*

PROOF. If $w$ is a relator of $\Phi_c(G_n)$ then $w \in \gamma_{c+1}(G_n)$ and we may choose $v = w$. Conversely, let $s(w) = \langle v^2\Theta\rangle_{c+1}$ for some $v \in \gamma_{c+1}(G_n)$. By Lemma 4, $\langle w^2\Theta\rangle_{c+1} = \langle v^2\Theta\rangle_{c+1}$ implies $2\langle \omega\Theta\rangle_{c+1} = 2\langle v\Theta\rangle_{c+1}$ and, in turn, $\langle w\Theta\rangle_{c+1} = \langle v\Theta\rangle_{c+1}$, $\langle wv^{-1}\Theta\rangle_{c+1} = 0$. By Lemma 2(i), $\langle w\Theta\rangle_c = 0$ and, by hypothesis, $\langle v\Theta\rangle_c = 0$. Thus $\langle wv^{-1}\Theta\rangle_c = 0$, and it follows by Lemma 2(v) that $wv^{-1} \in \gamma_{c+1}(G_n)$. Thus $w \in \gamma_{c+1}(G_n)$ as required.

As a corollary to Lemma 5 we obtain the following result:

LEMMA 6. *Let $w \in T_{n,c}$ be a relator for $\Phi_c(G_n)$. Then $\alpha_{ii}(s(w)) \equiv 0 \pmod 4$ for all $i = 1,\ldots,n$.*

PROOF. By Lemma 5, $s(w) = \langle v^2\Theta\rangle_{c+1}$ for some $v \in \gamma_{c+1}(G_n)$. By Lemma 2(vi), $\alpha_{ii}\langle v\Theta\rangle_{c+1} \equiv 0 \pmod 2$. Thus $\alpha_{ii}(s(w)) \equiv 0 \pmod 4$ as required.

For convenience, we will abbreviate $z = [f_i, f_j; f_i, f_j, p_1f_1, \ldots, p_nf_n]$ by $z = [i, j; p_1, \ldots, p_n]$. Further, for any $f_k$ we define

$$[z : q_kf_k] = \big[f_i, f_j; f_i, f_j, p_1f_1, \ldots, p_nf_n, q_kf_n\big].$$

Using this notation we obtain the following consequence of Lemma 3.

LEMMA 7. *Let $z = [i, j; p_1, \ldots, p_n]$ with $c = 4 + p_1 + \cdots + p_n$. Then for arbitrary $k$,*

(i) $$[z : 2f_k]^2 \equiv \prod_{i=1}^{n} [z : 2f_k : f_i]^{p_i}[z : 3f_k]^2 \mod \gamma_{c+4}(G_{n+1});$$

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(ii) $$\left\langle [z:2f_k]^2\Theta \right\rangle_{c+3} = \left(\langle z^2\Theta\rangle_{c+1} : 2x_k\right) + 2\left\langle [z:3f_k]\Theta \right\rangle_{c+3},$$

*where $\langle z^2\Theta\rangle_{c+1}$ is as determined in Lemma 4.*

[7(ii) is a direct consequence of 7(i).]

The next lemma gives a method for generating relations.

LEMMA 8. *If $w$ is a relator for $T_{n,c}$, then $[w:2f_k]$ is a relator for $T_{n+1,c+2}$ for any $k \leqslant n+1$.*

PROOF. By Lemma 5 it suffices to show that $s([w:2f_k]) = \langle u^2\Theta\rangle_{c+3}$ for some $u \in \gamma_{c+3}(G_{n+1})$. Since $w$ is a relator for $T_{n,c}$, by Lemma 5 $s(w) = \langle v^2\Theta\rangle_{c+1}$ for some $v \in \gamma_{c+1}(G_n)$. Thus

$$\begin{aligned} s([w:2f_k]) &= \left\langle [w:2f_k]^2\Theta \right\rangle_{c+3} \\ &= \left(\langle w^2\Theta\rangle_{c+1} : 2x_k\right) + 2\left\langle [w:3f_k]\Theta \right\rangle_{c+3} \quad \text{(by Lemma 7)} \\ &= \left(\langle v^2\Theta\rangle_{c+1} : 2x_k\right) + \left\langle [w:3f_k]^2\Theta \right\rangle_{c+3} \\ &= \left\langle [v:2f_k]^2\Theta \right\rangle_{c+3} + \left\langle [w:3f_k]^2\Theta \right\rangle_{c+3} \\ &= \left\langle u^2\Theta \right\rangle_{c+3}, \end{aligned}$$

with $u = [v:2f_k][w:3f_k] \in \gamma_{c+3}(G_{n+1})$.

It follows from Lemma 2(iv) that for $c$ even, $c \geqslant 6$, any relator $w$ of $T_{n,c}$ is a product of commutators of the form $z = [i, j: p_1, \ldots, p_n]$, $4 + p_1 + \cdots + p_n = c$. Since $z^2 \equiv 1 \bmod \gamma_{c+1}(G_n)$, $T_{n,c}$ is, in fact, generated by all such commutators. Thus to determine the relators of $T_{n,c}$ it suffices to assume that $n \leqslant c-2$ and that each $[i, j; p_1, \ldots, p_n]$ has weight at least one in each generator $f_1, \ldots, f_n$.

If $c \geqslant 8$ and $n = c-2$, it follows from the proof of Theorem B(i) of [2] that there is just one relator which involves all the $c-2$ variables, namely,

(1) $$u(c-2) = \prod_{\sigma} a(1\sigma, 2\sigma)$$

where $a(i, j) = [i, j; 1, 1, \ldots, \hat{i}, \hat{j}, \ldots, 1]$ ($\hat{k}$ indicates $k$ missing), and $\sigma$ runs over the permutations of $\{1, \ldots, c-2\}$ with $1\sigma < 2\sigma < \cdots < (c-2)\sigma$. Hence, we may assume that $n \leqslant c-3$. For each $i, j \in \{1, \ldots, c-3\}$, $i < j$, we define

(2) $$v(c, i, j) = \prod_{m=1}^{c-5} \left[f_i, f_j; f_i, f_j, g_1, \ldots, g_m, g_m, \ldots, g_{c-5}\right],$$

where $\{g_1, \ldots, g_{c-5}\} = \{f_1, \ldots, \hat{f_i}, \ldots, \hat{f_j}, \ldots, f_{c-3}\}$. By Lemma 3.2(i) of [2], $G_n$ satisfies the congruence $v(c, i, j) \equiv 1$. Using this fact and Lemma 8 we can reduce the generating set for $T_{n,c}$ as follows.

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LEMMA 9. *Let* $W = \{[i, j; p_1, \ldots, p_n], i < j\}$ *be the set of all commutators of weight* $c = 4 + p_1 + \cdots + p_k$ *such that* $p_k \leqslant 1$ *for all* $k < i$ *and the first nonzero integer reading left to right in the sequence* $p_n, p_{n-1}, \ldots, p_1$ *is odd. Then* $W$ *is a generating set for* $T_{n,c}$, $n \leqslant c - 3$, $c \geqslant 8$.

PROOF. The condition $i < j$ is clearly no restriction. The next condition, $p_k \leqslant 1$ for $k < i$ follows directly from Lemma 3.1(vii) of [2], by which for any $g$, $g_i$ in a centre-by-metabelian group $G$,

$$\begin{aligned}(3)\quad &[g, g_1; g, g_1, g_2, g_3, \ldots, g_m, g, g] \\ &\equiv [g, g_1; g, g_1, g_2, g_2, g_3, \ldots, g_m][g, g_2; g, g_2, g_1, g_1, g_3, \ldots, g_m]\end{aligned}$$

modulo $\gamma_{m+5}(G)$. The third condition, namely, the first non-zero integer in $p_n, \ldots, p_1$ is odd, may be seen as follows. As observed above $T_{n,c}$ is spanned by all commutators $z = [i, j; p_1, \ldots, p_n]$ with $p_k \leqslant 1$, $k < i$, $i < j$. Without loss of generality we may further assume that $z$ involves $f_{n-1}$ and $f_n$. Hence if both $p_{n-1}$ and $p_n$ are zero, this means that $i = n - 1$ and $j = n$ so $c - 2 = n$. Thus we may assume that at least one of $p_{n-1}, p_n$ is non-zero.

*Case* I. $p_n \neq 0$, $p_n$ even. Since $G_n$ satisfies the congruence $v(c, i, j) \equiv 1$ defined by (2), it follows that for $p_n \geqslant 2$,

$$[i, j; p_1, \ldots, p_n]^{p_n - 1} \prod_{m=1}^{n-1} [i, j; p_1, \ldots, p_m + 1, \ldots, p_n - 1]^{p_m} \equiv 1.$$

However, for $p_n$ even $[i, j; p_1, \ldots, p_n]^{p_n - 1} \equiv [i, j; p_1, \ldots, p_n]$, and the proof follows since each factor $[i, j; p_1, \ldots, p_m + 1, \ldots, p_n - 1]$ can be expressed as a product of commutators of the required form without affecting the oddness of the occurences of $f_n$.

*Case* II. $p_n = 0$, $p_{n-1}$ even. In this case $z = [i, n; p_1, \ldots, p_{n-1}, 0]$ and as before the law $v(c, i, n)$ gives

$$[i, n; p_1, \ldots, p_{n-1}, 0] \prod_{m=1}^{n-2} [i, n; p_1, \ldots, p_{m+1}, \ldots, p_{n-2}, p_{n-1} - 1, 0]^{p_m} \equiv 1.$$

Since each factor $[i, n; p_1, \ldots, p_m + 1, \ldots, p_{n-2}, p_{n-1} - 1, 0]$ reduces, by (3), to a product of commutators of the required form and of commutators covered by the Case I, the proof follows.

LEMMA 10. *For* $c \geqslant 8$, $n \leqslant c - 3$ *the set* $W$ *defined in Lemma* 9 *is a basis for* $T_{n,c}$.

PROOF. For $z = [i, j; p_1, \ldots, p_n]$, $z' = [i', j'; p'_1, \ldots, p'_1]$ we define $z < z'$ if $(i, j, p_1, \ldots, p_n) < (i', j', p'_1, \ldots, p'_n)$ in the lexicographic ordering of the $(n + 2)$-tuples. The proof of Lemma 9 shows that if $z \notin W$, then $z$ can be written as a product of elements of $W$ which are less than $z$ in the above ordering.

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By Lemma 9, $W$ generates $T_{n,c}$. Let $w = z_1 \cdots z_r$, $z_i \in W$, be a relator of $T_{n,c}$ with $z_1 < \cdots < z_r$. By Lemma 6, $\alpha_{kk}(s(w)) \equiv 0 \pmod 4$ for all $k$. Suppose $z_r = [i, j; p_1, \dots, p_n]$, $p_n$ odd, and let $\alpha_{jj}(s(z_r)) = x_j P_r x_j$. By Lemma 2(i), $P_r$ has a term

$$(4) \qquad x_1 \cdots x_{i-1} x_i^{p_i+2} x_{i+1}^{p_{i+1}} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n+1}$$

with coefficient $2p_n \not\equiv 0 \pmod 4$. Since $\alpha_{jj}(s(w)) = \Sigma_{k=1}^{r} \alpha_{jj}(s(z_k)) \equiv 0 \pmod 4$, it follows that for some $q < r$, $\alpha_{jj}(s(z_q)) = x_j P_q x_j$, where $P_q$ has a term (4). However, $z_q < z_r$ implies $z_q = [i', j'; p'_1, \dots, p'_n]$ with $(i', j', p'_1, \dots, p'_n) < (i, j, p_1, \dots, p_n)$. If $i' < i$ then each term in $P_q$ has degree at least 2 in $x'_i$, so none is of the form (4). If $p'_n < p_n$ then $p'_n \leqslant p_n - 2$ and ecah term of $P_q$ has degree at most $p'_n + 1 \leqslant p_n - 1$ in $x_n$, so $P_q$ has no term (4). Finally, if $p'_n = p_n$ and $p'_k < p_k$ for some $k < n$, then every term of $P_q$ with degree $p_n + 1$ in $x_n$ has degree $p'_i < p_k$ in $x_k$, and, again $P_q$ does not have a term (4). Thus, if $p_n \neq 0$, then $\alpha_{jj}(s(w)) \not\equiv 0 \pmod 4$, and $w$ is not a realtor of $T_{n,c}$.

Thus, we may assume that $w = z_1 \cdots z_r$ is a relator of $T_{n,c}$ with $z_r = [i, n; p_1, \dots, p_{n-1}, 0]$, $p_{n-1}$ odd. As above, if $\alpha_{nn}(s(z_r)) = x_n P_r x_n$, $P_r$ will have a term

$$(5) \qquad 2p_{n-1} x_1 \cdots x_{i-1} x_i^{p_i+2} x_{i+1}^{p_{i+1}} \cdots x_{n-1}^{p_{n-1}+1},$$

with $2p_{n-1} \not\equiv 0 \pmod 4$, and it follows, as above, that this term must occur in the expansion of some $P_q$, $q < r$. The same argument shows that this, too, is not possible. This completes the proof of the lemma.

We may summarise the above results as follows.

THEOREM 2. *For $c$ even, $c \geqslant 8$, the torsion subgroup $T_{n,c}$ of $\Phi_c(G_n)$ is an elementary abelian 2-group with a basis consisting of the set $W$ of all commutators $z = [i, j; p_1, \dots, p_n]$, $i < j$, with the following properties:*

(a) $c = 4 + p_1 + \cdots + p_n$, $p_k \geqslant 0$ *for all* $k \leqslant n$;

(b) $p_k \leqslant 1$ *for all* $k < i$;

(c) *the first nonzero integer reading from left to right in* $p_n, \dots, p_1$ *is odd*;

(d) *for any* $i(1) < \cdots < i(m)$, $m \leqslant n$, *$W$ does not contain* $[i(m-1), i(m); i(1), \dots, i(m-2)]$.

PROOF. The conditions (a), (b), (c) follow directly from Lemma 10, and (d) follows from the fact that if a relator contains any part of $u(c-2)$, as defined in (1), then it must contain all of $u(c-2)$.

For $c = 6$, essentially the same argument applies. However, $[f_i, f_j; f_i, f_j, f_k, f_k] \in \gamma_7(G_4)$ and as C. K. Gupta [1] has shown, $u(4) \notin \gamma_7(G_4)$. The basis for $T_{n,6}$ is given by the following theorem.

THEOREM 3. *For $c = 6$, $T_{n,6}$ has a basis consisting of all commutators* $[f_{i(1)}, f_{i(2)}; f_{i(1)}, f_{i(2)}, f_{i(3)}, f_{i(4)}]$, *with* $i(1) < i(2)$, $i(3) < i(4)$.

As with the case for odd $c$, a basis for $F_{n,c}$ must be chosen to account for the Jacobi congruence and the above structure of $T_{n,c}$. The argument is analogous to that for odd $c$ and we omit the details. The complete description of $\Phi_c(G_n)$, $c$ even, is given by the following theorem.

THEOREM 4. *For $c$ even, $c > 6$, let $\Phi_c(G_n) = F_{n,c} \times T_{n,c}$, where $T_{n,c}$ is the torsion subgroup of $\Phi_c(G_n)$ and $F_{n,c}$ is free abelian.*

(i) *A basis for $T_{n,c}$ is given by Theorems 2 and 3.*

(ii) *$F_{n,c}$ is generated by the simple basic commutators of weight $c$ and by the commutators of the form $[f_i, f_j; f_{k(1)}, f_{k(2)}, \ldots, f_{k(c-2)}]$ subject to $k(1) > k(2) \leqslant k(3) \leqslant \cdots \leqslant k(c-2)$; $i > j \leqslant k(3)$; $k(2) \leqslant j$ and such that $k(1) < i$ or $k(1) > i$ and $k(2) < j < i < k(1) < k(3) < \cdots < k(c-2)$ (no repeated entry) or $k(1) = i$, $k(2) < j$ and if $k(l)$ is another repeated entry $(3 \leqslant l \leqslant c-2)$, then $k(1) \leqslant k(l)$.*

CONCLUDING REMARKS. With the aid of Lemma 8 we can give a different description of $T_{n,c}$, $c$ even, as follows.

THEOREM 5. *For $c$ even, $c \geqslant 6$, the torsion subgroup $T_{n,c}$ of $\Phi_c(G_n)$ is generated by all commutators $z = [i, j; p_1, \ldots, p_n]$ with $4 + p_1 + \cdots + p_n = c$, $p_i \geqslant 0$, subject only to the relations*

$$[u(c'-2) : 2k_1 f_1 : \cdots : 2k_n f_n] = 1 \quad \text{and} \quad [v(c', i, j) : 2k_1 f_1 : \cdots : 2k_n f_n] = 1,$$

*where $c = c' + 2k_1 + \cdots + 2k_n$, $k_l \geqslant 0$ and $u(c'-2)$, $v(c', i, j)$ are as defined by* (1) *and* (2).

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