MODULAR REPRESENTATIONS OF THE SYMMETRIC GROUP

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Introduction. The theory of modular representations of the symmetric group was studied first by Nakayama (**5**, **6**), and later by Thrall and Nesbitt (**11**) and Robinson (**7**, **8**, **9**). Nakayama built up his elaborate theory of hooks for the express purpose of studying this problem, while Robinson's extensive work on the various phases of the relationship between Young diagrams, skew diagrams and star diagrams on the one hand, and representations of the symmetric group on the other, culminating in a set of relations among the degrees of the representations, serves as a starting point for this paper.

Brauer and Nesbitt (**2**) have shown in the general theory that, for a given prime p, the irreducible representations of a group may be separated into a number of p-blocks, each of which is characterized by the maximal power t of p which divides the degree of every representation of the block. If $g = p^a g'$, where g' is prime to p, then the equation $t + d = a$ relates t to the defect d of the block. If $t = a$, the block is of defect 0, while if $t = 0$, the block is of defect a. For the symmetric group Nakayama conjectured that the Young diagrams of all the representations of a single p-block had the same p-core after the removal of all their p-hooks. This conjecture was proved jointly in 1947 by Brauer and Robinson (**3**).

Brauer (**1**) also showed that the representations in a p-block of defect 1 can be arranged in a chain such that only adjacent members have a single modular component (with multiplicity 1) in common. For $n < 2p$, Nakayama succeeded in showing that in the case of the symmetric group S_n the ordering in the chain is precisely the natural order of the leg lengths r of the p-hooks, from $r = 0$ to $r = p - 1$, where each of the p distinct p-hooks is found in exactly one Young diagram.

The present paper extends Nakayama's result for blocks of defect 1 to values of $n \geqslant 2p$, and explains the derivation of a set of identities among the modular characters of the irreducible representations of a p-block of S_n. The nature of their linear dependence is studied in some detail. Notice is taken of the orthogonal relation between the coefficients in these identities and the columns of the matrix of decomposition numbers which gives the modular splitting of the irreducible representations of S_n, and this leads to an investigation of the nature of indecomposability in the regular representation of S_n. As a first step forward from Nakayama's one hook case, the indecomposables of the p-block of S_{2p} with zero p-core are obtained in a conclusive manner.

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1. Background. Let p be a rational prime and $\mathfrak{p}$ be a fixed prime ideal divisor of p in an algebraic number field K. Suppose a group G is represented by matrices whose coefficients are taken from the ring $\mathfrak{o}$ of $\mathfrak{p}$-integers of K (i.e., numbers of the form α/β, where α and β are integers of K and β is prime to $\mathfrak{p}$), and let $Z_1, Z_2, \ldots, Z_k$ be the distinct irreducible representations of G. If we let $\bar{K}$ be the residue class field of $\mathfrak{o}$ (mod $\mathfrak{p}$) and replace every coefficient of the Z_i by its residue class (mod $\mathfrak{p}$), then the resulting modular representations $\bar{Z}_i$ will, in general, be reducible and will split into irreducible modular representations F_κ with coefficients in $\bar{K}$. The splitting may be denoted by

$$\bar{Z}_i = \sum_{\kappa=1}^{k^*} d_{i\kappa} F_\kappa \qquad (i = 1, 2, \ldots, k),$$

where $d_{i\kappa}$ is the multiplicity with which F_κ appears in $\bar{Z}_i$. These rational integers $d_{i\kappa} \geqslant 0$ are called the *decomposition numbers* (mod p) of G.

If $U_1, U_2, \ldots, U_{k^*}$ are the distinct indecomposable components of $\bar{R}$, the regular representation of G with entries in $\bar{K}$, then

$$U_\kappa = \sum_{\lambda=1}^{k^*} c_{\kappa\lambda} F_\lambda \qquad (\kappa = 1, 2, \ldots, k^*),$$

where the $c_{\kappa\lambda}$, rational integers $\geqslant 0$, are the Cartan invariants of G (for p), and are related to the decomposition numbers via the equations

$$c_{\kappa\lambda} = \sum_{i=1}^{k} d_{i\kappa} d_{i\lambda}.$$

There exists a representation (U_κ) of G in K which, if taken (mod p), becomes similar to U_κ. We then have

$$(U_\kappa) = \sum_{j=1}^{k} d_{j\kappa} Z_j,$$

and it is to these representations (U_κ) that we shall be referring (without ambiguity) as the *indecomposable representations* of G. Such an indecomposable representation has the property that its character vanishes for all elements of G whose orders are divisible by p, i.e., for all p-singular elements. In the case of the symmetric group S_n a p-regular element is simply a permutation the lengths of whose cycles are all prime to p, while a p-singular element has at least one cycle of length p or a multiple of p.

Corresponding to the foregoing relations among the representations we have character relations which are valid for the p-regular elements of G. If we denote by $\eta^{(\kappa)}$ the character of U_κ, by $\Phi^{(\kappa)}$ that of F_κ, and by $\xi^{(i)}$ that of Z_i, these relations are

1.1 $$\xi^{(i)} = \sum_{\lambda=1}^{k^*} d_{i\lambda} \Phi^{(\lambda)}$$

1.2 $$\eta^{(\kappa)} = \sum_{i=1}^{k} d_{i\kappa} \xi^{(i)}.$$

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Regarding $(d_{i\kappa})$ as a matrix with i as row index and κ as column index, relations 1.1 and 1.2 indicate that the rows of this matrix give the splitting of the ordinary irreducible representations into their modular irreducible components, while the columns give the indecomposable representations of G in K as linear combinations of the ordinary irreducible representations with non-negative coefficients. It is this latter interpretation which will prove useful in the determination of the modular splitting of the irreducible representations of S_n.

A result in the modular theory which will also prove to be particularly useful is embodied in the following two formulae of Nakayama (**2**, p. 582):

1.3 $$\tilde{\eta}^{(\kappa)*} = \sum_{\lambda} a_{\kappa\lambda}\, \eta^{(\lambda)} \qquad \text{(for } p\text{-regular elements of } G),$$

1.4 $$\varphi^{(\lambda)} = \sum_{\kappa} a_{\kappa\lambda}\, \tilde{\varphi}^{(\kappa)} \qquad \text{(for } p\text{-regular elements of } H).$$

Observe that the same coefficients $a_{\kappa\lambda}$, which are positive integers or zero, appear in both formulae. The first states, in the notation of characters, that the representation of G induced by an indecomposable representation of a subgroup H of G, can be expressed as a linear combination of indecomposable representations of G, while the second states that if we restrict our attention to the element of a subgroup H of G, any modular irreducible representation of G becomes equivalent to a sum of modular irreducible representations of H.

A. Young showed that there exists a one-to-one correspondence between the irreducible representations of the symmetric group S_n and his tableaux or "diagrams", so that the same symbol can be used interchangeably for a Young diagram and for the corresponding irreducible representation. A generalization of the notion of a Young or *right* diagram is a *skew* diagram $[\alpha] - [\beta]$, introduced by Robinson (**9**), which consists of the nodes left after removing from the corner of a Young diagram $[\alpha]$ nodes which themselves make up a Young diagram $[\beta]$. If the skew diagram consists of disjoint constituents with no row or column in common, it is called a *disjoint* diagram. To every such diagram corresponds an induced representation; of particular significance are disjoint diagrams whose constituents are right diagrams. If, for example, there are two constituent right diagrams $[\beta]$ and $[\gamma]$, where $[\beta]$ is a representation of S_l and $[\gamma]$ a representation of S_m, then the resulting Kronecker product representation of the subgroup $S_l \times S_m$ is written $[\beta] \times [\gamma]$, and the representation of the symmetric group S_{l+m} on $l + m$ distinct symbols induced by this Kronecker product representation is written $[\beta] \cdot [\gamma]$. It is to $[\beta] \cdot [\gamma]$ that the forementioned diagram corresponds, and its reduction into irreducible components $[\alpha]$ of S_n $(n = l + m)$ takes the form

$$[\beta] \,.\, [\gamma] = \sum_{\alpha} {}_{\alpha}\lambda_{\beta}{}^{\gamma}\, [\alpha].$$

The ${}_{\alpha}\lambda_{\beta}{}^{\gamma}$ are obtained via the Littlewood-Richardson rule (**4**, p. 119) for writing down the irreducible components of $[\beta] \cdot [\gamma]$.

It seems unnecessary to summarize here the theory of *hooks* as developed

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in the papers of Nakayama and Robinson already referred to. We note in conclusion a recent paper by Nakayama and Osima (Nagoya Math. J. vol. 2 (1951), 111-117) in which an alternative proof is given of Nakayama's conjecture (cf. **3**).

2. The removal of the restriction $n<2p$ in Nakayama's one hook case. In studying the modular splitting of ordinary irreducible representations of S_n via his notion of a hook, Nakayama naturally started with the case of p-cores of n nodes and immediately reached the conclusion that the corresponding irreducible representations were also modular irreducible and that each of them formed by itself a block of defect 0. Further study along these lines led him to the result that each block of defect 1 contained exactly p representations, namely, those having Young diagrams with a given p-core of $n-p$ nodes and p-hooks of leg lengths $0, 1, 2, \ldots, p-1$, respectively—a result that he was able to prove only for the case $n < 2p$, but which followed directly for all n as soon as his conjecture (**6**, p. 423) was proved. For such a block he stated the following theorem (re-phrased):

2.1. *Let T_0 be a p-core of $n-p$ nodes and let $T_{0,r}$ be the (unique) diagram of n nodes with p-core T_0 and one p-hook of leg length r. Then the irreducible representation $[\beta]_r$ of S_n associated with $T_{0,r}$ possesses exactly one irreducible modular component (with multiplicity 1) in common with $T_{0,r+1} (r \neq p-1)$, one in common with $T_{0,r-1}$ $(r \neq 0)$, and none in common with $T_{0,s}$ $(s \neq r-1, r+1)$.*

To prove this theorem for $n<2p$, Nakayama utilized a result of Brauer (**1**) in the general modular theory concerning the arrangement in a chain of the representations in a block of defect 1, in which only neighbouring representations have a modular component (with multiplicity 1) in common, in order to identify his diagrams with the corresponding representations in the chain. His reason for considering values of $n < 2p$ was simply that only in this range could he be certain of having to contend only with blocks of defects 0 and 1.

To prove the theorem for all values of n, we accept the truth of the result for $n = p$ (Nakayama's proof covers this value), i.e., for a p-core of zero nodes. This means that the portion D_p of the D-matrix appropriate to the corresponding p-block of S_p is of the form

$$\begin{matrix} [\alpha]_0 \\ [\alpha]_1 \\ [\alpha]_2 \\ \cdot \\ [\alpha]_{p-2} \\ [\alpha]_{p-1} \end{matrix} \begin{bmatrix} 1 & 0 & 0 \ldots 0 & 0 \\ 1 & 1 & 0 \ldots 0 & 0 \\ 0 & 1 & 1 \ldots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \ldots 1 & 1 \\ 0 & 0 & 0 \ldots 0 & 1 \end{bmatrix} : D_p ,$$

where $[\alpha]_r$ is the representation of the block whose Young diagram is a p-hook of leg length r. The columns of D_p give the indecomposable components of

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the regular representation that belong to the block, so that in particular $[\alpha]_r$ appears in two indecomposable components, namely $[\alpha]_{r-1} + [\alpha]_r$ and $[\alpha]_r + [\alpha]_{r+1}$.

Let $[\beta_0]$ be the representation of S_{n-p} $(n \geqslant p + 1)$ whose Young diagram is the p-core T_0, so that $[\beta_0]$ is modular irreducible and also forms by itself a p-block of S_{n-p} of defect 0. Denote by $[\beta]_r$ and $[\beta]_{r+1}$ the two representations of the p-block of S_n of defect 1 with p-core $[\beta_0]$ whose Young diagrams have p-hooks of leg lengths r and $r + 1$ respectively. In order to show that $[\beta]_r + [\beta]_{r+1}$ is an indecomposable component of the regular representation of S_n, we prove first a preliminary lemma.

2.2. *If $[\beta_0]$ is a p-core of $n - p$ nodes, and $H_r = [p - r, 1^r]$ a p-hook of leg length r, then of all the diagrams $[\beta]$ of n nodes that can be obtained by building H_r on $[\beta_0]$ in accordance with the Littlewood-Richardson rule, there is only one which has $[\beta_0]$ as p-core, namely the (unique) diagram which contains the p-hook H_r and p-core $[\beta_0]$.*

Proof. Nakayama demonstrated the existence of exactly one diagram $[\beta]_r$ of n nodes which possesses the desired p-hook and p-core, but we need to show that it actually arises as a result of building in accordance with the Littlewood-Richardson rule. The nodes of the p-hook in question can be thought of as being added along the rim of the p-core $[\beta_0]$ so as to form a skew hook equivalent to the right hook H_r, and the only point that needs verification is that this building on $[\beta_0]$ does not violate any of the restrictions laid down in the Littlewood-Richardson rule.

We observe that the first and last nodes of a skew hook (starting from the top right and going to the bottom left) correspond respectively to the head and foot of the equivalent right hook H_r, so that there are exactly as many rows $(r + 1)$ and columns $(p - r)$ represented in the skew hook as in the right hook. Then, since no two added symbols from a given row of H_r may appear in the same column of the resultant diagram $[\beta]$, the $p - r$ nodes in the first row of H_r must be assigned, in order, one to each column of the skew hook. Likewise, since each node of the first column of H_r must appear in a later row of $[\beta]$ than its predecessors of that column, the nodes of the first column of H_r must be assigned, in order, one to each row of the skew hook.

Suppose that we designate the nodes of the right hook H_r in the following way:

$$\begin{array}{llll} C_1 & C_2 & C_3 \ldots & C_s \\ R_2 & & & \\ R_3 & & & \\ \cdot & & & \\ \cdot & & & \\ R_{r+1} & & & \end{array}$$

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where $s = p - r$ and $C_1 \equiv R_1$. Now consider the skew hook obtained by building on $[\beta_0]$ with H_r; from the preceding paragraph it is clear that there will be exactly one R per row ($C_1 \equiv R_1$) and exactly one C per column. Since none of the restrictions involved in the Littlewood-Richardson rule have been violated, $[\beta]_r$ can thus be obtained from this building process. To show that it appears only once, we notice that in the skew hook neither the C's nor the R's can be interchanged among themselves without violating the rule. Further no C can be interchanged with an R, for otherwise we would have two C's in a column of the product diagram. Finally $[\beta]_r$ is the only diagram which contains H_r as a p-hook and $[\beta_0]$ as its p-core.

To show that the process does not yield a diagram $[\beta]_t$ containing a p-hook H_t ($t \neq r$) and p-core $[\beta_0]$, we observe that such a diagram $[\beta]_t$ will contain a skew hook $\bar{H}_t$ equivalent to H_t, containing $t + 1$ rows and $p - t$ columns. If such a skew hook is to arise from building on $[\beta_0]$ with H_r, we shall have more than one C in at least one column of $\bar{H}_t$ if $t > r$, and more than one R in at least one row of $\bar{H}_t$ if $t < r$. In either case a restriction in our building process is violated, and hence such a diagram $[\beta]_t$ cannot arise. This proves the lemma.

Proof of 2.1 *for* $n > p$: Since $[\alpha]_r + [\alpha]_{r+1}$ is an indecomposable representation of S_p (mod p), and $[\beta_0]$ (a p-core) is a modular irreducible representation of S_{n-p}, then

$$([\alpha]_r + [\alpha]_{r+1}) \times [\beta_0]$$

is an indecomposable Kronecker product representation of the direct product subgroup $S_p \times S_{n-p}$ of S_n. Further, by Nakayama's formula (1.3), the corresponding induced representation of S_n

$$[\sigma]_r \cdot [\beta_0] + [\alpha]_{r+1} \cdot [\beta_0]$$

is a sum of indecomposable representations of S_n, whose irreducible components are obtained via the Littlewood-Richardson rule applied to the induced representations $[\alpha]_r \cdot [\beta_0]$ and $[\alpha]_{r+1} \cdot [\beta_0]$. Now the only components that we are interested in are those that belong to the p-block with p-core $[\beta_0]$, and the preliminary lemma tells us that there will be exactly two such irreducible representations, one obtained from $[\alpha]_r \cdot [\beta_0]$ and the other from $[\alpha]_{r+1} \cdot [\beta_0]$. Denoting the representations or the corresponding Young diagrams by $[\beta]_r$ and $[\beta]_{r+1}$, we observe that $[\beta]_r$ and $[\beta]_{r+1}$ each contain one p-hook, of leg length r and $r + 1$ respectively; hence, by the Murnaghan-Nakayama recursion formula, their characters cannot vanish for all elements of S_n of the type $P \,.\, V$, where P is a cycle of length p. Since the vanishing of the character

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for all p-singular elements of S_n is a necessary condition for indecomposability, it follows that neither $[\beta]_r$ nor $[\beta]_{r+1}$ is an indecomposable representation. However, since the p-hooks in $[\beta]_r$ and $[\beta]_{r+1}$ have parities of opposite signs, the character of the sum $[\beta]_r + [\beta]_{r+1}$ vanishes for all p-singular elements; inasmuch as these are the only representations in the block under consideration, the sum $[\beta]_r + [\beta]_{r+1}$ must be an indecomposable representation of S_n $(r \neq p - 1)$. It follows in exactly the same way that $[\beta]_{r-1} + [\beta]_r$ is an indecomposable representation of S_n $(r \neq 0)$. Hence for any p-core $[\beta_0]$ the ordering of the representations in the associated p-block of S_n of defect 1, such that only adjacent representations have a modular component in common, is the same as in the case of the $[\alpha]$'s; i.e., the part of the D-matrix corresponding to this block is again D_p. This completes the proof.

Before proceeding to investigate the modular splitting of representations whose Young diagrams contain two or more p-hooks, we deduce in the next section a number of relations among the characters of any particular block, which hold for all p-regular elements. It is these relations which play a vital role in our subsequent analysis.

3. Character relations for p-regular elements of S_n. In (**9**) Robinson obtained some relations among the degrees of irreducible representations $[\alpha]$ of S_n belonging to a p-block characterized by a p-core of zero nodes namely

3.1 $$\sum_{\alpha} x_\alpha \sigma \lambda = 0,$$

where x_α denotes the degree of $[\alpha]$; $\sigma = (-1)^{\Sigma r_i} = \pm 1$ is the product of the parities of the p-hooks removable from the diagram $[\alpha]$ to yield the zero p-core; and λ is an integer $\geqslant 0$ which gives the multiplicity with which the star diagram $[\alpha]^*_p$ of $[\alpha]$ contains a chosen representation $[b]$ as an irreducible component.

For each choice of $[b]$ there arises an identity 3.1. In a recent paper by Todd (**12**) the same identities appear in another form, namely, as the expansions of the "new multiplication" of two Schur or S-functions of degrees m and n in terms of S-functions of degree mn, where the S-functions of degree n are the characters of irreducible representations of order n of the full linear group.

One can show that 3.1 actually admits of a more general interpretation with the degree of $[\alpha]$ replaced by its character χ_α, so that 3.1 *becomes an identity among the modular components of the irreducible representations.* Furthermore, these identities also exist for p-blocks characterized by non-zero p-cores.

Robinson's line of attack, however, does not yield the larger set of relations which arise from a consideration of the removal of just one hook from each of the Young diagrams of a given p-block, where this hook may be of length

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p, $2p$, . . . , or bp. Suppose we start with the character relation

3.2 $$\sum_{\alpha} \chi_{\alpha}(R)\chi_{\alpha}(S) = 0,$$

where R and S do not belong to the same conjugate set of S_n. Let $R = V.P_k$, where P_k is a single cycle of length kp $(1 \leqslant k \leqslant b)$, and V is any permutation on the remaining $n - kp$ symbols. By the Murnaghan-Nakayama recursion formula

3.3 $$\chi_{\alpha}(V.P_k) = \sum_{\gamma_k} (-1)^{r_i}\chi_{\gamma_k}(V) = \sum_{\gamma_k} a_{\alpha\gamma_k}\chi_{\gamma_k}(V),$$

where the summation extends over all representations $[\gamma_k]$ of S_{n-kp} whose Young diagrams are obtainable from $[\alpha]$ by the removal of a single kp-hook H_i, and r_i is the leg length of H_i. Multiply 3.2 by $\chi_{\beta_k}(V)$, where $[\beta_k]$ is one of the irreducible representations of S_{n-kp} which appear in the right hand side of 3.3, and sum over all V:

$$\sum_{V} \chi_{\beta_k}(V) \sum_{\alpha} \chi_{\alpha}(S) \sum_{\gamma_k} a_{\alpha\gamma_k}\chi_{\gamma_k}(V) = 0.$$

Since the summation over V of the product $\chi_{\beta_k}(V) \,.\, \chi_{\gamma_k}(V)$ yields zero in all cases except when $[\gamma_k] = [\beta_k]$, we obtain

$$\sum_{\alpha} \chi_{\alpha}(S) \sum_{V} \chi_{\beta_k}(V) a_{\alpha\beta_k}\chi_{\beta_k}(V) = 0.$$

This gives

3.4 $$\sum_{\alpha} a_{\alpha\beta_k}\chi_{\alpha}(S) = 0 \qquad (k = 1, 2, \ldots, b),$$

where $a_{\alpha\beta_k}$ is the parity of the kp-hook which is removed from $[\alpha]$ to yield $[\beta_k]$, and $[\beta_k]$ ranges over all diagrams of S_{n-kp} which appear as residual diagrams of $[\alpha]$. *Observe that the* $[\beta_k]$ *are those diagrams of* S_{n-kp} *with the same* p*-core as the original block of* $[\alpha]$'s. For each $[\beta_k]$, 3.4 is a linear relation among the characters χ_{α} of a fixed p-block which holds for all p-regular elements S of S_n, i.e., an identity among the modular components of these characters. A similar procedure applied to each of the other p-blocks yields further identities of the same type.

Example. The representations of S_8 which belong to the 2-block with 2-core [0] are [8], [7,1], [6,2], $[6,1^2]$, [5,3], $[5,1^3]$, $[4^2]$, [4,3,1], $[4,2^2]$, $[4,2,1^2]$, $[4,1^4]$, $[3^2,2]$ $[3^2,1^2]$, $[3,2^2,1]$, $[3,1^5]$, $[2^4]$, $[2^3,1^2]$, $[2^2,1^4]$, $[2,1^6]$, $[1^8]$. The necessary information for producing the identities appropriate to this 2-block is contained in the following table, in which the column labels are the various $[\beta_k]$ which appear after the removal of hooks of length $2k$ $(k = 1, 2, 3, 4)$ from the row labels $[\alpha]$, and the entries are the parities of these hooks:

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[α]	k = 1										k = 2					k=3		k=4
	[6]	[5,1]	[4,2]	[4,1²]	[3²]	[3,1³]	[2³]	[2²,1²]	[2,1⁴]	[1⁶]	[4]	[3,1]	[2²]	[2,1²]	[1⁴]	[2]	[1²]	[0]
[8]	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1
[7,1]	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	−1
[6,2]	1	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	−1	.
[6,1²]	−1	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	1
[5,3]	.	1	.	.	1	.	.	.	.	.	.	.	−1	.	.	−1	.	.
[5,1³]	.	−1	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	−1
[4²]	.	.	1	.	−1	.	.	.	.	.	1	−1	.	.	.	.	.	.
[4,3,1]	.	.	.	1	.	.	.	.	.	.	−1	.	.	−1	.	1	.	.
[4,2²]	.	.	1	−1	.	.	1	.	.	.	.	.	.	.	.	.	1	.
[4,2,1²]	.	.	−1	.	.	.	.	1	.	.	1	.	.	.	−1	.	.	.
[4,1⁴]	.	.	.	−1	.	.	.	.	1	.	−1	.	.	.	.	.	.	1
[3²,2]	.	.	.	.	1	.	−1	.	.	.	.	−1	.	1	.	.	.	.
[3²,1²]	.	.	.	.	−1	1	.	−1	.	.	.	.	.	.	.	−1	.	.
[3,2²,1]	.	.	.	.	.	−1	.	.	.	.	.	1	.	.	1	.	−1	.
[3,1⁵]	.	.	.	.	.	−1	.	.	.	1	.	−1	.	.	.	.	.	−1
[2⁴]	.	.	.	.	.	.	1	−1	.	.	.	.	.	1	−1	.	.	.
[2³,1²]	.	.	.	.	.	.	−1	.	−1	.	.	.	1	.	.	.	1	.
[2²,1⁴]	.	.	.	.	.	.	.	−1	.	−1	.	.	−1	.	.	1	.	.
[2,1⁶]	.	.	.	.	.	.	.	.	−1	.	.	.	.	−1	.	−1	.	1
[1⁸]	.	.	.	.	.	.	.	.	.	−1	.	.	.	.	−1	.	−1	−1

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The columns of this table then give rise to the following identities:

$k = 1$:

(i) $[8] + [6,2] - [6,1^2] = 0$

(ii) $[7,1] + [5,3] - [5,1^3] = 0$

(iii) $[6,2] + [4^2] + [4,2^2] - [4,2,1^2] = 0$

(iv) $[6,1^2] + [4,3,1] - [4,2^2] - [4,1^4] = 0$

(v) $[5,3] - [4^2] + [3^2,2] - [3^2,1^2] = 0$

(vi) $[5,1^3] + [3^2,1^2] - [3,2^2,1] - [3,1^5] = 0$

(vii) $[4,2^2] - [3^2,2] + [2^4] - [2^3,1^2] = 0$

(viii) $[4,2,1^2] - [3^2,1^2] - [2^4] - [2^2,1^4] = 0$

(ix) $[4,1^4] - [2^3,1^2] - [2,1^6] = 0$

(x) $[3,1^5] - [2^2,1^4] - [1^8] = 0$

$k = 2$:

(xi) $[8] + [4^2] - [4,3,1] + [4,2,1^2] - [4,1^4] = 0$

(xii) $[7,1] - [4^2] - [3^2,2] + [3,2^2,1] - [3,1^5] = 0$

(xiii) $[6,2] - [5,3] + [2^3,1^2] - [2^2,1^4] = 0$

(xiv) $[6,1^2] - [4,3,1] + [3^2,2] + [2^4] - [2,1^6] = 0$

(xv) $[5,1^3] - [4,2,1^2] + [3,2^2,1] - [2^4] - [1^8] = 0$

$k = 3$:

(xvi) $[8]-[5,3]+[4,3,1]-[3^2,1^2]+[2^2,1^4]-[2,1^6] = 0$

(xvii) $[7,1] - [6,2] + [4,2^2] - [3,2^2,1] + [2^3,1^2] - [1^8] = 0$

$k = 4$:

(xviii) $[8] - [7,1] + [6,1^2] - [5,1^3] + [4,1^4] - [3,1^5] + [2,1^6] - [1^8] = 0$

In general the identities that we have just derived will not be linearly independent. To establish their linear dependence, consider the character of a representation $[\alpha]$ for an element $R = P_u \,.\, P_v \,.\, W$, where P_u is a cycle of length up, P_v is a second cycle (distinct from P_u) of length vp ($u \neq v$), and W is any permutation on the remaining $n - p(u + v)$ symbols. We assume that $[\alpha]$ contains hooks of length up and vp, and that $u + v \leqslant b$, where b is the number of successive p-hooks removable from $[\alpha]$ to yield its p-core. Applying the Murnaghan-Nakayama recursion formula twice, we obtain

$$\chi_\alpha(R) = \sum_{\beta_u} a_{\alpha\beta_u}\, \chi_{\beta_u}\, (P_v \,.\, W)$$

$$= \sum_{\beta_u} a_{\alpha\beta_u} \sum_{\beta_{u+v}} a'_{\beta_u\beta_{u+v}}\, \chi_{\beta_{u+v}}(W),$$

if we think of removing a hook of length up first, and

$$\chi_\alpha(R) = \sum_{\beta_v} a_{\alpha\beta_v}\, \chi_{\beta_v}\, (P_u \,.\, W)$$

$$= \sum_{\beta_v} a_{\alpha\beta_v} \sum_{\beta_{u+v}} a''_{\beta_v\beta_{u+v}}\, \chi_{\beta_{u+v}}\, (W),$$

if we remove a vp-hook first. Here $[\beta_u]$, $[\beta_v]$, $[\beta_{u+v}]$ are representations of S_{n-up}, S_{n-vp}, $S_{n-p(u+v)}$ respectively, and $a'_{\beta_u\beta_{u+v}}$ ($a''_{\beta_v\beta_{u+v}}$) is the parity of

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the hook which must be removed from $[\beta_u]$ ($[\beta_v]$) in order to yield $[\beta_{u+v}]$. Since these are expressions for the same character, we have, for each appropriate $[\alpha]$ of the p-block,

3.5 $$\sum_{\beta_u} a_{\alpha\beta_u} \sum_{\beta_{u+v}} a'_{\beta_u\beta_{u+v}}\, \chi_{\beta_{u+v}}(W) = \sum_{\beta_v} a_{\alpha\beta_v} \sum_{\beta_{u+v}} a''_{\beta_v\beta_{u+v}}\, \chi_{\beta_{u+v}}(W),$$

a linear relation among the ordinary irreducible characters of $S_{n-p(u+v)}$ for all elements W of $S_{n-p(u+v)}$. The linear independence of these characters then implies

3.6 $$\sum_{\beta_u} a'_{\beta_u\beta_{u+v}}\, a_{\alpha\beta_u} = \sum_{\beta_v} a''_{\beta_v\beta_{u+v}}\, a_{\alpha\beta_v}$$

for each $[\beta_{u+v}]$. Observe that, if $u = v$, no relation of this kind arise, since 3.5 becomes simply an identity. Multiplying through 3.6 by $\chi_\alpha(S)$, where S is a p-regular element of S_n, and summing over the $[\alpha]$'s of the block under consideration, we obtain

$$\sum_{\beta_u} a'_{\beta_u\beta_{u+v}} \sum_{\alpha} a_{\alpha\beta_u}\, \chi_\alpha(S) = \sum_{\beta_v} a''_{\beta_v\beta_{u+v}} \sum_{\alpha} a_{\alpha\beta_v}\, \chi_\alpha(S).$$

For each $[\beta_{u+v}]$ this is a relation among the identities arising from the $[\beta_u]$'s and those arising from the $[\beta_v]$'s, where $u \neq v$.

Referring to our previous example, the only values of u and v ($u \neq v$) satisfying $u + v \leqslant b$, where $b = 4$ in this particular block, are 1,2 and 1,3, so that the number of relations among our identities is simply the number of $[\beta_3]$'s and $[\beta_4]$'s, namely 3. The relation arising from [2] is obtained by multiplying (i), (ii), . . . , (x) by $1, 0, 0, 0, -1, 0, 0, 1, -1, 0$ respectively (namely, the parities of the 4-hooks which must be removed from [6], [5,1], . . . , $[1^6]$ to yield [2], and 0 if no such 4-hook exists); (xi), (xii,) . . . , (xv) by $1, 0, 1, -1, 0$ respectively (namely, the parities of the 2-hooks removable from [4], [3,1], . . . , $[1^4]$ to yield [2]); and equating the two linear combinations to yield

$$\text{(i)} - \text{(v)} + \text{(viii)} - \text{(ix)} = \text{(xi)} + \text{(xiii)} - \text{(xiv)}.$$

Similarly, corresponding to $[1^2]$ and [0], we obtain:

$$\text{(ii)} - \text{(iii)} + \text{(vii)} - \text{(x)} = \text{(xii)} - \text{(xiii)} - \text{(xv)},$$

$$\text{(i)} - \text{(ii)} + \text{(iv)} - \text{(vi)} + \text{(ix)} - \text{(x)} = \text{(xvi)} - \text{(xvii)}.$$

We should not assume, however, that every relation among the identities which arises in this way is distinct from every other one; it may happen that one relation is simply a restatement of two or more other relations. Consider an element of the type $P_u \,.\, P_v \,.\, P_w \,.\, Q$, where the P's are defined as before and Q is any permutation on the remaining $n - p(u + v + w)$ symbols, $u \neq v \neq w$. Assuming that $u + v + w \leqslant b$, we obtain, by the same reasoning as before, the relations

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$$\sum_{\beta_u} a_{\alpha\beta_u} \sum_{\beta_{u+v+w}} a'_{\beta_u\beta_{u+v+w}} \chi_{\beta_{u+v+w}}(Q)$$
$$= \sum_{\beta_v} a_{\alpha\beta_v} \sum_{\beta_{u+v+w}} a''_{\beta_v\beta_{u+v+w}} \chi_{\beta_{u+v+w}}(Q)$$
$$= \sum_{\beta_w} a_{\alpha\beta_w} \sum_{\beta_{u+v+w}} a'''_{\beta_w\beta_{u+v+w}} \chi_{\beta_{u+v+w}}(Q),$$

and once again the linear independence of the characters $\chi_{\beta_{u+v+w}}$ of $S_{n-p(u+v+w)}$ yields

$$\sum_{\beta_u} a'_{\beta_u\beta_{u+v+w}} a_{\alpha\beta_u} = \sum_{\beta_v} a''_{\beta_v\beta_{u+v+w}} a_{\alpha\beta_v} = \sum_{\beta_w} a'''_{\beta_w\beta_{u+v+w}} a_{\alpha\beta_w}$$

for each $[\beta_{u+v+w}]$. That is, for each $[\beta_{u+v+w}]$ only two of the three apparent relations which exist among the three sets of identities arising from the $[\beta_u]$'s, $[\beta_v]$'s, and $[\beta_w]$'s (namely, the relations between the sets taken in pairs) are distinct: the remaining relation is implied by the other two. The generalization of this to the case where we have $u_1, u_2, u_3, u_4, \ldots$, satisfying $u_1 + u_2 + u_3 + u_4 + \ldots \leqslant b$ and $u_1 \neq u_2 \neq u_3 \neq u_4 \ldots$, presents no added difficulty.

The following interpretation of the above relations among the identities may prove useful in understanding them. Since the number of modular irreducible characters of a group is less than the number of ordinary irreducible ones, there must exist a number of linear relations among the ordinary characters which hold for all p-regular elements (that is, identities among their modular components) in order to make up the difference. The number of modular characters of S_n being effectively the number of distinct partitions of n which contain neither p nor its multiples, the number of such identities must be the number of those partitions which contain p or its multiples, or the number of conjugate sets of p-singular elements. The identities that we have derived from all the blocks clearly correspond to those partitions of n which contain summands of length p, $2p$, $3p$, . . . or bp, and the fact that the latter classification is not mutually exclusive (i.e. a partition may contain more than one multiple of p) means that we have more identities than there are partitions of this category. The relations among the identities serve to remove the duplications: however, their independence requires further study.

In our previous example, there were 16 p-singular conjugate sets and 19 identities(the modular characters of $[5,2,1]$ and $[3,2,1^3]$ of the block with 2-core $[3,2,1]$ satisfy the remaining identity $[5,2,1] - [3,2,1^3] = 0$), so that the three relations among the identities (namely the three relations corresponding to $[2]$, $[1^2]$, and $[0]$) make up the difference. The three relations may be accounted for by the conjugate sets $[6,2]$, $[4,2^2]$, $[4,2,1^2]$, which in a sense give rise to two identities each.

An examination of the number of identities in those p-blocks of S_n for which b is fixed leads to the following conjecture[1]:

[1]This has now been proved by G. de B. Robinson.

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The number of indecomposables and the number of ordinary irreducible representations in a p-block of S_n characterized by a p-core of a nodes are the same as the corresponding numbers in a p-block of S_m with a p-core of a' nodes, where $n = a + bp$, $m = a' + bp$, i.e., where the same number of p-hooks are removable to yield a p-core. However, the corresponding blocks of the D-matrices will in general be different.

We conclude this section with two results which arise from the relations 3.4. Since these relations hold for all p-regular elements, we may replace the ordinary characters by their modular components Φ_λ and obtain

$$\sum_\alpha a_{\alpha\beta_k} \sum_\lambda d_{\alpha\lambda}\, \Phi_\lambda(S) = 0.$$

The linear independence of the Φ's then implies that

3.7 $$\sum_\alpha a_{\alpha\beta_k} d_{\alpha\lambda} = 0,$$

λ ranging over the modular characters of the block. Accordingly, if we think of the modular splitting of the $[\alpha]$'s as represented by a D-matrix (mod p) with the $[\alpha]$'s as row labels and the modular characters as column labels, we may state the following corollary to 3.7:

3.8. *The coefficients in the identities* 3.4 *are orthogonal to the columns of the D-matrix.*

Again, an indecomposable representation of the above block is a certain linear combination of ordinary irreducible representations, or, in terms of characters,

$$\eta_\lambda = \sum_\alpha d_{\alpha\lambda}\, \chi_\alpha.$$

Since the character χ_α for any element of the type $R = P_k \,.\, V$ takes the value $\sum_{\beta_k} a_{\alpha\beta_k}\, \chi_{\beta_k}(V)$, we have

$$\begin{aligned}\eta_\lambda(R) &= \sum_\alpha d_{\alpha\lambda} \sum_{\beta_k} a_{\alpha\beta_k}\, \chi_{\beta_k}(V)\\ &= \sum_{\beta_k} \Big(\sum_\alpha a_{\alpha\beta_k}\, d_{\alpha\lambda}\Big)\, \chi_{\beta_k}(V)\\ &= 0 \text{ by } 3.7.\end{aligned}$$

Since this holds for $k = 1, 2, \ldots, b$ and all p-singular elements of S_n are of the form $P_k \,.\, V$ for some integer k, we have a new proof of the following known result:

3.9. *The characters of the indecomposable representations of S_n vanish for all p-singular elements.*

4. The indecomposable representations of S_{2p}. We proceed with our investigation into the nature of the indecomposables for p-blocks of defects other

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than 0 and 1 by applying Nakayama's induction formula

$$\tilde{\eta}^{(k)*} = \sum_{\lambda} a_{\kappa\lambda}\eta^{(\lambda)},$$

which operates on an indecomposable of a subgroup of S_n to produce a sum of indecomposables of S_n, to the particular case where the subgroup in question is S_{2p-1} and S_n is S_{2p}. This serves to effect a passage from p-blocks of defects 0 and 1 to p-blocks of higher defects, characterized by more than one p-hook in their Young diagrams.

It is necessary to consider only the p-block of S_{2p} characterized by a p-core of zero nodes, inasmuch as the theory is now complete for the one p-hook and the p-core cases. We shall verify that all the indecomposable representations of such a p-block may be obtained, via the inducing process, from the following two types of indecomposables of S_{2p-1}:

(i) the indecomposables of the p-blocks of S_{2p-1} with p-core $[p - r, 1^{r-1}]$ $r = 1, 2, \ldots, p - 1$;

(ii) the indecomposable (and modular irreducible) representation $[p, 1^{p-1}]$.

The $p - 1$ indecomposables of the p-block of S_{2p-1} with p-core $[p - r, 1^{r-1}]$ are:

(1) $[2p - r, 1^{r-1}] + [p, p - r + 1, 1^{r-2}]$ $\quad (r \neq 1)$

(1a) $[2p - 1] + [(p - 1)^2, 1]$ $\quad (r = 1)$

(2) $[p - s, p - r + 1, 2^s, 1^{r-s-2}] + [p - s - 1, p - r + 1, 2^{s+1}, 1^{r-s-3}]$ $\quad (s = 0, 1, 2, \ldots, r - 3)$

(3) $[p - r + 2, p - r + 1, 2^{r-2}] + [(p - r)^2, 2^{r-1}, 1]$

(4) $[p - r, p - r - t, 2^{r-1}, 1^{1+t}] + [p-r, p-r-t-1, 2^{r-1}, 1^{2+t}]$ $\quad (t = 0, 1, 2, \ldots, p - r - 3)$

(5) $[p - r, 2^r, 1^{p-r-1}] + [p - r, 1^{p+r-1}]$ $\quad (r \neq p - 1)$

(5a) $[3, 2^{p-2}] + [1^{2p-1}]$ $\quad (r = p - 1)$

Neglecting all representations of S_{2p} except those belonging to the p-block with zero p-core, we can express the result of inducing on each of the above indecomposables of S_{2p-1} by the following notation:

(1)′ $[2p-r, 1^{r-1}] + [p, p-r+1, 1^{r-2}] \uparrow [2p-r+1, 1^{r-1}] + [2p-r, 1^r] + [p, p-r+2, 1^{r-2}] + [p, p-r+1, 1^{r-1}] = [a_1]+[b_1]+[c_1]+[d_1]$

(1a)′ $[2p-1]+[(p-1)^2,1] \uparrow [2p]+[2p-1, 1]+[p, p-1, 1]+[(p-1)^2,2]$

(2)′ $[p-s, p-r+1, 2^s, 1^{r-s-2}]+[p-s-1, p-r+1, 2^{s+1}, 1^{r-s-3}] \uparrow [p-s, p-r+2, 2^s, 1^{r-s-2}]+[p-s, p-r+1, 2^s, 1^{r-s-1}] + [p-s-1, p-r+2, 2^{s+1}, 1^{r-s-3}]+[p-s-1, p-r+1, 2^{s+1}, 1^{r-s-2}]$ $\quad (s = 0, 1, 2, \ldots, r - 3)$

(3)′ $[p-r+2, p-r+1, 2^{r-2}]+[(p-r)^2, 2^{r-1}, 1] \uparrow [(p-r+2)^2, 2^{r-2}] +[p-r+2, p-r+1, 2^{r-2}, 1]+[p-r+1, p-r, 2^{r-1}, 1]+[(p-r)^2, 2^r]$

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$$(4)' \quad [p-r, p-r-t, 2^{r-1}, 1^{1+t}]+[p-r, p-r-t-1, 2^{r-1}, 1^{2+t}]$$
$$\uparrow [p-r+1, p-r-t, 2^{r-1}, 1^{1+t}]+[p-r, p-r-t, 2^{r}, 1^{t}]$$
$$+[p-r+1, p-r-t-1, 2^{r-1}, 1^{2+t}]+[p-r, p-r-t-1, 2^{r}, 1^{1+t}]$$
$$(t = 0, 1, 2, \ldots, p-r-3)$$

$$(5)' \quad [p-r, 2^{r}, 1^{p-rp-1}]+[p-r, 1^{p+r-1}] \uparrow [p-r+1, 2^{r}, 1^{p-r-1}]$$
$$+[p-r, 2^{r+1}, 1^{p-r-2}]+[p-r+1, 1^{p+r-1}]+[p-r, 1^{p+r}]$$

$$(5a)' \quad [3, 2^{p-2}]+[1^{2p-1}] \uparrow [3^2, 2^{p-3}]+[3, 2^{p-2}, 1]+[2, 1^{2p-2}]+[1^{2p}]$$

The existence of an orthogonal relation 3.7 between the coefficients in the identities and the columns of the D-matrix suggests, as a first step towards building up our indecomposables from their irreducible components in any of the foregoing cases, the setting up of such a table as the following:

(1)′:

	$[p]$	$[p-r+1, 1^{r-1}]$	$[p-r, 1^{r}]$	$[0]$
$[a_1]$	.	0	.	$r-1$
$[b_1]$	.	.	0	r
$[c_1]$	$r-2$	1	.	.
$[d_1]$	$r-1$	.	1	.

For convenience in writing, the entries are not the parities $(-1)^{r_i}$, but the actual leg lengths r_i of the p-hooks and $2p$-hooks which are removable from the row labels to yield the column labels.

This differs from the table whose columns give us our identities for the p-block of representations of S_{2p} with p-core [0] in that the row labels of the latter table comprise all the representations of the p-block in question, while the table shown here contains only those representations of the p-block which arise from inducing on a single indecomposable of S_{2p-1}. Since 0 and the even integers (leg lengths) yield coefficients of +1 in the identities, and the odd integers coefficients of −1, it is clear that any linear combination of irreducible representations making up an indecomposable must contribute to each of these columns a number of odd integers equal to the number of even integers, if it contributes at all. The fact that the set of contributors to these columns is a sum of indecomposables means that the odd and even integers in each column do balance each other, so that the problem is to determine what further characterizations are needed to ensure that any subset of these contributors, possessing the same property, is actually an indecomposable.

Suppose we start by building the indecomposable to which $[a_1]$ belongs. The condition that we have imposed on our building process requires that $[c_1]$ be included in the combination in order that its 1 in column $[p-r+1, 1^{r-1}]$ balance the 0 of $[a_1]$. We say that these two p-hooks or, equivalently, the two entries which represent them, namely 0 and 1, are *linked* in column $[p-r+1, 1^{r-1}]$. Passing on to column [0], we observe that $[b_1]$ now must

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be included in order that its r balance the $r - 1$ of $[a_1]$; likewise in column $[p]$ we observe that $[d_1]$ must be included in order that its $r - 1$ balance the $r - 2$ of $[c_1]$. At the same time the entries in $[p - r, 1^r]$ are linked. Since this brings into the fold all the representations at our disposal, the result is that the combination $[a_1] + [b_1] + [c_1] + d_1]$ is not a sum of indecomposables of S_{2p}, but an indecomposable by itself.

An indecomposable, then, appears to possess the property of having complete *linkages* in the columns of what we shall henceforth call its *linkage matrix*. In the case where only one p-hook can be removed from each of the Young diagrams of a p-block of S_n, such a matrix for an indecomposable of this p-block consists of only one column (headed by the p-core of the block) and two rows, and the entries are of the form r and $r + 1$ for $r = 0, 1, 2, \ldots, p - 2$. Hence the linkage matrix is a generalization of this latter case, where two representations of the block can be combined to form an indecomposable if and only if their p-hooks are *linked*, i.e., have *consecutive* leg lengths. The generalization lies in the presence of the linkage property in more than one column, where these additional columns stem from the fact that, when we deal with p-blocks of representations with Young diagrams containing more than one p-hook, more than one residual diagram arise after the removal of an initial p-hook; also the removal of hooks of length kp $(k > 1)$ needs to be considered. So far as satisfying a necessary condition for indecomposability is concerned (namely, that the character of an indecomposable representation vanishes for all p-singular elements), it would be sufficient that the number of even leg lengths balance the number of odd in each column of the linkage matrix; the only justification for our definition that linkage takes place, not haphazardly between hooks of odd and even leg lengths, but between hooks of *consecutive* leg lengths, is that, so far as the indecomposables of S_{2p} are concerned, our p-hooks and $2p$-hooks occur only in such pairs.

Linkage matrices (with leg lengths as entries) similar to that for (1)′ set out above show that likewise in each of the remaining cases all of the irreducible components must be taken in order to form a combination which is orthogonal to the identities.

Inducing on the indecomposable representation $[p, 1^{p-1}]$ in (ii) yields

$$[p + 1, 1^{p-1}] + [p, 2, 1^{p-2}] + [p, 1^p] = [a]+[b]+[c]$$

with linkage matrix;

$$\begin{array}{c} \\ [a] \\ [b] \\ [c] \end{array}\begin{array}{c} \begin{array}{ccc} [p] & [1^p] & [0] \end{array} \\ \left[\begin{array}{ccc} . & 0 & p-1 \\ p-2 & 1 & . \\ p-1 & . & p \end{array}\right] \end{array}$$

so that $[a] + [b] + [c]$ clearly forms by itself an indecomposable of S_{2p}. Ob-

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serve that, whereas each of the preceding indecomposables had four irreducible components, this last one is composed of only three.

Thus a single indecomposable of the p-block with zero p-core arises in each of these cases. There is, however, a certain amount of duplication as r ranges over its integral values. For instance, $r = 3$ gives the same indecomposable for case (2), $s = 0$, as $r = 1$ for case (4), $t = 0$; similarly $r = 4$ yields the same indecomposable for case (2), $s = 1$, as $r = 2$ for case (4) $t = 0$, and the same indecomposable for case (2), $s = 0$, as $r = 1$ for case (4), $t = 1$. The number of duplications (regarding an indecomposable as a duplication on its second appearance) is $0, 0, 1, 2, 3, \ldots, p - 3$, as r takes the values $1, 2, 3, 4, 5, \ldots, p - 1$ respectively, or $\frac{1}{2}(p - 2)(p - 3)$ in all. Hence the number of distinct indecomposables belonging to the designated p-block that arise in this way is

$$(p - 1)^2 + 1 - \tfrac{1}{2}(p - 2)(p - 3) = \tfrac{1}{2}(p - 1)(p + 2).$$

The ordinary representations of the p-block in question include the following:

$$\begin{array}{l}
[2p], [2p-1, 1], \ldots, [p+1, 1^{p-1}];\\
[p^2], [p, p-1, 1], [p, p-2, 1^2], \ldots, [p, 1^p];\\
[(p-1)^2, 2], [p-1, p-2, 2, 1], [p-1, p-3, 2, 1^2], \ldots,\\
\qquad\qquad [p-1, 2^2, 1^{p-3}], [p-1, 1^{p+1}];\\
[(p-2)^2, 2^2], [p-2, p-3, 2^2, 1], [p-2, p-4, 2^2, 1^2], \ldots,\\
\qquad\qquad [p-2, 2^3, 1^{p-4}], [p-2, 1^{p+2}];\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
[3^2, 2^{p-3}], [3, 2^{p-2}, 1], [3, 1^{2p-3}];\\
[2^p], [2, 1^{2p-2}];\\
[1^{2p}];
\end{array}$$

so that their number is $p + p + (p - 1) + (p - 2) + \ldots + 3 + 2 + 1 = p + \frac{1}{2}p(p + 1) = \frac{1}{2}p(p + 3)$. The number of identities that the characters of these representations must satisfy for all p-regular elements is simply the number of diagrams of S_p with zero p-core (i.e., the number of distinct p-hooks) together with the diagram of 0 nodes, or $p + 1$ in all. That these are linearly independent follows from the fact that there is no solution in positive integers of the inequality $u + v \leqslant 2$, $u \neq v$, since only these values of u and v can give rise to relations among the identities. Hence the number of modular irreducible representations or, equivalently, the number of indecomposables, is $\frac{1}{2}p(p + 3) - (p + 1) = \frac{1}{2}(p^2 + p - 2) = \frac{1}{2}(p - 1)(p + 2)$. A comparison of this number with the number obtained by the induction process shows that we have obtained all the indecomposables.

The following is the D-matrix for the 5-block of S_{10} with 5-core [0]. As noted before, each column contains exactly four 1's except the seventh, which contains three.

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[10]	1	0	0	0	0	0	0	0	0	0	0	0	0	0
[9,1]	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$[8,1^2]$	0	1	1	0	0	0	0	0	0	0	0	0	0	0
$[7,1^3]$	0	0	1	1	0	0	0	0	0	0	0	0	0	0
$[6,1^4]$	0	0	0	1	0	0	1	0	0	0	0	0	0	0
$[5^2]$	0	1	0	0	1	0	0	0	0	0	0	0	0	0
[5,4,1]	1	1	1	0	1	1	0	0	0	0	0	0	0	0
$[5,3,1^2]$	0	0	1	1	0	1	0	1	0	0	0	0	0	0
$[5,2,1^3]$	0	0	0	1	0	0	1	1	0	0	1	0	0	0
$[5,1^5]$	0	0	0	0	0	0	1	0	0	0	1	0	0	0
$[4^2,2]$	1	0	0	0	0	1	0	0	0	1	0	0	0	0
[4,3,2,1]	0	0	0	0	1	1	0	1	1	1	0	0	0	0
$[4,2^2,1^2]$	0	0	0	0	0	0	0	1	1	0	1	1	0	0
$[4,1^6]$	0	0	0	0	0	0	0	0	0	0	1	1	0	0
$[3^2,2^2]$	0	0	0	0	1	0	0	0	1	0	0	0	0	1
$[3,2^3,1]$	0	0	0	0	0	0	0	0	1	1	0	1	1	1
$[3,1^7]$	0	0	0	0	0	0	0	0	0	0	0	1	1	0
$[2^5]$	0	0	0	0	0	0	0	0	0	1	0	0	1	0
$[2,1^8]$	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$[1^{10}]$	0	0	0	0	0	0	0	0	0	0	0	0	0	1

5. The indecomposable representations of S_n ($n > 2p$). In the section just concluded, the notion of a linkage has proved a useful tool in determining the indecomposables of S_{2p} belonging to the p-block with zero p-core, where a linkage was defined as taking place only between kp-hooks of consecutive leg lengths. Following this lead, we formulated a number of empirical rules regarding the use of linkages in constructing the indecomposables of S_n from those of S_{n-1}, and these rules produced (without apparent ambiguity) the indecomposables of S_n up to $n = 13$ for $p = 2, 3, 5$. The tables for most of these cases are contained in the author's thesis on file at the University of Toronto Library. A study of numerous examples led to a certain conjecture concerning the definition of an indecomposable and this will be the subject of a later paper.

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