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Ergodic Rotations of Nilmanifolds Conjugate to Their Inverses

J. P. Henniger

Abstract. In answer to a question posed in [3], we give sufficient conditions on a Lie nilmanifold so that any ergodic rotation of the nilmanifold is metrically conjugate to its inverse. The condition is that the Lie algebra be what we call quasi-graded, and is weaker than the property of being graded. Furthermore, the conjugating map can be chosen to be an involution. It is shown that for a special class of groups, the condition of quasi-graded is also necessary. In certain examples there is a continuum of conjugacies.

1 Introduction

It is a consequence of the discrete spectrum theorem of Halmos and von Neumann [4] that if T is an ergodic transformation with discrete spectrum on a standard Borel probability space, then T is isomorphic to its inverse T^{-1}, and if S is the conjugating automorphism, *i.e.*, $TS = ST^{-1}$, then S is an involution. In [3] the authors extend this result to the class of transformations with simple spectrum, and investigate the properties of S in more general situations. The authors pose the question as to whether an ergodic rotation of a nilmanifold is isomorphic to its inverse, and if so, whether there is a conjugating automorphism which is an involution. It is our purpose in this paper to give a condition sufficient for this to be so, and in a certain class of nilmanifolds also necessary.

Throughout this paper, G will denote a connected, simply connected nilpotent Lie group and Γ a uniform discrete subgroup, *i.e.*, such that the quotient space of right cosets $X = \Gamma\backslash G$ is compact. X is called a nilmanifold, and multiplication on the right of $\Gamma \setminus G$ by a fixed element $r \in G$ induces on X a measure preserving transformation, $\Gamma x \to \Gamma xr$, namely a rotation. We shall define the notion of quasi-graded nilpotent Lie groups (Definition 1), which includes some of the better known classes of groups like N_n, the class of upper triangular $n \times n$ matrices with ones along the diagonal, sufficient for any ergodic rotation to be metrically conjugate to its inverse, for all the uniform subgroups. We show also that the conjugating map can be chosen to be an involution (Corollary 2). On the other hand we shall find a subgroup of N_n for which ergodic rotations of its quotient spaces are not congugate to their inverses. For a special class of low-level groups G, the condition of quasi-graded is shown to be necessary (Section 5).

Of course, the group of rotations is not really G, but the quotient group $\Lambda \setminus G$ where Λ is the isotropy subgroup for e, the identity of G: $\Lambda = \{x \in G : \Gamma x = \Gamma\}$. $G' = \Lambda \setminus G$ then acts effectively on X, *i.e.*, for $g' \in G'$, $\Gamma x g' = \Gamma x$ for all $x \in G$

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implies $g' = e'$ where e' is the identity of G'. It is easy to show that $g \in \Lambda$ implies $xgx^{-1} = c \in \Gamma$ for all $x \in G$ which implies that g is in $\Gamma \cap Z$ where Z is the centre of G. Clearly any member of $\Gamma \cap Z$ is in Λ so we have that $\Lambda = \Gamma \cap Z$.

Parry [7] has shown that two unipotent affine transformations of nilmanifolds are metrically conjugate if and only they are algebraically conjugate. A transformation of $\Gamma \setminus G$ is affine if it is of the form $T(\Gamma x) = T(\Gamma)\phi(x)a$ where ϕ is a continuous homomorphism of G into itself such that $\phi(\Gamma) \subseteq \Gamma$ and $a \in G$. In fact ϕ is an automorphism of G'. (See the remarks (1.3)–(1.6) of [7]). It is unipotent if $(d\phi)_e$ is unipotent. Standard arguments show that an affine transformation is measure preserving. A rotation by $r \in G$ is seen to be an affine transformation T if we take $T(\Gamma) = \Gamma r$ and $\phi(x) = r^{-1}xr$. In this case $(d\phi)_e = \operatorname{Ad}\phi$ and that it is unipotent follows from the fact that the Lie algebra of G is nilpotent. Thus we know that our problem is reduced to the algebraic one of finding an affine transformation: two rotations of a given nilmanifold $X = \Gamma \setminus G$ given by r_1 and r_2 are conjugate if and only if there is an affine transformation ψ such that $\Gamma x\psi r_1 = \Gamma x r_2 \psi$ for all $x \in G$.

Writing $\psi = \phi a$ as above, this implies $(\Gamma x\phi)ar_1 = (\Gamma x\phi)(r_2\phi)a$ from which follows $ar_1 = (r_2\phi)a\ell$ where $\ell \in \Lambda$, since ϕ maps onto G. Thus $r_1 = a^{-1}\phi(r_2)a$ as an element of G'. Retracing our steps, we see that if we can find an automorphism ϕ of $\Lambda \setminus G$ and $a \in G$ such that $r_1 = a^{-1}\phi(r_2)a$, and we put $\psi = \phi a$, then $\Gamma x\psi r_1 = \Gamma x r_2\psi$ for all $x \in G$, *i.e.*, ψ is a conjugacy between the rotations given by r_1 and r_2. Of course the expression giving r_1 in terms of r_2 is itself an automorphism, but we shall find it natural to find it in stages, first ϕ and then a.

We will also make use of L. Green's criterion for a rotation to be ergodic—actually shown by Green [1, p. 65] for flows, and by Parry [8] for affine maps—namely: a rotation on X is ergodic if and only if the induced rotation on the 'maximal torus' $T = \Gamma K \setminus G$ is ergodic, where K is the commutator $[G, G]$ of G. It is a nilmanifold itself since $\Gamma K \setminus G \cong (K \setminus K\Gamma) \setminus (K \setminus G)$, and $K \setminus \Gamma K$ is a uniform discrete subgroup of $K \setminus G$.

2 Coordinates

We shall denote the descending central series in G by $\{G^{(j)}\}$ where $G^{(1)} = G$ and $G^{(j+1)} = [G, G^{(j)}]$ (where $[H, H_1]$ is the abstract group generated by the commutators $hh_1h^{-1}h_1^{-1}$, $h \in H$, $h_1 \in H_1$), $G^{(k)} \neq (e)$ and $G^{(k+1)} = (e)$. That is, G is assumed to be k-step nilpotent. Similarly in the Lie algebra $\mathfrak{g}$ of G, the descending central series $\{\mathfrak{g}^{(j)}\}$ will be denoted by $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(j+1)} = [\mathfrak{g}, \mathfrak{g}^{(j)}]$, where now the square brackets denote the Lie bracket. The ideal $\mathfrak{g}^{(j)}$ is the Lie algebra of $G^{(j)}$. Since G is nilpotent, connected and simply connected, the map $\exp\colon \mathfrak{g} \to G$ is an analytic diffeomorphism.

A basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ is a *strong Malcev basis* (a basis of the second kind, in Malcev's terms) if for each m, $\mathfrak{h}_m = \operatorname{sp}\{X_{(m+1)}, \dots, X_n\}$ is an ideal of $\mathfrak{g}$, and for each j, $j = 1, \dots, k$, $\mathfrak{h}_{m_j} = \mathfrak{g}^{(j)}$ (so that in particular $m_1 = 0$). The basis is *strongly based on the discrete subgroup* Γ if

$$(1) \qquad \Gamma = \{\exp a_1X_1 \cdot \exp a_2X_2 \cdot \cdots \cdot \exp a_nX_n : a_j \in \mathbf{Z}, 1 \le j \le m\}.$$

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Malcev [6] has shown that the uniformity of Γ is equivalent to the existence in $\mathfrak{g}$ of a strong Malcev basis strongly based on Γ. In this case $S = \{\exp t_1X_1 \cdot \exp t_2X_2 \cdot \cdots \cdot \exp t_nX_n : 0 \leq t_j < 1\}$ is a fundamental domain for $\Gamma \setminus G$ and the mapping $\beta\colon [0,1) \times \cdots \times [0,1) \to \Gamma \setminus G$ where $\beta(t_1,\ldots,t_n) = \Gamma \exp t_1X_1 \cdot \exp t_2X_2 \cdot \cdots \cdot \exp t_nX_n$ maps the Lebesgue measure dt to the G-invariant measure on $\Gamma \setminus G$. (See [2, Chapters 1 and 5].)

For the remainder of this paper we assume that we have chosen for $\mathfrak{g}$ a strong Malcev basis $\{X_1,\ldots,X_n\}$ strongly based on Γ. In these coordinates the maximal torus, $T = \Gamma K \setminus G$ where $K = G^{(2)}$, is the image under the exponential map of the additive group $\{(t_1,\ldots,t_{m_2})(\mathrm{mod}\,1)\}$. Using the abelian property of $K \setminus G$, one can see that the the action of a rotation $r = \exp \rho_1X_1 \cdot \cdots \cdot \exp \rho_nX_n$ on the maximal torus is isomorphic to that of the rotation of $[0,1)^{m_2}$ given by $(t_1,\ldots,t_{m_2}) \to (t_1 + \rho_1,\ldots,t_{m_2} + \rho_{m_2})$. By a well known criterion for the ergodicity of rotations of the torus, the criterion for r to be ergodic on $\Gamma\setminus G$ cited in the introduction now becomes:

> the rotation $r = \exp \rho_1X_1 \cdot \cdots \cdot \exp \rho_nX_n$ on $\Gamma \setminus G$ is ergodic if and only if the numbers $\{\rho_1,\ldots,\rho_{m_2},1\}$ are rationally independent.

3 Quasi-Graded Algebras and Groups

We now know that if the discrete subgroup Γ of G is uniform in G then the Lie algebra $\mathfrak{g}$ of G has a basis

$$\{X_1,\ldots,X_{m_2},X_{m_2+1},\ldots,X_{m_k+1},\ldots,X_n\}$$

strongly based on Γ, where $\mathrm{sp}\{X_{m_j+1},\ldots,X_n\} = \mathfrak{g}^{(j)}$. If we put $\mathfrak{h}_1 = \mathrm{sp}\{X_1,\ldots,X_{m_2}\}$, $\mathfrak{h}_2 = \mathrm{sp}\{X_{m_2+1},\ldots,X_{m_3}\}$, ..., $\mathfrak{h}_k = \mathrm{sp}\{X_{m_k+1},\ldots,X_n\}$, then $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_k$, and $[\mathfrak{h}_1,\mathfrak{h}_j] \subseteq \mathfrak{g}^{(j+1)}$. We introduce the following subspaces, which concern the notion of graded algebras:

$$\mathfrak{g}_1 = \mathfrak{h}_1, \quad \text{and for } j > 1, \quad \mathfrak{g}_j = [\mathfrak{g}_1,\mathfrak{g}_{j-1}]. \tag{2}$$

It is clear that $\mathfrak{g}_j \subseteq \mathfrak{g}^{(j)}$ for $j = 1,2,\ldots$.

Proposition 1 *If $\mathfrak{g}_j$ is as defined in (2), then for $p,q \in \mathbf{N}$, $[\mathfrak{g}_p,\mathfrak{g}_q] = \mathfrak{g}_{p+q}$.*

Proof $[\mathfrak{g}_p,\mathfrak{g}_1] = \mathfrak{g}_{p+1}$, for any $p \geq 1$ by the definition. Assume for the purpose of induction that the statement of the proposition is true for $q = j$ and any p. Then for each p

$$\begin{aligned}
[\mathfrak{g}_p,\mathfrak{g}_{j+1}] &= \big[\mathfrak{g}_p,[\mathfrak{g}_j,\mathfrak{g}_1]\big] \\
&= \big[\mathfrak{g}_j,[\mathfrak{g}_1,\mathfrak{g}_p]\big] + \big[\mathfrak{g}_1,[\mathfrak{g}_p,\mathfrak{g}_j]\big] \\
&= [\mathfrak{g}_j,\mathfrak{g}_{p+1}] + [\mathfrak{g}_1,\mathfrak{g}_{p+j}] \\
&= \mathfrak{g}_{(p+1)+j} + \mathfrak{g}_{(p+j)+1} \\
&= \mathfrak{g}_{p+(j+1)},
\end{aligned}$$

where the second line results from Jacobi's identity, and the third from the induction hypothesis. The proposition follows by induction. ■

Lemma 1 *$\mathfrak{h}_j \subseteq [\mathfrak{g}_1, \mathfrak{h}_{j-1}]$ for $j > 1$, and furthermore*

$$\mathfrak{h}_j \subseteq [\mathfrak{g}_1, \mathfrak{h}_{j-1}] \subseteq [\mathfrak{g}_1, \mathfrak{g}_{j-1}] = \mathfrak{g}_j. \tag{3}$$

Proof For $j > 1$, and letting $\sim$ denote difference of sets,

$$\begin{aligned}\mathfrak{h}_j &\subseteq \mathfrak{g}^{(j)} \sim \mathfrak{g}^{(j+1)} \\ &= [\mathfrak{g}, \mathfrak{g}^{(j-1)}] \sim \mathfrak{g}^{(j+1)} \\ &= [\mathfrak{h}_1 \oplus \mathfrak{g}^{(2)}, \mathfrak{g}^{(j-1)}] \sim \mathfrak{g}^{(j+1)}.\end{aligned}$$

By an elementary result, $[\mathfrak{g}^{(2)}, \mathfrak{g}^{(j-1)}] \subseteq \mathfrak{g}^{(j+1)}$, so we get $\mathfrak{h}_j \subseteq [\mathfrak{h}_1, \mathfrak{g}^{(j-1)}] \sim \mathfrak{g}^{(j+1)}$. Also since $\mathfrak{g}^{(j-1)} = \mathfrak{h}_{j-1} \oplus \mathfrak{g}^{(j)}$, by the same reasoning we get $\mathfrak{h}_j \subseteq [\mathfrak{h}_1, \mathfrak{h}_{j-1}] = [\mathfrak{g}_1, \mathfrak{h}_{j-1}]$. This proves the first statement.

$\mathfrak{h}_j \subseteq \mathfrak{g}_j$ for $j = 1$ by definition. Assuming for purposes of induction that $\mathfrak{h}_{j-1} \subseteq \mathfrak{g}_{j-1}$, we obtain from the above result

$$\mathfrak{h}_j \subseteq [\mathfrak{g}_1, \mathfrak{g}_{j-1}] = \mathfrak{g}_j.$$

and thus the second statement follows by induction. ■

A Lie algebra is said to be *graded* if for each $p \neq q$, $\mathfrak{g}_p \cap \mathfrak{g}_q = (0)$, or equivalently, $\mathfrak{g}_p = \mathfrak{h}_p$ for all p. For our purposes we require only the following weaker concept.

Definition 1 A Lie algebra $\mathfrak{g}$ will said to be quasi-graded if $\mathfrak{g}_p \cap \mathfrak{g}_q \neq (0)$ implies $p \equiv q \pmod 2$. A Lie group is quasi-graded if its Lie algebra is quasi-graded.

Example 1 Let N_n denote the nilpotent Lie group of $n \times n$ upper-diagonal matrices with ones along the main diagonal, whose Lie algebra is the set $\mathfrak{n}_n$ of strictly upper-diagonal matrices. A basis of $\mathfrak{n}_n$ is $\{e_{ij} : j > i, i = 1, \ldots, n-1\}$, where $e_{i,j}$ is the matrix with 1 in the i-th row and j-th column and zeros elsewhere, and $[A, B] = AB - BA$. Note that $[e_{ij}, e_{pq}] = e_{ij}e_{pq} - e_{pq}e_{ij} = e_{iq}\delta_{jp} - e_{pj}\delta_{qi}$.

The reader may check that in this case $\mathfrak{g}_j = \mathrm{sp}\{e_{i,i+j} : 1 \leq i \leq n-j\}$, and that since the $\mathfrak{g}_j$ are disjoint, N_n is graded, hence quasi-graded.

Other examples of graded nilpotent albebras are $\mathfrak{h}_n$, the $(2n+1)$-dimensional Heisenberg algebra, and $\mathfrak{k}_n$ (see Eg. 1.1.3 in [2]).

Example 2 Let $\mathfrak{s}$ denote the 8-dimensional sublalgebra of $\mathfrak{n}_5$ consisting of matrices of the form

$$\begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} & s_{15} \\ 0 & 0 & 0 & 0 & s_{25} \\ 0 & 0 & 0 & s_{34} & s_{35} \\ 0 & 0 & 0 & 0 & s_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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The basis elements e_{12}, e_{13}, e_{25}, e_{34}, e_{45} span $\mathfrak{s} \sim \mathfrak{s}^{(2)}$ so we must take $\mathfrak{s}_1$ to be the span of this set. One calculates that $\mathfrak{s}_2 = \operatorname{sp}\{e_{14}, e_{15}, e_{35}\}$, $\mathfrak{s}_3 = \operatorname{sp}\{e_{15}\}$, and $\mathfrak{s}_4 = (0)$. We have $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}_2 + \mathfrak{s}_3$ but e_{15} is in both $\mathfrak{s}_2$ and $\mathfrak{s}_3$. Thus $\mathfrak{s}$ is not quasi-graded.

Example 3 Let $\mathfrak{g}$ denote the 12-dimensional subalgbra of n_6 spanned by $\big\{ e_{pq} : (p,q) \in \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}\big\}$. By calculation one has

$$\mathfrak{g}_1 = \operatorname{sp}\big\{ e_{p,q} : (p,q) \in \{(1,2), (1,3), (2,6), (3,4), (4,5), (5,6)\}\big\},$$

$\mathfrak{g}_2 = \operatorname{sp}\big\{ e_{p,q} : (p,q) \in \{(1,6), (1,4), (3,5), (4,6)\}\big\}$, $\mathfrak{g}_3 = \operatorname{sp}\{e_{15}, e_{36}\}$, and $\mathfrak{g}_4 = \operatorname{sp}\{e_{16}\}$. $e_{1,6}$ is in both $\mathfrak{g}_2$ and $\mathfrak{g}_4$, and thus $\mathfrak{g}$ is quasi-graded, but not graded.

4 Main Results

We now restate the problem in the setting of the Lie algebra of G. From the remarks in the introduction we know that if we can find an automorphism ϕ of G and $a \in G$ such that $\phi(\Gamma) \subseteq \Gamma$ and $r^{-1} = a\phi(r)a^{-1}$, and we put $\psi = \phi a$, then ψ is a conjugacy between the rotations given by r and r^{-1}. (Such an automorphism induces an automorphism of $G' = \Lambda \setminus G$.) In $\mathfrak{g}$, to ϕ corresponds an isomorphism Φ of $\mathfrak{g}$ which maps $\gamma = \log(\Gamma)$ back into itself. To the inner automorphism $\alpha_x \colon x \to axa^{-1}$ corresponds the adjoint action of a on $\mathfrak{g}$, the automorphism of $\mathfrak{g}$ defined by $\exp\big((\operatorname{Ad} a)X\big) = \alpha_a\big(\exp(X)\big)$. We seek then an automorphism Φ of $\mathfrak{g}$ and $A \in \mathfrak{g}$ such that

$$(\operatorname{Ad}\exp A)\Phi(R) = -R \tag{4}$$

where $R = \log r$.

We shall make use of the formula

$$(\operatorname{Ad}\exp A)X = e^{\operatorname{ad} A}(X) = \sum_{k=0}^{\infty} \frac{1}{k!}(\operatorname{ad} A)^k X \tag{5}$$

for all $X \in \mathfrak{g}$, where $\operatorname{ad} AX = [A,X]$. (See [3, Section 2.13].) We note also that because G is nilpotent, the Campbell-Baker-Hausdorff formula holds on all of G [2, Section 2].

We shall relabel the basis $\{X_{m_p+1}, \dots, X_{m_{(p+1)}}\}$ of $\mathfrak{h}_p$ as $\{X_{p1}, \dots, X_{pn_p}\}$ (as ordered sets). By Campbell-Baker-Hausdorff formula, if $x = \exp x_1X_1 \cdot \exp x_2X_2 \cdot \dots \cdot \exp x_nX_n$, then

$$\log x = x_1X_1 + \dots + x_nX_n + \text{terms in } \mathfrak{g}^{(2)}.$$

Thus the component of $\log x$ in $\mathfrak{h}_1$ is $x_1X_1 + \dots + x_{m_2}X_{m_2}$. As in Section 1, if $R = R_1 + R_2 + \cdots$ is the decomposition of $R = \log r$ along the subsets $\mathfrak{h}_j$, then

$$R_1 = \rho_1X_1 + \dots + \rho_{m2}X_{m2}. \tag{6}$$

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Lemma 2 *If r is an ergodic rotation of $\Gamma \setminus G$, and R_1 is the component of $R = \log r$ in $\mathfrak{h}_1$, then* $\operatorname{ad} R_1$ *maps $\mathfrak{h}_p$ onto $\mathfrak{h}_{p+1}$ for each $p = 1, 2, \ldots$ (and $\mathfrak{g}_p$ onto $\mathfrak{g}_{p+1}$).*

Proof Using the notation of the preceeding paragraph, $X \in \mathfrak{h}_p$ has the expansion $X = \sum_{j=1}^{\infty} x_j X_{pj}$, and

$$\begin{aligned}(\operatorname{ad} R_1)(X) = [R_1, X] &= \Big[\sum_{i=1}^{m_1} \rho_i X_i, \sum_{j=1}^{n_p} x_j X_{pj}\Big] \\ &= \sum_{i=1}^{m_1}\sum_{j=1}^{n_p} \rho_i x_j [X_i, X_{pj}].\end{aligned}$$

Since $\mathfrak{h}_{p+1} \subseteq [\mathfrak{h}_1, \mathfrak{h}_p]$, the $[X_i, X_{pj}]$ span $\mathfrak{h}_{p+1}$, and since the ρ_j are rationally independent, they are non-zero, so as X varies, the double sum maps onto all of $\mathfrak{h}_{p+1}$. The proof for the $\mathfrak{g}_j$ is similar. ■

Proposition 2 *If $\mathfrak{g}$ is quasi-graded, and $\Phi\colon \mathfrak{g} \to \mathfrak{g}$ is defined by $\Phi(g) = (-1)^p g$ for $g \in \mathfrak{g}_p$, and extended to all of $\mathfrak{g}$ by linearity, then Φ is well-defined and defines an automorphism of $\mathfrak{g}$ which leaves* $\log \Gamma$ *unchanged.*

Proof Φ is well defined by the definition of quasi-graded, and on the basis elements of $\mathfrak{g}_p$ and $\mathfrak{g}_q$

$$[\Phi(X_{pi}), \Phi(X_{qj})] = [(-1)^p X_{pi}, (-1)^q X_{qj}] = (-1)^{p+q}[X_{pi}, X_{qj}],$$

and $\Phi([X_{pi}, X_{qj}]) = (-1)^{p+q}[X_{pi}, X_{qj}]$ since $[X_{pi}, X_{qj}] \in \mathfrak{g}_{(p+q)}$. Thus Φ extends a Lie algebra automorphism of $\mathfrak{g}$. If we let $\phi = \exp \circ \Phi$, then ϕ is an automorphism of G such that $\phi\big(\exp(X_j)\big) = \exp(\pm X_j)$. Thus ϕ has the effect of changing signs of the a_j in (1), which clearly leaves Γ unchanged, as required. ■

Note that in the above proof we can not assume that $\log \Gamma$ is a subalgebra, nor even an additive subgroup of G. In this case Γ would be called a *lattice* subgroup. An example where the uniform subgroup is not a lattice is $G = N_3$, with Γ the discrete subgoup with integral entries.

We now come to the first of our promised results.

Theorem 1 *Let $\mathfrak{g}$ be a quasi-graded nilpotent Lie algebra and Φ be as in Proposition 3. If $R = \log r$ where r is an ergodic rotation of $\Gamma \setminus G$, then there is an $A \in \mathfrak{g}$ such that* $(\operatorname{Ad} \exp A)\Phi(R) = -R$.

Proof In view of equations (4) and (5), we are required to solve

$$\Phi(R) + [T, \Phi(R)] + \frac{1}{2!}\big[T, [T, \Phi(R)]\big] + \cdots = -R \tag{7}$$

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for $T \in G$. We solve this recursively along the linear subspaces $\bigoplus_1^m \mathfrak{h}_j$. Let $T = T_1 + T_2 + \cdots$ and $R = R_1 + R_2 + \cdots$ where $T_j, R_j \in \mathfrak{h}_j$. Collecting terms in (7) in $\bigoplus_1^2 \mathfrak{h}_j$, and noting $\Phi(R_1) = -R_1$ we get $[T_1, -R_1] = -2R_2+$ terms in $\mathfrak{h}_j$ for $j > 2$. The part of this in $\bigoplus_1^2 \mathfrak{h}_j$ can be solved for T_1 by Lemma 2, thus satisfying equation (7) as far as terms in $\mathfrak{h}_2$ are concerned.

Now assume for purposes of induction that equation (8) has been solved for terms in $\bigoplus_1^m \mathfrak{h}_j$, yielding solutions for $T_1, \ldots, T_{m-1}$. The terms of equation (7) in $\mathfrak{h}_{m+1}$ are given by

$$\begin{aligned}\Phi(R_{m+1}) + \sum_{j=1}^{m} [T_j, \Phi(R_{(m-j+1)})]& \\ + \sum_{i+j+k=m+1} [T_i, [T_j, \Phi(R_k)] + \cdots + [T_1, [T_1, \ldots, [T_1, \Phi(R_1)] \cdots]& \\ = -R_{m+1},&\end{aligned}$$

where m T_1s appear in the last bracket. T_m appears only in the $[T_m, \Phi(R_1)]$ term and so T_m can be determined so that this equation is satisfied, again by Lemma 2. Thus (7) can be solved recursively for T, and we put $A = T$. ■

In view of the comments opening Section 4, we have:

Corollary 1 *If r is an ergodic rotation of nilmanifold $\Gamma \setminus G$ where G is a nilpotent Lie group, and Γ is a uniform subgroup, then r is metrically conjugate to its inverse rotation if G is quasi-graded.*

The conjugacy is given by $\Gamma x \to \Gamma x \phi a = \Gamma x \psi$ where $\phi(x) = \exp\big(\Phi(X)\big)$ and $a = \exp(A)$. Then $\psi^2 = \phi a \phi a = \phi^2 \phi(a) a = \phi(a) a$ since clearly Φ^2 is the identity on $\mathfrak{g}$, so ϕ^2 is the identity on G. Thus ψ is an involution if and only if $\phi(a) = a^{-1}$, or equivalently, $\Phi(A) = -A$.

Theorem 2 *Under the conditions of Theorem 1, A may be chosen so that $\Phi(A) = -A$.*

Proof A is defined by the equation

$$(8) \qquad (\operatorname{Ad} \exp A) \Phi(R) = -R.$$

From $\alpha_a^{-1} = \alpha_{a^{-1}}$ we see that applying $\operatorname{Ad} \exp(-A)$ to both sides of this equation gives

$$\big(\operatorname{Ad} \exp(-A)\big)(-R) = \Phi(R).$$

Applying the automorphism Φ to both sides of this yields

$$\Big(\operatorname{Ad} \exp\big(\Phi(-A)\big)\Big) \Phi(-R) = R.$$

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Since this equation is linear in R, we get

$$\Big(\mathrm{Ad}\exp\big(\Phi(-A)\big)\Big)\,\Phi(R) = -R. \tag{9}$$

Comparing (8) and (9), we conclude that $\Phi(A) = -A$ if the solution of (8) is unique when confined to a subset which is invariant under Φ.

To find A a series of linear equations $[T_j, R_1] = B_j$ were solved for the T_j recursively. The general solution for each j is $A_j + H_j$ where $H_j \in \mathfrak{s}_j$ where $\mathfrak{s}_j$ is the subspace $\{Y_j \in \mathfrak{g}_j : [Y_j, R_1] = 0\}$. Thus we may choose A_j to be in the orthocomplement of $\mathfrak{s}_j$ (regarding $\mathfrak{h}_j$ as $\mathbf{R}_{n_j}$ where n_j is the dimension of $\mathfrak{h}_j$). This orthocomplement is invariant under Φ, since on $\mathfrak{g}_j$, $\Phi = (-1)^j$ times the identity. Under this condition on A_j for each j, the solution of (8) is unique, and we get $\Phi(A) = -A$, as required. ■

In view of the remarks preceeding Theorem 2, we have therefore

Corollary 2 *The conjugacy between r and r^{-1} in Corollary 1 may be chosen to be an involution.*

Example 1 (continued) N_n is graded and thus Corollaries 1 and 2 apply. In this case, $\mathfrak{h}_j = \mathfrak{g}_j$ consists of those matrices with non-zero entries only along the j-th diagonal above the main diagonal. If one carries out the calculations in the proof of Theorem 1, solving for T_j leads to $(n - j - 1)$ equations in the $n - j$ variables in the j-th diagonal above the main diagonal of T_j. Each $\mathfrak{s}_j$ (in Theorem 2) is thus one dimensional for each $j = 1, \dots, n-1$. Thus Theorem 1 yields an an $(n-1)$-dimensional continuum of conjugacies in this case.

For the particularly simple case of $n = 3$, letting

$$R = \langle a, b, c\rangle = ae_{12} + be_{23} + ce_{13},$$

where e_{ij} is the matrix with 1 in the ij-th position and with all other entries 0, then the above procedure leads to

$$A = \left\langle \left(\frac{2cb}{a^2+b^2} + ta\right), \left(\frac{-2ac}{a^2+b^2} + tb\right), 0 \right\rangle \qquad (t \in \mathbf{R})$$

which yields a continuum of conjugacies which are involutions. But adding se_{13}, $s \in \mathbf{R}$, to A leads to a conjugacy ψ such that

$$\psi\big(\psi(\langle\!\langle x, y, z\rangle\!\rangle)\big) = \langle\!\langle x, y, z - 2s\rangle\!\rangle,$$

(where $\langle\!\langle x, y, z\rangle\!\rangle$ is the member of N_3 obtained from $\langle x, y, z\rangle$ by putting 1 in each main diagonal entry), so we also have a continuum of conjugacies which are of infinite order.

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5 Necessary Conditions

Our proof that, in at least a restricted subclass of algebras, the condition of being quasi-graded is also necessary is based on the following result. It is clear from the discussion at the beginning of Section 4 that condition (4) is also necessary for $r = \exp(R)$ to be conjugate to its inverse.

Proposition 3 *Suppose $r = \exp(R)$ is an ergodic rotation of $\Gamma \setminus G$ (where G is not necessarily quasi-graded). If Φ is an homomorphism of $\mathfrak{g}$ such that $(\operatorname{Ad}\exp A)\Phi(R) = -R$, then on $\mathfrak{h}_1$, $\Phi(X) = -X \pmod{\mathfrak{g}^{(2)}}$.*

Proof By looking at equation (5), one sees that $(\operatorname{Ad}\exp A)(X)$ leaves the $\mathfrak{h}_1$ component of X unchanged. Thus Φ must satisfy $\Phi(R_1) = -R_1 + Y$ where R_1 is the $\mathfrak{h}_1$-component of R and $Y \in \mathfrak{g}^{(2)}$. If we put $\Phi(X_i) = \sum_{j=1}^{n} c_{ij}X_j$, then

$$\Phi(R_1) = \Phi\Big(\sum_{i=1}^{m_2} \rho_i X_i\Big) = \sum_{i=1}^{m_2} \rho_i \sum_j c_{ij}X_j = \sum_j \Big(\sum_{i=1}^{m_2} \rho_i c_{ij}\Big) X_j.$$

We must have therefore $\sum_{i=1}^{m_2} \rho_i c_{ij} = -\rho_j$, $j = 1, \dots, m_2$, or

$$\sum_{i=1}^{m_2} \rho_i (c_{ij} + \delta_{ij}) = 0, \tag{10}$$

for each $j = 1, \dots, m_2$.

The requirement that ϕ map Γ into itself, so that $\Phi\big(\log(\Gamma)\big) \subseteq \log(\Gamma)$, forces the c_{ij}, $j = 1, \dots, m_2$ to be integral. For

$$\phi\big(\exp(X_i)\big) = \exp\big(\Phi(X_i)\big) = \exp\Big(\sum_{j=1}^{n} c_{ij}X_j\Big) = \prod_{j=1}^{m_2} \exp(c_{ij}X_j)\gamma_2$$

where $\gamma_2 \in G^{(2)}$, using the Campbell-Baker-Hausdorff formula, and the abelian property of $G/G^{(2)}$. This must be of the form $\prod_1^n \exp(b_j X_j)$, $b_j \in \mathbf{Z}$ (where $\prod_{m_2+1}^{n} \exp(b_j X_j) \in G^{(2)}$), so that the c_{ij}, $i, j = 1, \dots, m_2$, are integral.

The algebraic independence of the ρ_j in (10) then ensures that $c_{ij} = -\delta_{ij}$ for $i, j = 1, \dots, m_2$. Thus $\Phi(X_i) = -X_i + Y_i$ where $Y_i \in \mathfrak{g}^{(2)}$ for $i = 1, 2, \dots, m_2$, as required. ■

The algebras in our examples in Section 3 all had bases consisting of the elements e_{ij}, $i < j$, as did the subspaces $\mathfrak{g}_k$, and thus also their intersections. The basis elements can be formed by recursively taking products of basis elements with the basis

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elements $\{X_1, \ldots, X_{m_2}\}$ of $\mathfrak{g}_1$. Note that for a fixed $i = 1, \ldots, m_2$, the non-zero values of $[X_i, g]$ are distinct for different values of g taken from the basis vectors of $\mathfrak{g}$. Thus since $\mathfrak{g}_1 \cap \mathfrak{g}^{(2)} = (0)$, if $[X_i, X_j] \neq 0$,

$$[X_i, X_j] \notin [X_i, \mathfrak{g}^{(2)}] + [X_j, \mathfrak{g}^{(2)}]. \tag{11}$$

The class of algebras we delineate, for which the property of quasi-graded is also necessary, incorporates some of the above properties. Recall that an algebra is of level n if $\mathfrak{g}^{(n)} \neq (0)$, and $\mathfrak{g}^{(n+1)} = (0)$.

Theorem 3 *Let $\mathfrak{g}$ belong to the class of those Lie algebras of level 3, with $\mathfrak{g}_2 \cap \mathfrak{g}_3$ having a basis consisting of elements $[X_i, X_j]$ where $\{X_1, \ldots, X_{m_2}\}$ is the basis of $\mathfrak{g}_1$, and for which (11) holds for each i, j. If $\mathfrak{g}$ admits an automorphism Φ satisfying $\Phi(X) = -X$ $(\mathrm{mod}\ \mathfrak{g}^{(2)})$, then $\mathfrak{g}$ must be quasi-graded.*

Proof If $\mathfrak{g}$ is not quasi-graded, then $\mathfrak{g}_2 \cap \mathfrak{g}_3 \neq (0)$ so that there is a basis element $g = [X_i, X_j] \in \mathfrak{g}_2 \cap \mathfrak{g}_3$. By Proposition 3, $\Phi(X_i) = -X_i + Y_i$ where $Y_i \in \mathfrak{g}^{(2)}$. Thus

$$\begin{aligned} \Phi(g) &= [-X_i + Y_i, -X_j + Y_j] \\ &= g - [X_i, Y_j] + [X_j, Y_i], \end{aligned}$$

since $[Y_i, Y_j] \in \mathfrak{g}^{(4)} = (0)$. But since $g \in \mathfrak{g}^{(3)}$, similar reasoning shows that $\Phi(g) = -g$. From these two values for $\Phi(g)$, we get $g = ([X_i, Y_j] - [X_j, Y_i])/2$. Condition (11) implies that $g = 0$. Thus $\mathfrak{g}_2 \cap \mathfrak{g}_3 = (0)$ also, so $\mathfrak{g}$ is quasi-graded, as required. ■

Thus for groups whose algebra are of the class described in the theorem, an ergodic rotation of a nilmanifold is isomorphic to its inverse if and only if it is quasi-graded. As noted previously, Example 2 is in this class, but is not quasi-graded, and thus no rotation of a nilmanifold arising from it is conjugate to its inverse.

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Department of Mathematics
Trent University
Peterborough, Ontario
K9J 7B8
email: jhenniger@trentu.ca