PULSE DIFFRACTION BY AN IMPERFECTLY REFLECTING WEDGE

V. M. PAPADOPOULOS

(received 12 July 1960, revised 29 August 1960)

Abstract

A similarity method is used to develop a solution of the wave equation within a sector with mixed boundary conditions. In this manner the field which results from the diffraction of an incident pulse of step function time dependence is found.

1. Introduction

There have been two methods described recently by which the two dimensional problem of the diffraction of steady sinusoidal acoustic waves by an imperfectly reflecting infinite wedge may be examined. These methods, which were derived in the solution of problems involving waves in a shallow fluid, are those of Stoker [1] and Peters [2] (1952).

For the wave diffraction problem the precise nature of the disturbance within the imperfectly reflecting wedge has been ignored, and its effect replaced by the imposition of an impedance type of boundary condition at the surface. Stoker's method has been applied by Karp and others [3, 4, 5] and Peters' method by Senior [6] (1959) to the idealized problem which may be summarized as follows. A quantity $\varphi(r, \theta)$ is to be calculated which satisfies, in the region $\theta_0 < \theta < \theta_1$, the reduced wave equation

$$(\nabla^2 + k^2)\varphi(r, \theta) = 0,$$

under boundary conditions

$$\frac{\partial\varphi}{\partial\theta}\begin{cases} = ik\eta_0\,\varphi \text{ for } \theta = \theta_0, \\ = -ik\eta_1\,\varphi \text{ for } \theta = \theta_1, \end{cases}$$

for various incident fields. The quantities η which represent the reciprocal of the complex refractive index at the surface of the wedge are taken to be constant on the surface. The problem as stated is a little more general than that solved either by Senior, who specified that $\eta_0 = \eta_1$, or by Karp who restricted his calculation to right-angled wedges.

The present paper, which follows a number of others which utilize the same method, is intended to show that formal closed solutions which demonstrate the most important features of the diffracted field may be found more simply if the usual assumption of steady harmonic time dependence is not made. Rather we can consider an incident plane wave of step function time dependence and we can use the method of dynamic similarity to derive the solution for an incident pulse. This solution may then be converted, if necessary, to the steady solution.

The method of dynamic similarity is very closely related to the method of conical flows which has been described in detail by Goldstein and Ward [7]. Its application to certain transient two dimensional problems of supersonic flow was made by Craggs [7, 8]. It has also been applied to a series of problems of wave diffraction and refraction by the author [10, 11, 12]. Pulse diffraction by a perfectly conducting wedge was also examined by a similarity method by Keller and Blank [13].

The essence of the method of dynamic similarity is to consider in a two dimensional problem a function $\varphi(r, \theta, t)$ which satisfies the wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi(r, \theta, t) = 0, \tag{1}$$

and for which it is possible to assume that after a particular instant $t = 0$, the solution is of the specific form $\varphi(r/t, \theta)$. It will be seen that the scattered field which follows the arrival of a plane pulse at the vertex of the wedge is of this form. Following this assumption it is easily shown that for $r > ct$, under the transformation $r = ct \sec u$, φ satisfies the hyperbolic equation

$$-\frac{\partial^2\varphi}{\partial u} + \frac{\partial^2\varphi}{\partial\theta^2} = 0, \tag{2}$$

for which the general solution is

$$\varphi = f(u - \theta) + g(u + \theta), \tag{3}$$

where f and g are arbitrary functions which are constant on the characteristic lines $u + \theta = \text{const}$ and $u - \theta = \text{const}$ respectively.

If $r < ct$, under the transformation $r = ct \operatorname{sech}(-v)$, φ satisfies the elliptic equation

$$\frac{\partial^2\varphi}{\partial v^2} + \frac{\partial^2\varphi}{\partial\theta^2} = 0, \tag{4}$$

in a strip $\theta_0 < \theta < \theta_1$, $v < 0$, in the (v, θ) plane. The minus sign is introduced to define the correct branch for the inverse relationship, viz.

$$v = \ln\left\{\frac{ct}{r}\left(1 - \left[1 - \frac{r^2}{c^2t^2}\right]^{1/2}\right)\right\}.$$

The method is therefore seen to be one of reducing the diffraction problem to a problem in harmonic function theory. This was also the aim of Peters' method which was applied to the steady state problem. And this harmonic function problem has received further attention in an interesting series of papers by Van Dantzig [14] and Lauwerier [15].

2. The solution in the hyperbolic region $r > ct$

The problem is first set up as a problem in acoustics. In the linearized theory we may set up a velocity potential φ such that the velocity v $(= -\nabla\varphi)$ and the infinitesimal pressure change p $(= \rho\, \partial\varphi/\partial t)$ satisfy the momentum equation to the first order, where ρ is the density of the medium at rest. Then if c is the velocity of sound in the medium at rest, φ satisfies the wave equation (1).

We consider the walls of the wedge to be soft in the sense that the pressure is proportional to the normal velocity. If we take the boundaries to be the planes $\theta = 0$, and $\theta = \psi$, the appropriate conditions have the form

$$\frac{\partial\varphi}{\partial t}(r, 0, t) = \frac{\alpha}{r}\frac{\partial\varphi}{\partial\theta} \qquad \text{for } \theta = 0,$$

and

$$\frac{\partial\varphi}{\partial t}(r, \psi, t) = \frac{-\beta}{r}\frac{\partial\varphi}{\partial\theta} \qquad \text{for } \theta = \psi, \tag{5}$$

where α and β are constants which give a measure of the softness of the boundaries. For the purpose of generality we do not take α and β to be equal.

We now consider the case of a plane pulse of potential φ which is travelling towards the vertex of the wedge within a region of angle ψ. We take the pulse to have a step function profile so that it corresponds to a plane pressure impulse. Depending on the angle of incidence γ of the pulse and the angle ψ of the wedge there may be a number of reflected pulses which accompany the incident pulse. The determination of the complete incident field which satisfies the correct boundary conditions is not a difficult matter since the situation is a steady one with no diffraction effects present.

For simplicity, however, we will examine only the case as depicted in fig. 1 where the incident pulse does not meet the walls of the wedge until after it arrives at the vertex at time $t = 0$. We can now assert that the subsequent disturbance has a potential which depends only on the variables r/t and θ, and for this disturbance we can use the transformation described in the introduction. The hyperbolic region $r > ct$ is one in which the solution takes the form given in equation (3), and since the initial field at $t = 0$ determines the field as $r/t \to \infty$ we are able to determine the field in the whole region $r > ct$. The field structure is shown in figure 2, where the polar co-ordinates r/t and θ are used. In this co-ordinate system the characteristic lines, $u \pm \theta =$ constant, are the half-tangents to the circle $r = ct$, which being the envelope of characteristics is itself a characteristic separating the

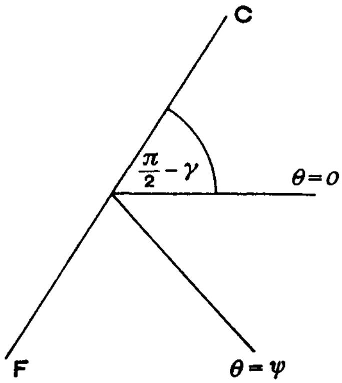

Fig. 1. Initial state with pulse FC moving steadily to the right.

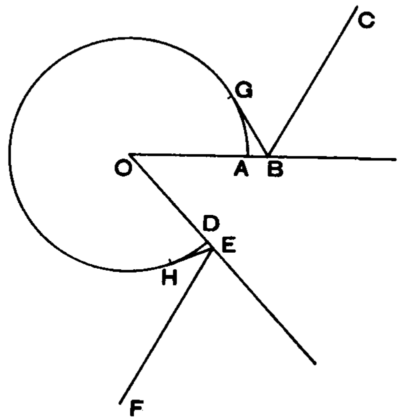

Fig. 2. Subsequent state showing the field structure.

hyperbolic from the elliptic region. The initial condition of an incident pulse just reaching the vertex fixes the value of the potential at $r/t = \infty$, and this immediately leads to the result that the lines BC, EF, BG and EH are the characteristic lines which separate distinct regions of constant potential. To the right of the lines BC and EF the potential is 0, and to the left it is 1, except within the regions ABG where it is $1 + R_0$ and within DEH where it is $1 + R_\psi$. These constants R_0 and R_ψ are the reflexion coefficients to be expected when a pulse is reflected under the boundary condition (5) on an infinite plane surface.

In particular near the surface $\theta = 0$ we have the result

$$\varphi = U(\theta + \gamma - u) + R_0 U(\gamma - \theta - u),$$

where $U(z)$ is the unit step function, and from (5) the boundary condition to be satisfied is that

$$\frac{c}{\sin u}\frac{\partial \varphi}{\partial u} + \alpha \frac{\partial \varphi}{\partial \theta} = 0.$$

It follows that

$$R_0 = \frac{\alpha \sin \gamma - c}{\alpha \sin \gamma + c}. \tag{6}$$

Similarly near the surface $\theta = \psi$

$$\varphi = U(-\gamma + 2\pi - u - \theta) + R_\psi U(\theta + 2\pi - u - 2\psi - \gamma);$$

since the boundary condition on $\theta = \psi$ is

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$$\frac{c}{\sin u}\frac{\partial\varphi}{\partial u} - \beta\frac{\partial\varphi}{\partial\theta} = 0,$$

it follows that

$$R_\psi = \frac{\beta \sin\,(2\pi - \psi - \gamma) - c}{\beta \sin\,(2\pi - \psi - \gamma) + c}. \qquad (7)$$

We may note in passing that when the angle of incidence at the surface $\theta = \theta_0$ is $\sin^{-1} c/\alpha$, the reflexion coefficient R_0 vanishes, and if the angle of incidence at the other surface is $\sin^{-1} c/\beta$, R_ψ vanishes; this is a situation when an incident pulse is completely absorbed at one surface or the other.

The principal result, however, is that just outside the circle $r = ct$ the potential is piecewise constant, and therefore in particular that the tangential derivative $\partial\varphi/\partial\theta$ vanishes on this circle except at the singular points $\theta = \gamma$ and $\theta = 2\psi - \gamma + 2\pi$.

3. The solution in the elliptic region $r < ct$

The calculations so far have been of a rather trivial nature. They have served, however, to separate the main reflection effects of the surfaces of the wedge from the diffraction effects of the vertex.

Within the sonic circle, we have seen that the potential is an harmonic function in v and θ. Because this circle, being the envelope of the characteristics in the hyperbolic region, is also a characteristic, the only link which can subsist between the elliptic and the hyperbolic region is that *the tangential derivative of* φ (or the tangential velocity component) *must be continuous.* On introducing a conjugate harmonic we are led to consider a complex potential W with $Rl(W) = \varphi$. Conditions to be satisfied by W which are relevant to the solution may be derived as follows:

The situation as already depicted leads us to expect singularities on the sonic circle at the points, G and H, and in addition we can expect singularities only at the vertex of the wedge and possibly at the points A and D. Within the strip $0 < \theta < \psi$, $v < 0$ of the complex $(v + i\theta)$-plane, W and its complex derivatives must be regular functions, since all the possible singularities are on the boundary. The Cauchy Riemann conditions enable us to make the identity

$$\frac{dW}{dv} = \frac{\partial\varphi}{\partial v} - i\frac{\partial\varphi}{\partial\theta},$$

and it follows that on the boundary with $v = 0$, $0 < \theta < \psi$, dW/dv is real, with simple poles at $\theta = \gamma$, $\theta = 2\pi + 2\psi - \gamma$, where φ has a step discontinuity, the residues at these poles being proportional to the jumps in φ.

The boundary conditions on the absorbing wedge likewise take the form

$$\frac{c}{\sinh v}\frac{\partial\varphi}{\partial v} = \begin{cases} +\alpha\dfrac{\partial\varphi}{\partial\theta} & \text{on } \theta = 0 \\ -\beta\dfrac{\partial\varphi}{\partial\theta} & \text{on } \theta = \psi \end{cases} \tag{8}$$

Hence on $\theta = 0$

$$\text{Rl}\,\frac{dW}{dv}\left[\frac{c}{\sinh v} - i\alpha\right] = 0,$$

or

$$\arg\frac{dW}{dv} = -\arc\tan c/\alpha \sinh v, \tag{9}$$

while on $\theta = \psi$

$$\text{Rl}\,\frac{dW}{dv}\left[\frac{c}{\sinh v} + i\beta\right] = 0,$$

or

$$\arg\frac{dW}{dv} = +\arc\tan c/\beta \sinh v. \tag{10}$$

It is convenient in writing down the solution to map the strip in the complex $(v + i\theta)$-plane into the upper half ζ $(= \xi + i\eta)$ plane by the transformation $\zeta = \text{sech}\, -\pi(v + i\theta)/\psi$ or $(v + i\theta)\pi/\psi = \ln\,[1 - (1 - \zeta^2)^{1/2}]/\zeta$. The conditions to be satisfied on the ξ-axis by the derivative $dW/d\zeta$ are that

1. $dW/d\zeta$ is imaginary on the segments $\xi < -1$, $\xi > 1$ except at simple poles at the points $\xi = \sec\gamma\pi/\psi$ and $\sec(\gamma - 2\pi)\pi/\psi$. These conditions follow from the vanishing of $\partial\varphi/\partial\theta$ on the sonic circle, as mentioned above.

2. On the segment $0 < \xi < 1$, which corresponds to the boundary $\theta = 0$ $r < ct$, it follows from equation 9 that

$$\arg\frac{dW}{d\zeta} =$$

$$-\arc\tan\left\{\frac{2c}{\alpha}\zeta^{\psi/\pi} \Big/ [1-(1-\zeta^2)^{1/2}]^{\psi/\pi} - (1 + (1-\zeta^2)^{1/2}]^{\psi/\pi}\right\} = -L(\zeta, \alpha).$$

3. On the segment $-1 < \xi < 0$ which corresponds to the boundary $\theta = \psi$, $r < ct$ it follows from equation 10 that

$$\arg\frac{dW}{d\zeta} = +L(\zeta, \beta).$$

4. The point at infinity in the ζ-plane which corresponds to the point $\theta = \psi/2$ on the sonic circle can only be an ordinary point so that both W and $dW/d\zeta$ must be regular at infinity. This restricts the solution to the class of those for which as $|\zeta| \to \infty$, $dW/d\zeta = 0(\zeta^{-1-\delta})$ with δ a positive constant.

5. The singularity at the origin is one which cannot be regarded as a

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source of energy. It follows that as $\zeta \to 0$, $dW/d\zeta$ is $0(\zeta^{-1+\varepsilon})$ with ε positive.

It is the conditions (2) and (3) to which we must turn first. We may examine the subsidiary potential problem of finding an expression for $dW/d\zeta$ which has the required argument on the two segments $-1 < \xi < 0$ and $0 < \xi < 1$ and which is real on the remainder of the real ξ-axis. If we write $\ln dW/d\zeta$ in the form of a Cauchy integral

$$\ln \frac{dW}{d\zeta} = \ln \left|\frac{dW}{d\zeta}\right| + i \arg \frac{dW}{d\zeta} = \frac{1}{\pi}\int_{-1}^{+1} \frac{f(z)}{z-\zeta}\, dz$$

then it is easily seen that when ζ approaches the real axis on the segment from above with $f(z)$ a real function, the imaginary part of the integral is $f(\zeta)$, and if ζ approaches real values outside the segment $-1 < \xi < 1$ the imaginary part is zero. Accordingly we can identify the function $f(z)$ with the argument defined in conditions (2) and (3), so that we may write

$$\frac{dW}{d\zeta} = A(\zeta) \exp\left[-\frac{1}{\pi}\int_0^1 \frac{L(t,\alpha)}{t-\zeta}\, dt + \frac{1}{\pi}\int_{-1}^0 \frac{L(t,\beta)}{t-\zeta}\, dt\right] = A(\zeta)\exp T(\zeta), \text{say}, \quad (11)$$

where we may choose $A(\zeta)$ in order to satisfy the remaining conditions. The exponential factors are bounded both at $\zeta = 0$ and as $|\zeta| \to \infty$. The presence of these factors enables us to take $A(\zeta)$ to be real on the segment $-1 < \xi < 1$. We may combine the remaining conditions under the assertion that if

$$A(\zeta) = \frac{B(\zeta)}{(1-\zeta^2)^{1/2}}\left\{\frac{L}{\zeta - \sec \gamma\pi/\psi} + \frac{M}{\zeta - \sec(\gamma - 2\pi)\pi/\psi}\right\}$$

L and M being constants, then $B(\zeta)$ is a function which is real and bounded on the whole of the real axis with no singularities on the axis, as well as being regular in the whole upper half ζ-plane. Accordingly by an extension of Liouville's theorem, following a reflection in the real axis of the upper into the lower half plane, $B(\zeta)$ must be a constant. Since we already have available the constants L and M, this may be taken to be 1. The constants L and M may now be found since the residue of $dW/d\zeta$ at the poles $\zeta = \sec \pi\gamma/\psi$ and $\sec(2\pi - \gamma)\pi/\psi$ must be proportional to the discontinuity in potential. It follows that L and M are defined by the identities

$$R_0 = \pi L \cot \gamma\pi/\psi \exp T[\sec(\gamma\pi/\psi)],$$

and

$$R_\psi = -\pi M \cot(\gamma - 2\pi)\pi/\psi \exp T[\sec(\gamma - 2\pi)\pi/\psi].$$

Thus the complete solution for the complex derivative is that

$$\pi\frac{dW}{d\zeta} = \frac{\exp T(\zeta)}{(1-\zeta^2)^{1/2}}\left\{\frac{R_0 \tan \gamma\pi/\psi \exp - T(\sec \gamma\pi/\psi)}{\zeta - \sec \gamma\pi/\psi} - \frac{R_\psi \tan(\gamma - 2\pi)\pi/\psi \exp - T(\sec(\gamma - 2\pi)\pi/\psi)}{\zeta - \sec(\gamma - 2\pi)\pi/\psi}\right\}, \quad (12)$$

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with R_0 and R_ψ defined in (6) and (7) and $T(\zeta)$ in (11). This result is sufficient to determine the pressure and the velocity components everywhere within the sonic circle since we have the identities

$$rp = -\rho s^2 \partial\varphi/\partial s, \quad rV_r = -s\partial\varphi/\partial s \text{ and } rV_\theta = -\partial\varphi/\partial\theta,$$

where $s = r/t$ and V_r, V_θ are the velocity components. The derivatives $\partial\varphi/\partial s$ and $\partial\varphi/\partial\theta$ are given by the equations

$$\frac{\partial\varphi}{\partial s} = \frac{\pi}{\psi s\left(1 - \dfrac{s^2}{c^2}\right)^{1/2}} \mathrm{Rl}\left[\frac{dW}{d\zeta}\zeta(1-\zeta^2)^{1/2}\right],$$

and

$$\frac{\partial\varphi}{\partial\theta} = \frac{\pi}{\psi}\mathrm{Rl}\left[i\frac{dW}{d\zeta}\zeta(1-\zeta^2)^{1/2}\right],$$

ζ being given explicitly by the relation

$$\zeta = \frac{2\left(\dfrac{r}{ct}\right)^{\pi/\psi}}{\exp(i\theta\pi/\psi)[1-(1-r^2/ct^2)^{1/2}]^{\pi/\psi} + \exp -i\theta\pi/\psi[1+(1-r^2/ct^2)^{1/2}]^{\pi/\psi}}.$$

4. The results for incident plane waves of steady sinusoidal time dependence

The results given in the previous sections are associated with plane waves of potential which have a step function profile. What is more to the point, since we have more explicit information about the pressure, the incident field is a plane impulse of pressure of delta function profile. Since the diffraction problem is a linear one the superposition principle may be used to set up an incident plane wave of arbitrary form, but since there is considerable interest in the case of steady sinusoidal waves we shall only consider this case. The means of deriving the scattered field which corresponds to the incident field

$$p = \rho c\delta[ct - r\cos(\theta+\gamma)]$$

is the Laplace transform. Given the corresponding scattered field p_{sc} as derived from equation 12, and its Laplace transform $\int_0^\infty p_{sc} \exp -\sigma t\, dt$ it follows that the scattered field which corresponds to an incident field of form $\exp iw[tc - r\cos(\theta+\gamma)]/c$ is

$$\frac{\exp iwt}{\rho c}\int_0^\infty p_{sc}\exp -iwt\, dt$$

and this may be transformed on putting $s = r/t$ in the integrand into the formula

$$-\frac{\exp iwt}{\rho c}\int_0^\infty \frac{\partial\varphi}{\partial s}\exp(-iwr/s)ds$$

And since this integral is essentially an integral over the range of t^{-1}, keeping r and θ fixed, the integral will still be correct if we regard the constants α and β to be complex with a non-negative real part, this being a case of interest in electromagnetic theory. Indeed we may also take the quantity c to be a complex constant without altering the fact that we have a solution of the reduced wave equation. Thus we have a formal solution which holds for a medium with finite conductivity in the presence of a complex impedance boundary condition.

5. Electromagnetic theory

Although the solution has been given for a problem in acoustics, it is exceedingly simple to make the change to electromagnetic theory. For the two dimensional problem of diffraction by a wedge we have the usual separation into E and H polarizations with independent fields linearly dependent on the electric and magnetic field components parallel to the edge of the wedge.

Thus if we take a vector potential with on the z-component non-zero and equal to φ we can construct either the electrical polarization with

$$\mathbf{k}\cdot\mathbf{E} = E_z = \partial\varphi/\partial t, \qquad \mathbf{B} = -\nabla\times\mathbf{k}\varphi,$$

or the magnetic polarization with

$$c^2\mathbf{k}\cdot\mathbf{B} = c^2B_z = \partial\varphi/\partial t, \qquad \mathbf{E} = \nabla\times\mathbf{k}\varphi.$$

The corresponding boundary conditions in the two cases depend on the surface resistance R and the magnetic permeability μ of the region $0 < \theta < \psi$; they have the form

$$\left.\begin{aligned} B_r &= -\mu E_z/R_1 \\ B_z &= \mu E_r/R_1 \end{aligned}\right\} \text{ on } \theta = 0,$$

and

$$\left.\begin{aligned} B_r &= \mu E_z/R_2 \\ B_z &= -\mu E_r/R_2 \end{aligned}\right\} \text{ on } \theta = 0.$$

For the electric polarization then we have the conditions that $\alpha = R_1/\mu$, $\beta = R_2/\mu$ and for the magnetic polarization the conditions are that $\alpha = \mu c^2/R_1$, $\beta = \mu c^2/R_2$; the results follow directly from the solution of the acoustic problem.

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Brown University,
Providence, R.I.

now at University of Melbourne,
Victoria, Australia.

The research reported in this paper has been sponsored in part by the Air Force Cambridge Research Center under contract AF 19 (604)–4561, and in part by the Office of Naval Research and the David W. Taylor Model Basin under contract Nonr — 562(24).

https://doi.org/10.1017/S1446788700026392 Published online by Cambridge University Press