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N-SERIES AND FILTRATIONS OF THE AUGMENTATION IDEAL

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1. A problem of Lazard. Let G be a group. Denote by ZG the group ring of G over the integers and by $\Delta = \Delta(G)$ the augmentation ideal of ZG, that is, the kernel of the augmentation map $\epsilon : ZG \to Z$ defined by $\sum \alpha(g)g \mapsto \sum \alpha(g)$. Then Δ is a free abelian group with a free basis $\{g - 1 : g \in G, g \neq 1\}$. A *filtration* of Δ is a sequence

$$\Delta = I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$$

of ideals of Δ with the property $I_n I_m \subseteq I_{n+m}$ for all n, m. In particular,

$$\Delta \supseteq \Delta^2 \supseteq \Delta^3 \supseteq \ldots \supseteq \Delta^n \supseteq \ldots$$

is a filtration of Δ and $\Delta^n \subseteq I_n$ for any filtration $\{I_n\}_{n=1}^{\infty}$ of Δ. Thus $\{\Delta^n\}_{n=1}^{\infty}$ is the most rapidly descending filtration of Δ.

For any ideal I of ZG define

$$\partial(I) = \{x \in G : x - 1 \in I\}.$$

It is easily verified that $\partial(I)$ is a normal subgroup of G. If $\{I_n\}_{n=1}^{\infty}$ is a filtration of Δ then we obtain a descending series

$$G = \partial(I_1) \geqq \partial(I_2) \geqq \partial(I_3) \geqq \ldots \geqq \partial(I_n) \geqq \ldots$$

of normal subgroups of G with the property $[\partial(I_n), \partial(I_m)] \leqq \partial(I_{n+m})$ for all n and m: for if $x \in \partial(I_n)$ and $y \in \partial(I_m)$ then

$$[x, y] - 1 = x^{-1}y^{-1}((x - 1)(y - 1) - (y - 1)(x - 1)) \in I_{n+m}.$$

A sequence $G = H_1 \geqq H_2 \geqq H_3 \geqq \ldots \geqq H_n \geqq \ldots$ of subgroups of G is an *N-series* for G of $[H_n, H_m] \leqq H_{n+m}$ for all n, m. Thus if $\{I_n\}_{n=1}^{\infty}$ is a filtration of $\Delta(G)$ then $\{\partial(I_n)\}_{n=1}^{\infty}$ is an N-series for G.

In [**4**] Lazard has raised the following question. Let $\mathscr{H} = \{H_n\}_{n=1}^{\infty}$ be an N-series for G. Does there exist a filtration $\{I_n\}_{n=1}^{\infty}$ of $\Delta(G)$ such that $H_n = \partial(I_n)$ for all n?

Denote by $\gamma_n(G)$ the nth term of the lower central series of G. Since $\{\gamma_n(G)\}_{n=1}^{\infty}$ is the most rapidly descending N-series for G and $\{\Delta^n\}_{n=1}^{\infty}$ is the most rapidly descending filtration of $\Delta(G)$, an affirmative answer to Lazard's question would imply an affirmative answer to the *dimension conjecture*: $\partial(\Delta^n) = \gamma_n(G)$ for all n. However, Rips [**9**] has exhibited a finite 2-group G for which $\partial(\Delta^4) >$

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$\gamma_4(G)$. Hence, in general, the answer to Lazard's question is "no". We are still left with the problem: For what classes of groups or what classes of N-series does Lazard's problem have an affirmative answer?

In the following sections we will develop some machinery for dealing with certain cases of Lazard's problem. In particular it will be shown that Lazard's problem has an affirmative answer for N-series in abelian groups and in groups of prime exponent. In §5 we will deal with a more specialized form of the problem: For a given N-series $\{H_j\}_{j=1}^{\infty}$ and a given $n \geqq 1$ when does there exist a filtration $\{I_j\}_{j=1}^{\infty}$ of $\Delta(G)$ such that $H_n = \partial(I_n)$? The answers obtained are that such a filtration always exists for $n \leqq 3$ and exists for $n = 4$ if G is a finite group of odd order. Rips' example is a counterexample to the case $n = 4$, $|G|$ even.

The results of §5 when specialized yield many of the known results on the dimension problem. For example, Theorem 6 yields a new proof of a result of Passi [**8**]: If G is a finite group of odd order then $\partial(\Delta^4) = \gamma_4(G)$. In fact, we obtain a stronger result: If G is a finite group of odd order then the canonical homomorphism $\varphi_3 : \gamma_3(G)/\gamma_4(G) \rightarrow \Delta^3/\Delta^4$ is a split monomorphism (see §2 for the definition of φ_n).

2. A canonical filtration. Let $\mathscr{H} = \{H_j\}_{j=1}^{\infty}$ be an N-series for G. Any filtration $\mathscr{I} = \{I_j\}_{j=1}^{\infty}$ of $\Delta(G)$ with the property that $\partial(I_j) = H_j$ for all j will be called a *Lazard filtration of* $\Delta(G)$ *relative to* $\mathscr{H}$.

The N-series $\mathscr{H} = \{H_j\}_{j=1}^{\infty}$ induces a *weight* function w on G:

$$w(x) = \begin{cases} k, \text{ if } x \in H_k \backslash H_{k+1} \\ \infty, \text{ if } x \in \bigcap_j H_j. \end{cases}$$

Since $\mathscr{H}$ is an N-series, $w([x, y]) \geqq w(x) + w(y)$. Define a family $\{\Lambda_j\}_{j=0}^{\infty}$ of Z-submodules of ZG as follows: Λ_k is spanned over Z by all products $(g_1 - 1)(g_2 - 1) \ldots (g_s - 1)$ with $\sum_{j=1}^{s} w(g_j) \geqq k$. Note that the empty product $1 \in \Lambda_0$. It is easily seen that $\Lambda_0 = ZG$, $\Lambda_1 = \Delta(G)$ and $\Lambda_i\Lambda_j \subseteq \Lambda_{i+j}$ for all $i, j \geqq 0$. Thus each Λ_k is an ideal of ZG and $\{\Lambda_j\}_{j=1}^{\infty}$ is a filtration of $\Delta(G)$. The filtration $\mathscr{L} = \mathscr{L}(\mathscr{H}) = \{\Lambda_j\}_{n=1}$ is called the *canonical filtration of* $\Delta(G)$ *with respect to* $\mathscr{H}$.

LEMMA 1. *Let $\mathscr{H} = \{H_j\}_{j=1}^{\infty}$ be an N-series for G. If there exists a Lazard filtration $\mathscr{I} = \{I_j\}_{j=1}^{\infty}$ of $\Delta(G)$ relative to $\mathscr{H}$ then $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}_{j=1}^{\infty}$ is also a Lazard filtration of $\Delta(G)$ relative to $\mathscr{H}$ and $\Lambda_j \subseteq I_j$ for all j.*

Proof. Let $\mathscr{I} = \{I_j\}$ be a Lazard filtration of $\Delta(G)$ relative to $\mathscr{H}$, that is, $\partial(I_j) = H_j$ for all j. Let $g_1, g_2, \ldots, g_s \in G$, $\sum_{j=1}^{s} w(g_j) \geqq k$. Then $(g_1 - 1)(g_2 - 1) \ldots (g_s - 1) \in I_{w(g_1)} I_{w(g_2)} \ldots I_{w(g_s)} \subseteq I_k$. Hence $\Lambda_k \subseteq I_k$ for all k. If $x \in H_k$ then $w(x) \geqq k$ and so $x - 1 \in \Lambda_k$. Thus

$$H_k \leqq \partial(\Lambda_k) \leqq \partial(I_k) = H_k$$

and therefore $\partial(\Lambda_k) = H_k$. Hence $\mathscr{L}(\mathscr{H})$ is the "smallest" Lazard filtration of $\Delta(G)$ relative to $\mathscr{H}$.

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The following lemma, in the context of the dimension conjecture, is due to G. Higman.

LEMMA 2. *Let* $\mathscr{H} = \{H_j\}_{j=1}^{\infty}$ *be an* N*-series for* G *and* $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}_{j=1}^{\infty}$ *the canonical filtration of* $\Delta(G)$ *with respect to* $\mathscr{H}$. *Let* $\bar{G} = K/L$ *be a finite* p*-subquotient of* G (p *prime*) *with* K *finitely generated and let* $\bar{H}_j = (H_j \cap K)L/L$. *Then* $\bar{\mathscr{H}} = \{\bar{H}_j\}_{j=1}^{\infty}$ *is an* N*-series for* $\bar{G}$. *Let* $\mathscr{L}(\bar{\mathscr{H}}) = \{\bar{\Lambda}_j\}_{j=1}^{\infty}$ *be the canonical filtration of* $\Delta(\bar{G})$ *with respect to* $\bar{\mathscr{H}}$. *If for each such* $\bar{G}$, $\partial(\bar{\Lambda}_n) = \bar{H}_n$ *then* $\partial(\Lambda_k) = H_k$.

Proof. Suppose $\partial(\Lambda_k) \neq H_k$. Then there exists $x \in G$ such that $x - 1 \in \Lambda_k$ and $x \notin H_k$. We can write

$$x - 1 = \sum_{i=1}^{r} \alpha_i(g_{i1} - 1)(g_{i2} - 1) \ldots (g_{is(i)} - 1)$$

where, for each i, $\sum_j w(g_{ij}) \geqq k$. Set $K = \langle g_{ij} : i = 1, 2, \ldots, r; j = 1, 2, \ldots, s(i)\rangle$ and

$$K^* = K/H_k \cap K, H_i^* = (H_i \cap K)(H_k \cap K)/H_k \cap K.$$

Then $\mathscr{H}^* = \{H_i^*\}$ is an N-series for K^* and $H_k^* = 1$. Thus K^* is a finitely generated nilpotent group. By a result of Gruenberg [**2**], K^* is residually a finite prime power order group. Let $L^* = L/H_k \cap K$ be a normal subgroup of K^* such that K^*/L^* is a finite p-group and x^*, the image of x in K^*, does not lie in L^*. Then $K/L \simeq K^*/L^*$ is a finite p-group and $x \notin L$. Now set $\bar{G} = K/L$ and $\bar{H}_i = (H_i \cap K)L/L$ for all i. For any $g \in K$, set $\bar{g} = gL$. Now $\bar{H}_k = 1$ (since $H_k \cap K \subseteq L$) and $\bar{x} \neq 1$. Hence $\bar{x} \notin \bar{H}_k$. But

$$\bar{x} - 1 = \sum_i \alpha_i(\bar{g}_{i1} - 1)(\bar{g}_{i2} - 1) \ldots (\bar{g}_{is(i)} - 1) \in \bar{\Lambda}_k$$

where $\bar{\Lambda}_k$ is the kth term in the canonical filtration of $\Delta(\bar{G})$ relative to $\bar{\mathscr{H}} = \{\bar{H}_j\}_{j=1}^{\infty}$. Thus $\partial(\bar{\Lambda}_k) \neq \bar{H}_k$.

Lemma 2 allows us to reduce many questions about arbitrary groups to questions about finite p-groups.

Let $\mathscr{H} = \{H_j\}$ be an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. The mapping $H_k \rightarrow \Lambda_k/\Lambda_{k+1}$ defined by $x \rightarrow (x - 1) + \Lambda_{k+1}$ is a homomorphism with kernel $\partial(\Lambda_{k+1})$. Since $H_{k+1} \leqq \partial(\Lambda_{k+1})$ there is an induced homomorphism

$$\varphi_k : H_k/H_{k+1} \rightarrow \Lambda_k/\Lambda_{k+1}.$$

The map φ_k will be called the *canonical homomorphism* of H_k/H_{k+1} into Λ_k/Λ_{k+1}. Note that $\partial(\Lambda_{k+1}) = H_{k+1}$ if and only if φ_k is a monomorphism. For if $H_{k+1} = \partial(\Lambda_{k+1})$ then the kernel of the map $H_k \rightarrow \Lambda_k/\Lambda_{k+1}$ is $\partial(\Lambda_{k+1}) = H_{k+1}$ and so the induced map φ_k is a monomorphism. On the other hand, suppose φ_k is a monomorphism and $x \in \partial(\Lambda_{k+1})$. Then $\varphi_k(\bar{x}) = (x - 1) + \Lambda_{k+1} = \Lambda_{k+1}$ and so $\bar{x} \in \operatorname{Ker}(\varphi_k) = \{\bar{1}\}$, i.e. $x \in H_{k+1}$. Thus $\partial(\Lambda_{k+1}) \leqq H_{k+1}$. Since $H_{k+1} \leqq \partial(\Lambda_{k+1})$ by definition, $\partial(\Lambda_{k+1}) = H_{k+1}$.

Each ideal Λ_k in the canonical filtration $\mathscr{L}(\mathscr{H})$ relative to an N-series $\mathscr{H}$ is a free abelian group (since it is a subgroup of the free abelian group ZG). For any free abelian group A denote its rank by $\operatorname{rank}_Z(A)$.

LEMMA 3. *Let G be a group, $\mathscr{H}$ an N-series for G and $\mathscr{L}(\mathscr{H})$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Assume G/G' has exponent m. Then $m^{k-1}\Delta(G) \subseteq \Lambda_k$ for all $k \geqq 1$. Thus $\operatorname{rank}_Z(\Lambda_k) = \operatorname{rank}_Z(\Delta(G))$ for all $k \geqq 1$. In particular if G is finite then $\operatorname{rank}_Z(\Lambda_k) = |G| - 1$ for all $k \geqq 1$.*

Proof. Using the well-known isomorphism $\Delta/\Delta^2 \simeq G/G'$ (cf. [**2**]) it follows that if G/G' has exponent m then $m\Delta \subseteq \Delta^2$. Thus, by induction, $m^{k-1}\Delta \subseteq \Delta^k \subseteq \Lambda_k \subseteq \Delta$ for all $k \geqq 1$. Since Δ and $m^{k-1}\Delta$ are isomorphic as abelian groups

$$\operatorname{rank}_Z(\Delta) = \operatorname{rank}_Z(m^{k-1}\Delta) \leqq \operatorname{rank}_Z(\Lambda_k) \leqq \operatorname{rank}_Z(\Delta).$$

The result follows.

3. Some machinery. Throughout this section G is a finite nilpotent group and $\mathscr{H} : G = H_1 \geqq H_2 \geqq \ldots \geqq H_{n+1} = 1$ a finite N-series for G. Let w be the weight function on G relative to $\mathscr{H}$ as defined above. Then $w(x) = \infty$ if and only if $x = 1$. If $x \neq 1$ define $\mathscr{O}^*(x)$ to be the order of the coset $x\,H_{w(x)+1}$ in $H_{w(x)}/H_{w(x)+1}$, that is, $\mathscr{O}^*(x)$ is the least positive integer d such that $x^d \in H_{w(x)+1}$. Each quotient H_k/H_{k+1} is a finite abelian group and hence there exist elements $\bar{x}_{k1}, \bar{x}_{k2}, \ldots, \bar{x}_{k\lambda(k)}$ (where $\bar{x}_{ki} = x_{ki}H_{k+1}$) such that each element $\bar{g} \in H_k/H_{k+1}$ can be written uniquely in the form

$$\bar{g} = \bar{x}_{k1}{}^{e(1)}\,\bar{x}_{k2}{}^{e(2)} \ldots \bar{x}_{k\lambda(k)}{}^{e(\lambda(k))}$$

where $0 \leqq e(j) < \mathscr{O}^*(x_{kj})$ for all j. Moreover we choose the x_{kj} so that $\mathscr{O}^*(x_{kj}) | \mathscr{O}^*(x_{k,j+1})$. Set

$$\Phi = \{x_{kj} : k = 1, 2, \ldots, n; j = 1, 2, \ldots, \lambda(k)\}.$$

Order Φ by setting $x_{ij} < x_{kl}$ if $i < k$ or $i = k$ and $j < l$. Let $|\Phi| = m$. Index the elements of Φ by the integers $1, 2, \ldots, m$ so that $x_i < x_j$ if and only if $i < j$. Then every element $g \in G$ can be written uniquely in the form

$$g = x_1{}^{e(1)}\,x_2{}^{e(2)} \ldots x_m{}^{e(m)} \tag{1}$$

where $0 \leqq e(i) < \mathscr{O}^*(x_i)$ for all i. The ordered set $\Phi = \{x_1, x_2, \ldots, x_m\}$ is called a *uniqueness basis for G relative to $\mathscr{H}$*.

Let Φ be a fixed uniqueness basis for G relative to the N-series $\mathscr{H}$. By an *m-sequence* $\alpha = (e(1), e(2), \ldots, e(m))$ we mean an ordered m-tuple of non-negative integers. The set $\mathscr{S}_m$ of all m-sequences is ordered lexicographically, that is, if $\alpha = (e(1), e(2), \ldots, e(m))$ and $\beta = (f(1), f(2), \ldots, f(m))$ are m-sequences then $\alpha < \beta$ if there exists j, $1 \leqq j \leqq m$, such that $e(i) = f(i)$ for all $i < j$ and $e(j) < f(j)$. Under this ordering $\mathscr{S}_m$ is well-ordered. An

m-sequence $\alpha = (e(1), e(2), \ldots, e(m))$ is *basic* (with respect to Φ) if $0 \leqq e(i) < \mathscr{O}^*(x_i)$ for all i. It follows from the uniqueness of the expression (1) that there is a one-to-one correspondence between the elements of G and the basic m-sequences.

The *weight* of an m-sequence $\alpha = (e(1), e(2), \ldots, e(m))$ is defined to be

$$W(\alpha) = \sum_{j=1}^{m} w(x_j)e(j).$$

Given an m-sequence $\alpha = (e(1), e(2), \ldots, e(m))$ we define the *proper product* $P(\alpha) \in ZG$ by

$$P(\alpha) = \prod_{i=1}^{m} (x_i - 1)^{e(i)}$$

where the factors occur in order of increasing i from left to right. If $W(\alpha) \geqq k$ then $P(\alpha) \in \Delta_k$. If α is a basic m-sequence then $P(\alpha)$ is called a *basic product.* Note that if $\alpha = (0, 0, \ldots, 0)$ then $P(\alpha) = 1$.

Since

$$x_i^{e(i)} = (1 + (x_i - 1))^{e(i)} = 1 + \sum_{j=1}^{e(i)} \binom{e(i)}{j} (x_i - 1)^j$$

we see from (1) that

$$\text{(2)} \quad g = 1 + e(1)(x_1 - 1) + e(2)(x_2 - 1) + \ldots + e(m)(x_m - 1)$$
$$+ \text{ a } Z\text{-linear combination of basic products of higher degree.}$$

We can rewrite (2) as

$$\text{(2')} \quad g - 1 = e(1)(x_1 - 1) + e(2)(x_2 - 1) + \ldots + e(m)(x_m - 1)$$
$$+ \text{ a } Z\text{-linear combination of basic products of higher degree.}$$

It follows from (2) (respectively (2′)) that the basic products (respectively basic products $\neq 1$) span ZG (respectively $\Delta(G)$). Since the number of basic products is $|G| = \operatorname{rank}_Z(ZG)$ and the number of basic products distinct from 1 is $|G| - 1 = \operatorname{rank}_Z(\Delta(G))$, we have

LEMMA 4. *The basic products are a free basis for ZG; the basic products distinct from* 1 *are a free basis for* Δ.

LEMMA 5. *Let $x_{i(1)}, x_{i(2)}, \ldots, x_{i(s)} \in \Phi$, $r = \sum_{j=1}^{s} w(x_{i(j)})$ and $\mu = \min\{i(1), i(2), \ldots, i(s)\}$. Then the product*

$$(*) \quad (x_{i(1)} - 1)(x_{i(2)} - 1) \ldots (x_{i(s)} - 1)$$

can be written as a Z-linear combination of proper products $P(\alpha)$, $\alpha = (e(1), e(2), \ldots, e(m))$, such that

(i) $W(\alpha) \geqq r$,

(ii) $e(j) = 0$ *for all* $j < \mu$.

Proof. If all $x_{i(j)} \in H_n$ then the factors $x_{i(j)} - 1$ commute pairwise and the product (*) is equal to $(x_{k(1)} - 1)(x_{k(2)} - 1) \ldots (x_{k(s)} - 1)$ where $k(1) \leqq k(2) \leqq \ldots \leqq k(s)$ are the integers $i(1), i(2), \ldots, i(s)$ in increasing order. Assume the lemma holds whenever all $x_{i(j)} \in H_{k+1}$. Let $x_{i(1)}, x_{i(2)}, \ldots, x_{i(s)} \in H_k$ and suppose d of the $x_{i(j)} \in H_k \backslash H_{k+1}$. If $d = 0$ then, by the induction hypothesis, we are done. So assume $d > 0$ and the lemma holds for products (*) with fewer than d of the $x_{i(j)}$ in $H_k \backslash H_{k+1}$ and the remaining $x_{i(j)}$ in H_{k+1}. Suppose $w(x_{i(j+1)}) = k$ and $x_{i(j)} > x_{i(j+1)}$. Put $x = x_{i(j)}$, $y = x_{i(j+1)}$. Replace $(x - 1)(y - 1)$ in (*) using the identity

$$(x-1)(y-1) = (y-1)(x-1) + (y-1)(x-1)([x,y]-1) + (y-1)([x,y]-1) + (x-1)([x,y]-1) + ([x,y]-1)$$

and then replace $[x, y] - 1$ by its basic form (2′). Since $w([x, y]) \geqq w(x) + w(y)$ each of the terms arising from these replacements have weight $\geqq r$. Moreover no x_j with $j < \mu$ is introduced. The new terms arising from the last three terms of the above identity each have fewer than d occurrences of factors $x_i - 1$ with $w(x_i) = k$ and, hence, can be written in the desired form. Thus, by repeating this process sufficiently often, we are reduced to the consideration of products of the form

$$(\substack{*\\*}) \qquad (x_{i(1)} - 1) \ldots (x_{i(d)} - 1) \ldots (x_{i(s)} - 1)$$

where $i(1) \leqq i(2) \leqq \ldots \leqq i(d)$, $w(x_{i(j)}) = k$ if $1 \leqq j \leqq d$, $w(x_{i(j)}) > k$ if $j > d$ and $\sum_j w(x_{i(j)}) \geqq r$. By the first induction hypothesis, $(x_{i(d+1)} - 1) \ldots (x_{i(s)} - 1)$ may be written as a Z-linear combination of proper products $P(\beta)$, $\beta = (e(1), e(2), \ldots, e(m))$ with $W(\beta) \geqq r - dk$ and $e(j) = 0$ for $j < \min\{i(d + 1), \ldots, i(s)\}$. But then $(x_{i(1)} - 1) \ldots (x_{i(d)} - 1)P(\beta)$ is a proper product of the type specified in the conclusion of the lemma. Therefore, the lemma holds if d of the $x_{i(j)}$ have weight k. Thus, by induction, the lemma holds if all $x_{i(j)} \in H_k$. It follows then that the lemma is valid for all $x_{i(j)} \in H_1 = G$.

COROLLARY. *Let $g_1, g_2, \ldots, g_s \in G$ and let $\sum w(g_i) \geqq k$. Then $(g_1 - 1)(g_2 - 1) \ldots (g_s - 1)$ can be written as a Z-linear combination of proper products $(x_{i(1)} - 1)(x_{i(2)} - 1) \ldots (x_{i(t)} - 1)$ with $\sum_j w(x_{i(j)}) \geqq k$.*

Proof. Express each $g_j - 1$ in its basic form (2′). Then $(g_1 - 1)(g_2 - 1) \ldots (g_s - 1)$ is expressed as a Z-linear combination of products $(x_{i(1)} - 1)(x_{i(2)} - 1) \ldots (x_{i(t)} - 1)$ with $x_{i(j)} \in \Phi$ and $\sum_j w(x_{i(j)}) \geqq k$. Now apply lemma 5 to each of these products.

From this corollary and the definition of the canonical filtration we obtain:

THEOREM. *Let G be a finite nilpotent group, $\mathscr{H}$ a finite N-series for G, Φ a uniqueness basis for G relative to $\mathscr{H}$ and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration* *of $\Delta(G)$ relative to $\mathscr{H}$. Then the ideal Λ_k is spanned over Z by the proper products $P(\alpha)$ with $W(\alpha) \geqq k$.*

Many of the proofs in the following sections will be carried out by induction over the well ordered set $\mathscr{S}_m$ of m-sequences. In these arguments the process of *straightening*, that is, replacing a non-proper product $(x_{i(1)} - 1)(x_{i(2)} - 1) \ldots (x_{i(s)} - 1)$ by a Z-linear combination of proper products, as in Lemma 5, will play an important role. Usually the notion will be used in the following way. Let $P(\alpha)$ be a non-basic proper product; then $P(\alpha)$ is of the form $P(\alpha_1)(x-1)^d P(\alpha_2)$ where $P(\alpha_1)$ and $P(\alpha_2)$ are suitable proper products, $x \in \Phi$ and $d = \mathscr{O}^*(x)$. Now

$$x^d = (1 + (x-1))^d = 1 + \sum_{j=1}^{d-1} \binom{d}{j} (x-1)^j + (x-1)^d$$

and so

$$(x-1)^d = - \sum_{j=1}^{d-1} \binom{d}{j} (x-1)^j + (x^d - 1).$$

Therefore

$$P(\alpha) = - \sum_{j=1}^{d-1} \binom{d}{j} P(\alpha_1)(x-1)^j P(\alpha_2) + P(\alpha_1)(x^d - 1)P(\alpha_2).$$

Each of the products $P(\alpha_1)(x-1)^j P(\alpha_2)$, $1 \leqq j \leqq d-1$ is a proper product $P(\beta)$ with $\beta < \alpha$. Since $w(x^d) > w(x)$ we can replace $x^d - 1$ by its basic form (2′) without introducing any new factors $x_i - 1$ with $x_i \leqq x$; say $x^d - 1 = \sum_k a_k P(\gamma_k)$, the γ_k basic. Now apply Lemma 5 to each of the products $P(\gamma_k) P(\alpha_2)$, say

$$P(\gamma_k)P(\alpha_2) = \sum_r b_{kr} P(\delta_{kr}).$$

Then $P(\alpha_1)(x^d - 1)P(\alpha_2) = \sum_{k,r} a_k b_{kr} P(\alpha_1)P(\delta_{kr})$ where each product $P(\alpha_1)P(\delta_{kr})$ is proper and (by (ii) of Lemma 5) is of the form $P(\beta)$ with $\beta < \alpha$. Thus $P(\alpha)$ has been expressed as a Z-linear combination of proper products $P(\beta)$, $\beta < \alpha$.

4. Some special cases. The machinery of § 3 can be used to establish an affirmative answer to the Lazard problem for abelian groups and groups of prime exponent.

THEOREM 2. *Let G be an abelian group and $\mathscr{H} = \{H_j\}_{j=1}^{\infty}$ a descending series of subgroups of G. Let $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}_{j=1}^{\infty}$ be the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Then $\partial(\Lambda_j) = H_j$ for all j. Moreover, if G is finite then for each k the canonical homomorphism*

$$\varphi_k : H_k/H_{k+1} \rightarrow \Lambda_k/\Lambda_{k+1}$$

is a split monomorphism.

Proof. By Lemma 2 we may assume G is finite. By passing to quotients by H_{k+1}, we may assume $H_{k+1} = 1$. Thus $\varphi_k : H_k \to \Lambda_k/\Lambda_{k+1}$ by $\varphi_k : x \to (x - 1) + \Lambda_{k+1}$. Let Φ be a uniqueness basis for G relative to $\mathscr{H}$. Since $\Delta(G)$ has a free basis consisting of the non-identity basic products, in order to define a homomorphism of $\Delta(G)$ into an abelian group it suffices to define it on the basic products. We thus define $\tau : \Delta(G) \to G$ by

$$\tau(x_i - 1) = x_i, \quad \text{for all } x_i \in \Phi,$$
$$\tau(P(\alpha)) = 1, \quad \text{for all other basic } \alpha.$$

We now observe the following four properties.

(1) $\tau(g - 1) = g$ for all $g \in G$: We write g in the form (1), $g = x_1{}^{e(1)}x_2{}^{e(2)} \ldots x_m{}^{e(m)}$. Then in the basic form (2′),

$$g - 1 = e(1)(x_1 - 1) + \ldots + e(m)(x_m - 1) + \text{basic products of higher degree.}$$

Therefore $\tau(g - 1) = x_1{}^{e(1)} \ldots x_m{}^{e(m)} = g$.

(2) $\tau(P(\alpha)) = 1$ for all proper products $P(\alpha)$ not of the form $x_i - 1, x_i \in \Phi$: Let $P(\alpha)$ be a proper product not of the form $x_i - 1$, $x_i \in \Phi$. Assume that for all $\beta < \alpha$, $\tau(P(\beta)) = 1$ if β is not of the form $x_i - 1$, $x_i \in \Phi$. If α is basic then $\tau(P(\alpha)) = 1$ by definition. If α is not basic then $P(\alpha)$ is of the form $P(\alpha_1)(x - 1)^d P(\alpha_2)$ where $x \in \Phi$, $d = \mathscr{O}^*(x)$. Replace $(x - 1)^d$ in $P(\alpha)$ by $-\sum_{j=1}^{d-1} \binom{d}{j}(x - 1)^j + (x^d - 1)$, replace $x^d - 1$ by its basic form (2′) and straighten the resulting terms (which in this case is just rearrangement of the factors into the proper order). This expresses $P(\alpha)$ as a Z-linear combination of proper products $P(\beta)$, $\beta < \alpha$. If either $P(\alpha_1)$ or $P(\alpha_2)$ is not 1 then, by the induction hypothesis, each such $P(\beta)$ is mapped onto 1 by τ and, therefore, $\tau(P(\alpha)) = 1$. If $P(\alpha_1) = P(\alpha_2) = 1$ then

$$P(\alpha) = (\mathrm{x} - 1)^d = -d(x - 1) - \binom{d}{2}(x - 1)^2 - \ldots - d(x - 1)^{d-1} + (x^d - 1)$$

and so $\tau(P(\alpha)) = x^{-d} \cdot x^d = 1$. Thus, by induction over the well-ordered set $\mathscr{S}_m$ of m-sequences, assertion (2) is proved.

(3) τ maps Λ_k into H_k: By Theorem 1 it suffices to show that $\tau(P(\alpha)) \in H_k$ whenever $W(\alpha) \geqq k$. If $W(\alpha) \geqq k$ and $P(\alpha) \neq x_i - 1$, $x_i \in \Phi$, then $\tau(P(\alpha)) = 1 \in H_k$. If $W(\alpha) \geqq k$ and $P(\alpha) = x_i - 1$, $x_i \in \Phi$, then $w(x_i) = W(\alpha) \geqq k$ and $x_i \in H_k$; thus $\tau(P(\alpha)) = x_i \in H_k$.

(4) τ vanishes on Λ_{k+1}: It suffices to show that $\tau(P(\alpha)) = 1$ whenever $W(\alpha) \geqq k + 1$. Let $W(\alpha) \geqq k + 1$. If $P(\alpha) \neq x_i - 1$, $x_i \in \Phi$, then $\tau(P(\alpha)) = 1$; if $P(\alpha) = x_i - 1$, $x_i \in \Phi$, then $\tau(P(\alpha)) = x_i \in H_{k+1} = 1$.

It follows from (1), (3) and (4) that τ induces a homomorphism $\tau_k : \Lambda_k/\Lambda_{k+1} \to H_k$ such that $\tau_k : (g - 1) + \Lambda_{k+1} \to g$ for all $g \in H_k$. Thus $\tau_k\varphi_k$ is the identity on H_k and, therefore, φ_k is a split monomorphism. In particular this means $\partial(\Lambda_{k+1}) = 1$. Dropping the assumption that $H_{k+1} = 1$ this implies $\partial(\Lambda_{k+1}) \leqq H_{k+1}$ and, hence, $\partial(\Lambda_{k+1}) = H_{k+1}$.

THEOREM 3. *Let G be a group of prime exponent p, let $\mathscr{H} = \{H_j\}$ be an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Then $\partial(\Lambda_k) = H_k$ for all k.*

Proof. We may assume G is finite. Suppose there exists an integer k and an element $g \in G$ such that $g - 1 \in \Lambda_k$, $g \notin H_k$. Passing to quotients by H_k, we may assume $H_k = 1$, $g \neq 1$. Let Φ be a uniqueness basis for G relative to $\mathscr{H}$. Since every non-identity element of G has order p, $\mathscr{O}^*(x) = p$ for all $x \neq 1$ and

$$(x - 1)^p = - \sum_{j=1}^{p-1} \binom{p}{j} (x - 1)^j.$$

Express g in the form (1), $g = x_{i(1)}{}^{e(1)} x_{i(2)}{}^{e(2)} \ldots x_{i(s)}{}^{e(s)}$, $i(1) < i(2) < \ldots < i(s)$, $0 < e(j) < p$, $j = 1, 2, \ldots, s$. Then in the basic form (2′)

$$g - 1 = e(1)(x_{i(1)} - 1) + e(2)(x_{i(2)} - 1) + \ldots + e(s)(x_{i(s)} - 1) + \ldots .$$

Suppose we are given an expression for $g - 1$ as a Z-linear combination of proper products, say $g - 1 = \sum_{t=1}^{n} a_t P(\alpha_t)$. Let b be the coefficient of the proper product $x_{i(1)} - 1$ in this expression; b may be 0. Suppose some $P(\alpha_u)$ occurring with non-zero coefficient a_u is not basic; then

$$P(\alpha_u) = P(\beta_1)(x - 1)^p P(\beta_2) = - \sum_{j=1}^{p-1} \binom{p}{j} P(\beta_1)(x - 1)^j P(\beta_2)$$

for some $x \in \Phi$. This will give a new expression for $g - 1$ as a Z-linear combination of proper products. What is the coefficient b' of $x_{i(1)} - 1$ in this new expression for $g - 1$? We will have $b' = b$ if $P(\beta_1) \neq 1$ or $P(\beta_2) \neq 1$ or $x \neq x_{i(1)}$. If $P(\beta_1) = P(\beta_2) = 1$ and $x = x_{i(1)}$ then $b' = b - a_u p$. In any case $b' \equiv b \bmod p$. We can repeat this procedure to obtain yet a new expression for $g - 1$. After finitely many such repetitions we must obtain an expression for $g - 1$ in which all the terms are integer multiples of *basic* products. This must, of course, be the expression (2′). Therefore $b \equiv e(1) \bmod p$.

Since $g - 1 \in \Lambda_k$ there exists an expression for $g - 1$ as a Z-linear combination of proper products each of weight $\geqq k$, say $g - 1 = \sum_{t=1}^{n} a_t P(\alpha_t)$. The coefficient b of $x_{i(1)} - 1$ in this expression satisfies $b \equiv e(1) \bmod p$; since $0 < e(1) < p$, $b \neq 0$. Hence $W(x_{i(1)} - 1) = w(x_{i(1)}) \geqq k$, that is, $x_{i(1)} \in H_k = 1$, a contradiction.

5. On the general case. The following theorem generalizes the well-known result that $g - 1 \in \Delta(G)^2$ if and only if $g \in \gamma_2(G)$.

THEOREM 4. *Let G be a group, $\mathscr{H} = \{H_j\}$ an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Then $\partial(\Lambda_2) = H_2$. Moreover, the canonical homomorphism $\varphi_1 : G/H_2 \rightarrow \Delta(G)/\Lambda_2$ is an isomorphism.*

Proof. It is sufficient to show that φ_1 is an isomorphism. Passing to quotients by H_2 we may assume $H_2 = 1$. Then G is abelian and $\Lambda_k = \Delta^k$ for all $k \geqq 1$. By the well-known isomorphism $G \simeq \Delta/\Delta^2 = \Delta/\Lambda_k$ for abelian G. The result follows.

LEMMA 6. *Let G be a finite nilpotent group, $\mathscr{H}$ a finite N-series for G, $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Then Λ_2 has a free basis consisting of*

(i) $(x_i - 1)^{d(i)}, w(x_i) = 1, d(i) = \mathscr{O}^*(x_i)$,

(ii) $P(\alpha), \alpha$ basic, $W(\alpha) \geqq 2$.

Proof. Clearly all elements of type (i) and (ii) are in Λ_2 and there are a total of $|G| - 1$ such elements. In view of Lemma 3 it suffices to show that these elements span Λ_2. Let $\gamma \in \Lambda_2$. By Lemma 4, we can write

$$\gamma = a(1)(x_1 - 1) + a(2)(x_2 - 1) + \ldots + a(k)(x_k - 1) \\ + \text{basic products of weight} \geqq 2$$

where $w(x_1) = w(x_2) = \ldots = w(x_k) = 1$. Since $\Lambda_2 = \operatorname{Ker} \tau$ where $\tau : \Delta(G) \to G/H_2$ is the homomorphism defined in the proof of Theorem 4,

$$\tau(\gamma) = \prod_{i=1}^{k} x_i^{a(i)} H_2 = H_2$$

and so $\Pi_{i=1}^{k} x_i^{a(i)} \in H_2$. By the uniqueness of the expression (1) it follows that for each i, $a(i) = b_i\mathscr{O}^*(x_i) = b_i d(i)$ for some integer b_i. Thus

$$\gamma = b_1 d(1)(x_1 - 1) + b_2 d(2)(x_2 - 1) + \ldots + b_k d(k)(x_k - 1) \\ + \text{basic products of weight} \geqq 2.$$

Now

$$(x_i - 1)^{d(i)} = -\sum_{j=1}^{d(i)} \binom{d(i)}{j} (x_i - 1)^j + (x_i^{d(i)} - 1).$$

If we replace $x_i^{d(i)} - 1$ by its basic expression (2′) then we see that

$$d(i)(x_i - 1) = -(x_i - 1)^{d(i)} + \text{basic products of weight} \geqq 2$$

and, hence,

$$\gamma = -b_1(x_i - 1)^{d(1)} - b_2(x_2 - 1)^{d(2)} - \ldots - b_k(x_k - 1)^{d(k)} \\ + \text{basic products of weight} \geqq 2,$$

that is, γ is a Z-linear combination of proper products of types (i) and (ii).

The next result for the case $\mathscr{H} = \{\gamma_i(G)\}$ is attributed by Gruenberg [**1**] to G. Higman and D. Rees, independently. Another proof is due to Hoare [**3**]. The fact that φ_2 splits in that case is due to Sandling [**10**].

THEOREM 5. *Let G be a group, $\mathscr{H}$ an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Then $\partial(\Lambda_3) = H_3$. Moreover, if G is*

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finite then the canonical homomorphism $\varphi_2 : H_2/H_3 \to \Lambda_2/\Lambda_3$ *is a split monomorphism.*

Proof. In [**6**] we give a proof of this result for the special case that $\Lambda_k = \Delta^k$. With appropriate modifications that proof goes through in the present case.

THEOREM 6. *Let* G *be a finite group,* $\mathscr{H} = \{H_j\}$ *an* N*-series for* G *and* $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ *the canonical filtration of* $\Delta(G)$ *relative to* $\mathscr{H}$. *Then the kernel of the canonical homomorphism* $\varphi_3 : H_3/H_4 \to \Lambda_3/\Lambda_4$ *is an elementary abelian* 2*-group. If* G *has odd order then* φ_3 *is a split monomorphism and* $\partial(\Lambda_4) = H_4$.

Proof. By passing to quotients by H_4 we may assume $H_4 = 1$. Since $[H_2, H_2] \leqq H_4$, H_2 is abelian. Let Φ be a uniqueness basis for G relative to $\mathscr{H}$. Define a homomorphism $\tau : \Lambda_2 \to H_2$ by specifying its values on the basis for Λ_2 given in Lemma 6 as follows:

$$
\begin{array}{lll}
(x_i - 1)^{d(i)} & \to 1 & w(x_i) = 1, d(i) = \mathscr{O}^*(x_i) \\
x_i - 1 & \to x_i^2 & w(x_i) = 2 \\
(x_i - 1)(x_j - 1) & \to [x_i, x_j] & w(x_i) = w(x_j) = 1 \\
x_i - 1 & \to x_i^2 & w(x_i) = 3 \\
(x_i - 1)(x_j - 1) & \to [x_i, x_j] & w(x_i) = 1, w(x_j) = 2 \\
(x_i - 1)(x_j - 1)(x_k - 1) & \to [x_i, x_k, x_j] & w(x_i) = w(x_j) = w(x_k) = 1 \\
P(\alpha) & \to 1 & \alpha \text{ basic, } W(\alpha) \geqq 4.
\end{array}
$$

(1) $\tau(g - 1) = g^2$ for all $g \in H_2$: Write $g = x_{i(1)}{}^{e(1)} x_{i(2)}{}^{e(2)} \dots x_{i(s)}{}^{e(s)}$ in the unique form (1) with $w(x_{i(j)}) \geqq 2$ for all j. Then

$$g - 1 = e(1)(x_{i(1)} - 1) + \dots + e(s)(x_{i(s)} - 1) + \text{basic products of weight} \geqq 4.$$

Hence $\tau(g - 1) = x_{i(1)}{}^{2e(1)} x_{i(2)}{}^{2e(2)} \dots x_{i(s)}{}^{2e(s)} = g^2$.

(2) If $x_i \in \Phi$, $w(x_i) = 1$, and $w(g) \geqq 2$ then $\tau((x_i - 1)(g - 1)) = [x_i, g]$: As in (1) write

$$g - 1 = e(1)(x_{i(1)} - 1) + \dots + e(s)(x_{i(s)} - 1) + \text{basic products of weight} \geqq 4.$$

Then

$$(x_i - 1)(g - 1) = e(1)(x_i - 1)(x_{i(1)} - 1) + \dots + e(s)(x_i - 1)(x_{i(s)} - 1) + \text{basic products of weight} \geqq 5.$$

Therefore

$$\tau((x_i - 1)(g - 1)) = [x_i, x_{i(1)}]^{e(1)}[x_i, x_{i(2)}]^{e(2)} \dots [x_i, x_{i(s)}]^{e(s)} = (x_i, x_{i(1)}{}^{e(1)} x_{i(2)}{}^{e(2)} \dots x_{i(s)}{}^{e(s)}] = [x_i, g]$$

since $H_4 = 1$.

(3) τ vanishes on Λ_4: Let α be an m-sequence, $W(\alpha) \geqq 4$, and suppose

$\tau(P(\beta)) = 1$ for all $\beta < \alpha$, $W(\beta) \geqq 4$. If α is basic then $\tau(P(\alpha)) = 1$. So assume α is not basic; then $P(\alpha) = P(\alpha_1)(x-1)^d P(\alpha_2)$ for some $x \in \Phi$, $d = \mathcal{O}^*(x)$ and suitable $P(\alpha_1), P(\alpha_2)$. Arguing as in Theorem 5, $\tau(P(\alpha)) = 1$ if $W(\alpha_1) + W(\alpha_2) \geqq 3$. Thus we may assume $W(\alpha_1) + W(\alpha_2) \leqq 2$.

$W(\alpha_1) + W(\alpha_2) = 0$: In this case $P(\alpha) = (x-1)^d$. If $w(x) = 1$ then $\tau(P(\alpha)) = 1$ by definition. If $w(x) \geqq 2$ then $P(\alpha) \equiv -d(x-1) + (x^d - 1)$ mod Ker τ and so $\tau(P(\alpha)) = (x^2)^{-d}.\ (x^d)^2 = 1$.

$W(\alpha_1) + W(\alpha_2) = 1$: Suppose first $P(\alpha) = (x-1)^d(y-1)$, $w(y) = 1$. Then $w(x) = 1$ and $d \geqq 3$. Thus

$$P(\alpha) \equiv -d(x-1)(y-1) - \binom{d}{2}(x-1)^2(y-1) + (x^d-1)(y-1)$$

modulo Ker τ. Now

$$(x^d-1)(y-1) = (y-1)(x^d-1) + (y-1)(x^d-1)([x^d,y]-1) \\ + (y-1)([x^d,y]-1) + (x^d-1)([x^d,y]-1) + ([x^d,y]-1)$$

and the middle three terms, when straightened, are Z-linear combinations of $P(\beta)$'s with $\beta < \alpha$, $W(\beta) \geqq 4$, and so belong to Ker τ. Therefore

$$P(\alpha) \equiv -d(x-1)(y-1) - \binom{d}{2}(x-1)^2(y-1) + (y-1)(x^d-1) \\ + ([x^d,y]-1)$$

modulo Ker τ. Hence

$$\tau(P(\alpha)) = [x,y]^{-d}[x,y,x]^{-\binom{d}{2}}[y,x^d][x^d,y]^2 = 1$$

since

$$[x^d,y] = [x,y]^d[x,y,x]^{\binom{d}{2}} \text{ mod } \gamma_4(G).$$

It remains to consider the case $P(\alpha) = (y-1)(x-1)^d$, $w(y) = 1$. If $w(x) = 1$ then $d \geqq 3$ and

$$P(\alpha) \equiv -d(y-1)(x-1) - \binom{d}{2}(y-1)(x-1)^2 + (y-1)(x^d-1)$$

modulo Ker τ. Thus

$$\tau(P(\alpha)) = [y,x]^{-d}[y,x,x]^{-\binom{d}{2}}[y,x^d] = 1$$

since $[y,x^d] \equiv [y,x]^d[y,x,x]^{\binom{d}{2}}$ mod $\gamma_4(G)$. If $w(x) \geqq 2$ then $P(\alpha) \equiv -d(y-1)(x-1)$ mod Ker τ and $\tau(P(\alpha)) = [y,x]^{-d} = [y,x^d]^{-1} = 1$ since $\gamma_4(G) \leqq H_4 = 1$.

$W(\alpha_1) + W(\alpha_2) = 2$. There are five cases to consider:

(a) $P(\alpha) = (x-1)^d(y-1)$, $w(y) = 2$

(b) $P(\alpha) = (y-1)(x-1)^d$, $w(y) = 2$

(c) $P(\alpha) = (x-1)^d(y-1)(z-1)$, $w(y) = w(z) = 1$

(d) $P(\alpha) = (y-1)(x-1)^d(z-1)$, $w(y) = w(z) = 1$

(e) $P(\alpha) = (y-1)(z-1)(x-1)^d$, $w(y) = w(z) = 1$.

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Case (a): If $w(x) \geqq 2$ then $\tau(P(\alpha)) = 1$. If $w(x) = 1$ then $P(\alpha) \equiv -d(x-1)(y-1) \bmod \operatorname{Ker} \tau$ and $\tau(P(\alpha)) = [x, y]^{-d} = [x^d, y]^{-1} = 1$ (since $H_4 = 1$). A similar argument works in Case (b).
Case (c): $P(\alpha) \equiv -d(x-1)(y-1)(z-1) \bmod \operatorname{Ker} \tau$ and, thus, $\tau(P(\alpha)) = [x, z, y]^{-d} = (x^d, z, y]^{-1} = 1$. Similar arguments work in Cases (d) and (e).

We have thus established by induction over $\mathscr{S}_m$ that $\tau(P(\alpha)) = 1$ for all α such that $W(\alpha) \geqq 4$. Hence τ vanishes on Λ_4.

(4) $\tau(\Lambda_3) \leqq H_3$: Let α be an m-sequence, $W(\alpha) \geqq 3$, and suppose $\tau(P(\beta)) \in H_3$ for all $\beta < \alpha$, $W(\beta) \geqq 3$. If α is basic then, from the definition of τ, $\tau(P(\alpha)) \in H_3$. So assume α is not basic; then $P(\alpha) = P(\alpha_1)(x-1)^d P(\alpha_2)$, $d = \mathscr{O}^*(x)$. Arguing as in Theorem 5, we may assume $W(\alpha_1) + W(\alpha_2) \leqq 1$. If $W(\alpha_1) + W(\alpha_2) = 0$ then $P(\alpha) = (x-1)^d$. If $w(x) > 1$ then $W(\alpha) \geqq 4$ and, by (3), $\tau(P(\alpha)) = 1 \in H_3$. If $w(x) = 1$ then $\tau(P(\alpha)) = 1 \in H_3$ by definition. Suppose $W(\alpha_1) + W(\alpha_2) = 1$; say $P(\alpha) = (x-1)^d(y-1)$, $w(y) = 1$. Then

$$P(\alpha) = -\sum_{j=1}^{d-1} \binom{d}{j} (x-1)^j(y-1) + (x^d-1)(y-1).$$

If we straighten the last term on the right-hand side then all terms except $-d(x-1)(y-1)$ are of the form $P(\beta)$, $\beta < \alpha$, $W(\beta) \geqq 3$, and hence have images in H_3. But $\tau(d(x-1)(y-1)) = [x, y]^d \equiv [x^d, y] \equiv 1 \bmod H_3$. Thus $\tau(P(\alpha)) \in H_3$. The case $P(\alpha) = (y-1)(x-1)^d$ is handled in a similar manner.

It follows from (1), (3) and (4) that τ induces a homomorphism $\tau_3 : \Lambda_3/\Lambda_4 \rightarrow H_3/H_4$ such that $\tau_3 : (g-1) + \Lambda_4 \rightarrow g^2$ for all $g \in H_3$. Thus $\tau_3\varphi_3 : g \rightarrow g^2$ for all $g \in H_3$. Hence if $g \in \operatorname{Ker} \varphi_3$ then $g^2 = 1$ and so $\operatorname{Ker} \varphi_3$ is an elementary abelian 2-group. If G has odd order then $g \rightarrow g^2$ is an automorphism of H_3 and, therefore, φ_3 is a split monomorphism.

In [**8**] Passi has given a proof of the special case $\partial(\Delta^4) = \gamma_4(G)$ for a finite p-group, $p \neq 2$, involving cocycle calculations. His proof does not yield the fact that the canonical map φ_3 splits.

COROLLARY. *Let G be a group, $\mathscr{H} = \{H_j\}$ an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. If for every finite 2-subquotient $\bar{G}$ of G and every N-series $\bar{\mathscr{H}} = \{\bar{H}_i\}$ of $\bar{G}$, $\partial(\bar{\Lambda}_4) = \bar{H}_4$, where $\mathscr{L}(\bar{\mathscr{H}}) = \{\bar{\Lambda}_j\}$ is the canonical filtration of $\Delta(\bar{G})$ relative to $\bar{\mathscr{H}}$, then $\partial(\Lambda_4) = H_4$.*

COROLLARY. *Let G be a finite 2-group, $\mathscr{H} = \{H_j\}$ an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. Suppose that in the decomposition of G/H_2 into a direct sum of cyclic subgroups there is at most one cycle of order > 2. Then $\partial(\Lambda_4) = H_4$.*

Proof. Let $x_1, x_2, \ldots, x_t$ be the elements of Φ in $G \backslash H_2$. Then

$$\mathscr{O}^*(x_1) = \mathscr{O}^*(x_2) = \ldots = \mathscr{O}^*(x_{t-1}) = 2 \text{ and } \mathscr{O}^*(x_t) \geqq 2.$$

Define $\tau' : \Lambda_2 \rightarrow H_2$ to be the same as τ above except that $\tau' : x_i - 1 \rightarrow x_i$,

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$w(x_i) \geqq 2$. The proof then follows exactly as the proof of Theorem 6. Note, however, that the case $P(\alpha) = (x-1)^d(y-1)$, $w(y) = 1$, occurring in (3) of that proof, cannot occur under the hypothesis of the corollary for, since $d = 2$, $W(\alpha) = 3$.

Can the methods used above be pushed any further? Suppose $\mathscr{H} = \{H_j\}$ is an N-series for G and $\mathscr{L}(\mathscr{H}) = \{\Lambda_j\}$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$. We would like a result of the sort that, with certain possible exceptions, $\partial(\Lambda_5) = H_5$. Passing to quotients by H_5 we may assume $H_5 = 1$. Then H_3 is abelian. There is a homomorphism $\sigma : H_3 \to \Lambda_3/\Lambda_5$ defined by $\sigma : g \to (g-1) + \Lambda_5$. The kernel of σ is $\partial(\Lambda_5)$. Thus we would like to show that σ is a monomorphism and, hence, $\partial(\Lambda_5) = 1$. If we can find a homomorphism $\tau : \Lambda_3 \to H_3$ which vanishes on Λ_5 then we obtain an induced homomorphism $\tau^* : \Lambda_3/\Lambda_5 \to H_3$. An examination of the composite $\tau^*\sigma$ may enable us to determine conditions on G and $\mathscr{H}$ under which σ is a monomorphism. In order to define such a homomorphism τ it would be useful to have a free basis for Λ_3. The next result exhibits such a free basis. Unfortunately, except in very special cases, we have not been able to find suitable maps τ with which to carry out the above program.

LEMMA 7. *Let G be a finite nilpotent group, $\mathscr{H}$ a finite N-series for G, $\mathscr{L}(\mathscr{H})$ the canonical filtration of $\Delta(G)$ relative to $\mathscr{H}$ and Φ a uniqueness basis for G relative to $\mathscr{H}$. Then Λ_3 has a free basis consisting of*

(i)	$(x_i-1)^{d(i)}$	$w(x_i) = 1,\ d(i) = \mathscr{O}^*(x_i) \geqq 3$
(ii)	$(x_i-1)^{d(i)}$	$w(x_i) = 2,\ d(i) = \mathscr{O}^*(x_i)$
(iii)	$d(i)(x_i-1)(x_j-1)$	$w(x_i) = w(x_j) = 1,\ d(i) = \mathscr{O}^*(x_i),\ i \leqq j$
(iv)	$P(\alpha)$	*α basic,* $W(\alpha) \geqq 3$.

Proof. Elements of types (i), (ii) and (iv) clearly lie in Λ_3. If $w(x_i) = w(x_j) = 1$ then

$$(x_i-1)^{d(i)}(x_j-1) = -\sum_{k=1}^{d-1}\binom{d}{k}(x_i-1)^k(x_j-1) + (x_i^{d(i)}-1)(x_j-1)$$

is in Λ_3. Each of the terms $(x_i-1)^k(x_j-1)$, $k \geqq 2$, and $(x_i^{d(i)}-1)(x_j-1)$ belong to Λ_3 and, therefore, $d(i)(x_i-1)(x_j-1) \in \Lambda_3$.

We can show that the number of elements (i) - (iv) is equal to $\operatorname{rank}_Z(\Lambda_3) = |G| - 1$ by setting up a one-one correspondence between these elements and the non-identity basic products as follows:

$$\begin{array}{lll}
(x_i-1)^{d(i)} & \to x_i - 1 & w(x_i) = 1,\ d(i) \geqq 3 \\
(x_i-1)^{d(i)} & \to x_i - 1 & w(x_i) = 2 \\
d(i)(x_i-1)(x_j-1) & \to \begin{cases} x_i - 1 & i = j,\ w(x_i) = 1,\ d(i) = 2 \\ (x_i-1)(x_j-1) & \text{otherwise} \end{cases} & \\
P(\alpha) & \to \mathrm{P}(\alpha) & \alpha \text{ basic},\ W(\alpha) \geqq 3.
\end{array}$$

Thus if we can show that the elements of types (i) - (iv) span Λ_3 over Z then they do, in fact, constitute a Z-basis for Λ_3.

Let α be an m-sequence, $W(\alpha) \geqq 3$, and suppose for all $\beta < \alpha$, $W(\beta) \geqq 3$, $P(\beta)$ is a Z-linear combination of elements of types (i) - (iv). If α is basic then $P(\alpha)$ is of type (iv) and we are done. So assume α is not basic; then $P(\alpha) = P(\alpha_1)(x-1)^d P(\alpha_2)$, $x \in \Phi$, $d = \mathcal{O}^*(x_i)$. Replace $(x-1)^d$ by

$$-\sum_{j=1}^{d-1} \binom{d}{j} (x-1)^j + (x^d - 1),$$

replace $x^d - 1$ by its basic form (2′) and straighten the resulting terms. Then $P(\alpha)$ is expressed as a Z-linear combination of proper products $P(\beta)$, $\beta < \alpha$. If $W(\beta) \geqq 3$ for each such β then, by the induction hypothesis we are done. This will occur if $W(\alpha_1) + W(\alpha_2) \geqq 2$ or $w(x) \geqq 3$. So we may assume $W(\alpha_1) + W(\alpha_2) \leqq 1$ and $w(x) \leqq 2$. If $W(\alpha_1) + W(\alpha_2) = 0$ then $P(\alpha) = (x-1)^d$. If $w(x) = 1$ then $d \geqq 3$ and $P(\alpha)$ is of type (i); if $w(x) = 2$ then $P(\alpha)$ is of type (ii). Suppose $W(\alpha_1) + W(\alpha_2) = 1$, say $P(\alpha) = (x-1)^d(y-1)$, $w(y) = 1$. Then

$$P(\alpha) = -\sum_{j=1}^{d-1} \binom{d}{j} (x-1)^j(y-1) + (x^d-1)(y-1).$$

Replacing $x^d - 1$ by its basic form (2′) and straightening the resulting terms it follows from the induction hypothesis that

$$P(\alpha) = -d(x-1)(y-1) + \text{a } Z\text{-linear combination of products of types (i) - (iv).}$$

But $d(x-1)(y-1)$, $d = \mathcal{O}^*(x)$, is of type (iii) and we are done. There remains the case $P(\alpha) = (y-1)(x-1)^d$, $w(y) = 1$. In this case

$$P(\alpha) = -\sum_{j=1}^{d-1} \binom{d}{j} (y-1)(x-1)^j + (y-1)(x^d-1).$$

Replacing $x^d - 1$ by its basic form (2′), straightening the resulting terms and applying the induction hypothesis we have

$$P(\alpha) = -d(y-1)(x-1) + \text{a } Z\text{-linear combination of products of types (i) - (iv)}$$

where $d = \mathcal{O}^*(x)$. If $w(x) = 2$ then $(y-1)(x-1)$ is of type (iv); if $w(x) = 1$ then, since $\mathcal{O}^*(y) | \mathcal{O}^*(x) = d$, $d(y-1)(x-1)$ is an integral multiple of $\mathcal{O}^*(y)\ (y-1)(x-1)$, a product of type (iii).

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