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Equivariant Tamagawa Numbers and Galois Module Theory I

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Abstract. Let L/K be a finite Galois extension of number fields. We use complexes arising from the étale cohomology of $\mathbb{Z}$ on open subschemes of Spec $\mathcal{O}_L$ to define a canonical element of the relative algebraic K-group $K_0(\mathbb{Z}[\mathrm{Gal}(L/K)], \mathbb{R})$. We establish some basic properties of this element, and then use it to reinterpret and refine conjectures of Stark, of Chinburg and of Gruenberg, Ritter and Weiss. Our results precisely explain the connection between these conjectures and the seminal work of Bloch and Kato concerning Tamagawa numbers. This provides significant new insight into these important conjectures and also allows one to use powerful techniques from arithmetic algebraic geometry to obtain new evidence in their favour.

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Introduction

Let L/K be a finite Galois extension of number fields of group G. For any integral domain R, with field of fractions E, let $K_0(R[G], E)$ denote the Grothendieck group of the fibre category of the functor $-\otimes_R E$ from the category of finitely generated projective left $R[G]$-modules to the category of finitely generated left $E[G]$-modules.

If M is any motive which is defined over K, then the base change motive $M_L := h^0(\mathrm{Spec}\, L) \otimes_{h^0(Spec K)} M$ admits a natural left action of $\mathbb{Q}[G]$. In [3] it is shown that for each prime p the cohomological methods introduced by Bloch and Kato (cf. [2, 19, 20]) and Fontaine and Perrin-Riou (cf. [15]) allow one to attach to each motive M_L (which satisfies certain standard conjectures) a canonical element $T\Omega^p(L/K, M)$ of $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$. After making certain additional assumptions on M_L (again conjectured to hold in all cases) one can show that $T\Omega^p(L/K, M) = 0$ for almost all primes p and so one can define an element

$$T\Omega(L/K, M) := \prod_p T\Omega^p(L/K, M) \in K_0(\mathbb{Z}[G], \mathbb{Q}).$$

As a natural generalisation of the conjectures of [6, 8] the 'Equivariant Tamagawa Number Conjecture' formulated in [3] asserts that in all cases

$$T\Omega(L/K, M) = 0 \in K_0(\mathbb{Z}[G], \mathbb{Q}). \tag{1}$$

For many motives M this equality would be of considerable interest since $T\Omega(L/K, M)$ encapsulates information about the G-structure of lattices in the de-Rham, Betti and motivic cohomology spaces of M_L. Indeed, it is shown in [4] that if $H^0(K, M) = H^0(K, M^*(1)) = 0$, then the image of $T\Omega(L/K, M)$ under the connecting homomorphism $K_0(\mathbb{Z}[G], \mathbb{Q}) \to K_0(\mathbb{Z}[G])$ of relative K-theory can be interpreted in the spirit of classical Galois module theory (as surveyed for example in [9]). Recall that the category $\mathcal{M}_K$ of motives which are defined over K is expected to be semisimple. In principal therefore, to extend the results of [4] to cover all objects of $\mathcal{M}_K$ one need only deal with the motives $\mathbb{Q}(0) := h^0(\mathrm{Spec}\ K)$ and $\mathbb{Q}(1) := h^0(\mathrm{Spec}\ K)(1)$. In addition, for these motives the conjectural equalities (1) are of particular interest since they are related to existing conjectures in classical Galois module theory. In this series of papers we shall discuss these special cases in some detail. In this first paper we focus on the motive $\mathbb{Q}(0)$. By extending the approach of [7, 8] we define a canonical element $T\Omega(L/K, 0)$ of $K_0(\mathbb{Z}[G], \mathbb{R})$ and then reinterpret the Stark Conjecture (as formulated by Tate in [27]), the Strong Stark Conjecture (as formulated by Chinburg in [10]), the Ω_3-Conjecture of Chinburg (cf. [11, 12]) and the recently formulated Lifted Root Number Conjecture of Gruenberg, Ritter and Weiss (cf. [17, 18]) in terms of $T\Omega(L/K, 0)$. We also show that if Stark's Conjecture is true for L/K, then

$$T\Omega(L/K, 0) = T\Omega(L/K, \mathbb{Q}(0)) \in K_0(\mathbb{Z}[G], \mathbb{Q}). \tag{2}$$

These reinterpretations provide important new insight into the above 'classical' conjectures. For example, the central conjecture of [17, 18] (which is the finest of the aforementioned conjectures) is equivalent to an equality in $K_0(\mathbb{Z}[G], \mathbb{Q})$ and by comparing this to the equality (2) we are able to give a canonical interpretation (in terms of p-adic étale cohomology) of each p-primary component of the conjectures of loc. cit. This interpretation is new and likely to be crucial in any systematic study of the conjectures of loc. cit. (In fact our approach also leads to alternative proofs of many of the main results of loc. cit. – cf. Remark 2.3.4.) At the same time, the comparison results proved here provide useful clarification of the rather complicated constructions of [3] and, more concretely, allow one to interpret the extensive body of work in support of the conjectures of Chinburg et al. as evidence for the general conjectural equality (1). This is important since there are still very few explicit examples in which (1) has been completely verified for any motive (see [1] for some explicit results in this direction).

In [5] we consider the motive $\mathbb{Q}(1)$. More precisely, we relate the conjectural equality (1) in this case to the Ω_1-Conjecture formulated in [12], and we use the

Artin-Verdier Duality Theorem to show that the Ω_2-Conjecture of loc. cit. arises naturally when investigating the compatibility of (1) for $\mathbb{Q}(0)$ and $\mathbb{Q}(1)$ with the functional equations of the Artin L-functions associated to L/K. In effect, this result gives an interpretation in terms of arithmetical duality of the 'remarkable parallelism' of behaviour between 'additive' and 'multiplicative' results in classical Galois module theory which has often been commented upon by both Fröhlich and Chinburg (cf. [9]). In conjunction with results of this paper the results of [5] also give an affirmative answer to Question 1.54 of [6].

In further papers we will show that the approach described here leads to significant new evidence in favour of the central conjectures of [10–12, 17, 18], both in the setting of finite abelian extensions of $\mathbb{Q}$ and also in the setting of finite Galois extensions of global function fields.

The basic contents of this paper is as follows:

1. Algebraic Preliminaries

In this section we recall for the reader's convenience some of the algebraic preliminaries which underlie the constructions to be used in subsequent sections.

Throughout this manuscript, all modules are, unless explicitly stated otherwise, left modules.

1.1. RELATIVE K_0-GROUPS AND REDUCED DETERMINANTS

Let $\mathcal{A}$ be a $\mathbb{Z}$-order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A. For any field extension F of $\mathbb{Q}$ we set $A_F := A \otimes_{\mathbb{Q}} F$, and for each prime p we set $A_p := A_{\mathbb{Q}_p}$ and $\mathcal{A}_p := \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We let $\mathfrak{P}(\mathcal{A})$ and $\mathfrak{P}(A_F)$ denote the categories of finitely generated projective $\mathcal{A}$-modules and of finitely generated A_F-spaces, respectively. We write $K_0(\mathcal{A})$ and $K_0(A_F)$ for the Grothendieck groups of the categories $\mathfrak{P}(\mathcal{A})$ and $\mathfrak{P}(A_F)$, and we let $K_0(\mathcal{A}, F)$ denote the Grothendieck group of the fibre category of the functor $-\otimes_{\mathbb{Z}} F$ from $\mathfrak{P}(\mathcal{A})$ to $\mathfrak{P}(A_F)$. (An explicit description of the latter group in terms of generators and relations is given on p. 215 of [25].) We make much

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use of the following commutative diagram

$$
\begin{array}{ccccccccc}
\zeta(A)^{\times +} & \overset{\subseteq}{\to} & \zeta(A_F)^{\times +} & & & & & & \\
\uparrow \mathrm{nr}_{\mathcal{A}} & & \uparrow \mathrm{nr}_{A_F} & & & & & & \\
K_1(\mathcal{A}) & \to & K_1(A_F) & \overset{\partial^1_{\mathcal{A},F}}{\to} & K_0(\mathcal{A},F) & \overset{\partial^0_{\mathcal{A},F}}{\to} & K_0(\mathcal{A}) & \to & K_0(A_F) \\
\uparrow = & & \uparrow & & \uparrow & & \uparrow = & & \\
K_1(\mathcal{A}) & \to & K_1(A) & \overset{\partial^1_{\mathcal{A},\mathbb{Q}}}{\to} & K_0(\mathcal{A},\mathbb{Q}) & \overset{\partial^0_{\mathcal{A},\mathbb{Q}}}{\to} & K_0(\mathcal{A}) & \to & K_0(A) \\
 & & \downarrow \varepsilon & & \downarrow \gamma & & & & \\
 & & \coprod_p K_1(A_p)/\mathrm{Im}(\iota_{\mathcal{A}_p}) & \overset{\delta}{\to} & \coprod_p K_0(\mathcal{A}_p,\mathbb{Q}_p) & & & &
\end{array}
\tag{1.1.1}
$$

In this diagram we have used the following notation: $\zeta(R)$ denotes the centre of any ring R, $\mathrm{nr}_A\colon K_1(A) \to \zeta(A)^\times$ and $\mathrm{nr}_{A_F}\colon K_1(A_F) \to \zeta(A_F)^\times$ are the respective reduced norm maps, $\zeta(A)^{\times +}$ and $\zeta(A_F)^{\times +}$ denote the images of nr_A and nr_{A_F} respectively, $\mathrm{nr}_{\mathcal{A}}$ is the composite of nr_A and the natural scalar extension morphism $K_1(\mathcal{A}) \to K_1(A)$, $\partial^0_{\mathcal{A},F}([P, g, Q]) = (P) - (Q)$ for each object P and Q of $\mathfrak{B}(\mathcal{A})$ and A_F-equivariant isomorphism $g\colon P \otimes F \to Q \otimes F$, $\partial^1_{\mathcal{A},F}((A^n_F, \phi)) = [\mathcal{A}^n, \phi, \mathcal{A}^n]$ for each natural number n and $\phi \in \mathrm{Aut}_{A_F}(A^n_F)$, ε is the morphism induced by the maps $(A^n, \phi) \mapsto (A^n_p, \phi \otimes \mathbb{Q}_p)$, $\iota_{\mathcal{A}_p}$ is the scalar extension morphism $K_1(\mathcal{A}_p) \to K_1(A_p)$, δ is induced by the morphisms $\partial^1_{\mathcal{A}_p,\mathbb{Q}_p}\colon K_1(A_p) \to K_0(\mathcal{A}_p, \mathbb{Q}_p)$ which are defined in the same way as $\partial^1_{\mathcal{A},\mathbb{Q}}$, and γ is induced by the morphisms

$$[P, g, Q] \longmapsto [P \otimes \mathbb{Z}_p, g \otimes \mathbb{Q}_p, Q \otimes \mathbb{Z}_p].$$

One knows that γ is bijective, that the second and third rows of the diagram are exact and that δ is injective (cf. [25], Th. 15.5).

Let R and E denote either $\mathbb{Z}$ and $\mathbb{Q}$, or $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for some prime p. If $\mathcal{A}$ is an R-order in a finite dimensional semisimple E-algebra A, then for any field extension F of E the map nr_{A_F} is injective and we set

$$\hat{\partial}^1_{\mathcal{A},F} := \partial^1_{\mathcal{A},F} \circ \mathrm{nr}^{-1}_{A_F}\colon \zeta(A_F)^{\times +} \to K_0(\mathcal{A}, F).$$

In the case that $F = E$ we shall write $\partial^i_{\mathcal{A}}$ and $\hat{\partial}^1_{\mathcal{A}}$ in place of $\partial^i_{\mathcal{A},E}$ and $\hat{\partial}^1_{\mathcal{A},E}$ respectively.

There are two alternative descriptions of the groups $K_0(R[G], E)$ which we shall occasionally find convenient. To recall the first we let $\mathfrak{H}(R[G])$ denote the category of finite $R[G]$-modules which have finite projective dimension. Then for any object X of $\mathfrak{H}(R[G])$ there is an exact sequence of $R[G]$-modules $0 \to P^{-1} \overset{\phi}{\to} P^0 \to X \to 0$ with P^{-1} and P^0 objects of $\mathfrak{B}(R[G])$, and the association $X \mapsto [P^{-1}, \phi \otimes_R E, P^0]$ induces a well defined isomorphism

$$t_{R[G]}\colon K_0T(R[G]) \overset{\sim}{\to} K_0(R[G], E) \tag{1.1.2}$$

where $K_0T(R[G])$ is the Grothendieck group of $\mathfrak{H}(R[G])$ (with relations given by short exact sequences).

To describe the second alternative we need a little more notation. For any field F of characteristic 0, we fix an algebraic closure F^c of F and set $\Gamma(F) := \mathrm{Gal}(F^c/F)$. We write $R_F(G)$ for the ring of F-valued virtual characters of G, and let $R_F^+(G)$ denote the subset of F-valued characters. We let $\mathrm{Det}_{F[G]}: K_1(F[G]) \xrightarrow{\sim} \mathrm{Hom}_{\Gamma(F)}(R_{F^c}(G), F^{c\times})$ denote the isomorphism induced by

$$(X, f) \longmapsto \big[\chi \mapsto \det_{F^c}\big(f \mid \mathrm{Hom}_{F^c[G]}(V_\chi, F^c \otimes_F X)\big)\big], \tag{1.1.3}$$

where here X is a finitely generated $F[G]$-space, $f \in \mathrm{Aut}_{F[G]}(X)$, and for each character $\chi \in R_{F^c}^+(G)$ we have made a choice of $F^c[G]$-space V_χ which affords χ.

If $\mathcal{A} = \mathbb{Z}[G]$, then the map δ in (1.1.1) is bijective (cf. [13], Rem. 49.11(iv)). Taken in conjunction with the lower commuting square of diagram (1.1.1), the maps $\mathrm{Det}_{\mathbb{Q}_p[G]}$ therefore induce isomorphisms

$$h_{G,p}: \frac{\mathrm{Hom}_{\Gamma(\mathbb{Q}_p)}(R_{\mathbb{Q}_p^c}(G), \mathbb{Q}_p^{c\times})}{\mathrm{Det}_{\mathbb{Q}_p[G]}(\mathrm{Im}(\iota_{\mathbb{Z}_p[G]}))} \xrightarrow{\sim} K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \tag{1.1.4}$$

and, writing $J(\mathbb{Q}^c)$ for the group of ideles of $\mathbb{Q}^c$, also

$$h_G: \frac{\mathrm{Hom}^+_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J(\mathbb{Q}^c))}{\prod_p \mathrm{Det}_{\mathbb{Q}_p[G]}(\mathrm{Im}(\iota_{\mathbb{Z}_p[G]}))} \xrightarrow{\sim} K_0(\mathbb{Z}[G], \mathbb{Q}). \tag{1.1.5}$$

The numerator of the left hand side of (1.1.5) is the subset of $\mathrm{Hom}_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J(\mathbb{Q}^c))$ consisting of those functions h such that $h(\chi)_v$ is a strictly positive real number for each symplectic character χ and archimedean place v of $\mathbb{Q}^c$, and the product in the denominator includes $p = \infty$ where one sets $\mathbb{Q}_\infty = \mathbb{Z}_\infty = \mathbb{R}$. The 'Hom-descriptions' of (1.1.4) and (1.1.5) originate with Fröhlich and can be very useful computationally. More details can be found in [16] (or, perhaps more conveniently in this instance, in Appendix A of [18]).

We end this section by quickly reviewing the formalism of 'reduced determinants' used in [3].

Let E be any field and A a finite dimensional central simple E-algebra. For any A-spaces V and W we let $\mathrm{Is}_A(V, W)$ denote the set of A-equivariant isomorphisms from V to W. For any field extension E' of E and any A-space V we set $V' := V \otimes_E E'$. If E' splits A, then for any finitely generated A-spaces V and W, and any indecomposable idempotent e' of the E'-algebra A' we define an E'-line

$$\delta_{A'}(V', W') := \det_{E'}(\mathrm{Hom}_{E'}(\mathrm{Hom}_{A'}(A'e', V'), \mathrm{Hom}_{A'}(A'e', W'))).$$

This line is independent (modulo canonical isomorphisms) of the choice of e' and has a canonical action of $\mathrm{Gal}(E'/E)$. The E-line

$$\delta_A(V, W) := H^0(\mathrm{Gal}(E'/E), \delta_{A'}(V', W'))$$

is then independent of the choice of E'. Moreover, each isomorphism $\phi \in \mathrm{Is}_A(V, W)$

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gives rise to an element

$$\theta(\phi, e') \in \mathrm{Hom}_{E'}(\mathrm{Hom}_{A'}(A'e', V'), \mathrm{Hom}_{A'}(A'e', W'))$$

which is induced by mapping a homomorphism f to $(\phi \otimes_E E') \circ f$, and the 'reduced determinant' of ϕ

$$\det_A(\phi) := \det_{E'}(\theta(\phi, e'))$$

belongs to $\delta_A(V, W)$ and is independent of the choices of both E' and e'. We set

$$\delta_A^+(V, W) := \{\det_A(\phi): \phi \in \mathrm{Is}_A(V, W)\} \subset \delta_A(V, W).$$

If $V \neq 0$, then the E-line $\delta_A(V, V)$ naturally identifies with E, and we write $\delta_A^+(V)$ in place of $\delta_A^+(V, V) \subseteq E$. If $\alpha \in \mathrm{Aut}_A(V)$, then $\det_A(\alpha) = \mathrm{nr}_{\mathrm{End}_A(V)}(\alpha)$ and so $\delta_A^+(V) = \zeta(A)^{\times +}$. In accordance with the usual convention, we set $\delta_A(\{0\}, \{0\}) = E$ and $\delta_A^+(\{0\}) = \{1\}$.

If V and W are isomorphic finitely generated A-spaces, then for each element $\tau \in \delta_A^+(V, W)$ we write $\Phi(\tau)$ for its fibre under the canonical map $\det_A$: $\mathrm{Is}_A(V, W) \to \delta_A^+(V, W)$. For any element $\tau \in \delta_A^+(V_1, V_2)$ we write τ^{-1} for the element of $\delta_A^+(V_2, V_1)$ which is equal to $\det_A(\phi^{-1})$ for each $\phi \in \Phi(\tau)$.

Let now V_1, V_2 and V_3 be pairwise isomorphic finitely generated A-spaces. If ϕ_1' and ϕ_2' are E'-linear maps from $\mathrm{Hom}_{A'}(A'e', V_1')$ to $\mathrm{Hom}_{A'}(A'e', V_2')$ and from $\mathrm{Hom}_{A'}(A'e', V_2')$ to $\mathrm{Hom}_{A'}(A'e', V_3')$ respectively, then the association $\det_{E'}(\phi_2') \otimes_{E'} \det_{E'}(\phi_1') \mapsto \det_{E'}(\phi_2' \circ \phi_1')$ induces a $\mathrm{Gal}(E'/E)$-equivariant identification

$$\delta_{A'}(V_2', V_3') \otimes_{E'} \delta_{A'}(V_1', V_2') \xrightarrow{\sim} \delta_{A'}(V_1', V_3')$$

(where $\mathrm{Gal}(E'/E)$ acts diagonally on the left-hand side), and, hence, also an identification of E-lines

$$\delta_A(V_2, V_3) \otimes_E \delta_A(V_1, V_2) \xrightarrow{\sim} \delta_A(V_1, V_3). \tag{1.1.6}$$

For any elements $\tau_1 \in \delta_A^+(V_1, V_2)$ and $\tau_2 \in \delta_A^+(V_2, V_3)$ we shall write $\tau_2 \circ \tau_1$ for the element of $\delta_A^+(V_1, V_3)$ which is equal to $\det_A(\phi_2 \circ \phi_1)$ for any choice of isomorphisms $\phi_1 \in \Phi(\tau_1)$ and $\phi_2 \in \Phi(\tau_2)$. With respect to the identification (1.1.6) one therefore has $\tau_2 \otimes_E \tau_1 = \tau_2 \circ \tau_1$.

If A is now any finite-dimensional semisimple E-algebra, then reduced determinants are defined via its Wedderburn decomposition. To be specific, we suppose that $A = \prod_{i \in I} A_i$ with each A_i a central simple E_i-algebra for a suitable field extension E_i of E. Then the centre $\zeta(A)$ of A identifies with the product $\prod_{i \in I} E_i$, and the image of the reduced norm map nr_A is equal to $\zeta(A)^{\times +} = \prod_{i \in I} \zeta(A_i)^{\times +}$. For any finitely generated A-spaces V and W there are corresponding decompositions $V = \oplus_{i \in I} V_i$ and $W = \oplus_{i \in I} W_i$ and hence also $\mathrm{Is}_A(V, W) \xrightarrow{\sim} \oplus_{i \in I} \mathrm{Is}_{A_i}(V_i, W_i)$. For any isomorphism $\phi \in \mathrm{Is}_A(V, W)$ we let ϕ_i denote its component in $\mathrm{Is}_{A_i}(V_i, W_i)$ and then set $\det_A(\phi) := \prod_{i \in I} \det_{A_i}(\phi_i)$. We let $\delta_A^+(V, W)$ denote the image of the set $\mathrm{Is}_A(V, W)$ under $\det_A(-)$, and we write $\delta_A(V, W)$ for the associated $\zeta(A)$-line. A 'trivialisation', resp. 'trivialisation fibre', of $\mathrm{Is}_A(V, W)$ is an element of $\delta_A^+(V, W)$, resp. a fibre of the map $\det_A: \mathrm{Is}_A(V, W) \to \delta_A^+(V, W)$. For each $\tau \in \delta_A^+(V, W)$ we write $\Phi(\tau)$ for the corresponding trivialisation fibre of $\mathrm{Is}_A(V, W)$, and trivialisation inverses τ^{-1} and compositions $\tau_2 \circ \tau_1$ are defined componentwise. For any field extension F of E we write $\tau \otimes F$ for the image of τ under the natural inclusion $\delta_A^+(V, W) \subseteq \delta_{A \otimes_E F}^+(V \otimes_E F, W \otimes_E F)$. We recall that the terminology of 'trivialisations' is motivated by ([3], Ex. 1.1.2).

Let R be a Dedekind domain with quotient field E, and let F be a field extension of E. For each finitely generated R-module, respectively E-module, X we write X_F for $X \otimes_R F$, respectively $X \otimes_E F$. For any morphism ϕ of finitely generated R-modules, respectively E-modules, we write ϕ_F for the associated morphism of F-modules $\phi \otimes_R F$, respectively $\phi \otimes_E F$.

In the remainder of this section, we let $\mathcal{A}$ be a $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A, and F a subfield of $\mathbb{C}$. We assume that A has a Wedderburn decomposition $\prod_{i \in I} A_i$ as described above.

LEMMA 1.1.1. *Let P and Q be objects of $\mathfrak{B}(\mathcal{A})$ and τ an element of $\delta_{A_F}^+(P_F, Q_F)$. Then there is a unique element $[P, \tau, Q]$ of $K_0(\mathcal{A}, F)$ which is equal to $[P, \phi, Q]$ for any (and therefore every) $\phi \in \Phi(\tau)$.*

Proof. Let ϕ and ϕ' be any elements of $\mathrm{Is}_{A_F}(P_F, Q_F)$, and set $\phi'' := \phi' \circ \phi^{-1} \in \mathrm{Aut}_{A_F}(Q_F)$. In $K_0(\mathcal{A}, F)$ one has

$$\begin{aligned}[P, \phi', Q] - [P, \phi, Q] &= [Q, \phi'', Q] \\ &= \partial^1_{\mathcal{A},F}((Q_F, \phi'')) \\ &= \hat{\partial}^1_{\mathcal{A},F}(\det_{A_F}(\phi'')).\end{aligned}$$

The commutativity of diagram (1.1.1) implies that this element is 0 if and only if $\det_{A_F}(\phi'')$ belongs to $\mathrm{Im}(\mathrm{nr}_{\mathcal{A}}) \subset \zeta(A_F)^{\times +}$. This implies the stated result since $\det_{A_F}(\phi'') = \det_{A_F}(\phi') \otimes_{\zeta(A_F)} \det_{A_F}(\phi)^{-1}$ with respect to the identifications (1.1.6). □

LEMMA 1.1.2. *Let F be any subfield of $\mathbb{R}$.*

(i) *For any finitely generated isomorphic A-spaces V and W one has $\delta_{A_F}^+(V_F, W_F) \cap \delta_A(V, W) = \delta_A^+(V, W)$.*
(ii) *Let P and Q be objects of $\mathfrak{B}(\mathcal{A})$, and τ an element of $\delta_{A_F}^+(P_F, Q_F)$. Then $[P, \tau, Q]$ belongs to $K_0(\mathcal{A}, \mathbb{Q})$ if and only if* τ belongs to *$\delta_A(P_{\mathbb{Q}}, Q_{\mathbb{Q}})$.*

Proof. Claim (i) is a straightforward consequence of the Hasse–Schilling–Maass Norm Theorem, and is proved in ([3], Lem. 1.1.1(v)).

To prove (ii) we assume first that $\tau \in \delta_{A_F}^+(P_F, Q_F) \cap \delta_A(P_{\mathbb{Q}}, Q_{\mathbb{Q}})$. Then (i) implies that $\tau \in \delta_A^+(P_{\mathbb{Q}}, Q_{\mathbb{Q}})$ and so we may choose an isomorphism $\phi \in \mathrm{Is}_A(P_Q, Q_{\mathbb{Q}})$ such that $\det_A(\phi) = \tau$. Using Lemma 1.1.1 we deduce that $[P, \tau, Q] = [P, \phi, Q]$, and this element clearly belongs to $K_0(\mathcal{A}, \mathbb{Q})$.

To prove the converse we choose $\phi \in \Phi(\tau)$ and assume that the element $[P, \phi, Q] \in K_0(\mathcal{A}, F)$ belongs to the subgroup $K_0(\mathcal{A}, \mathbb{Q})$. Then ([25], Lem. 15.6) implies that there are objects P' and Q' of $\mathfrak{B}(\mathcal{A})$ and an isomorphism $\phi' \in \mathrm{Is}_A(P'_{\mathbb{Q}}, Q'_{\mathbb{Q}})$ such that $[P, \phi, Q] = [P', \phi', Q'] \in K_0(\mathcal{A}, F)$. By Lemma 15.8 of loc. cit. this equality implies that there are objects N and N' of $\mathfrak{B}(\mathcal{A})$, isomorphisms $\theta_1 : P \oplus N \xrightarrow{\sim} P' \oplus N'$ and $\theta_2 : Q \oplus N \xrightarrow{\sim} Q' \oplus N'$ in $\mathfrak{B}(\mathcal{A})$, and an A_F-equivariant automorphism μ of $(Q \oplus N)_F$ such that $\det_{A_F}(\mu) = 1$ and the following diagram commutes

$$\begin{array}{ccc} (P \oplus N)_F & \xrightarrow{\mu \circ [\phi \oplus (1_N)_F]} & (Q \oplus N)_F \\ \downarrow{\scriptstyle \theta_{1,F}} & & \downarrow{\scriptstyle \theta_{2,F}} \\ (P' \oplus N')_F & \xrightarrow{(\phi' \oplus 1_{N'})_F} & (Q' \oplus N')_F. \end{array} \tag{1.1.7}$$

The isomorphisms θ_1 and θ_2 together induce a commuting diagram

$$\begin{array}{ccc} \delta^+_A(P_{\mathbb{Q}}, Q_{\mathbb{Q}}) & \xrightarrow{\iota_A} & \delta^+_A(P'_{\mathbb{Q}}, Q'_{\mathbb{Q}}) \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\ \delta^+_{A_F}(P_F, Q_F) & \xrightarrow{\iota_{A_F}} & \delta^+_{A_F}(P'_F, Q'_F) \end{array} \tag{1.1.8}$$

in which the vertical maps are the natural inclusions, the map ι_A is the bijection induced by mapping

$$\det_A(g) \longmapsto \det_A\big(\theta_2 \circ (g \oplus 1_N) \circ \theta_1^{-1}\big) = \det_A(g')$$

where here $g \in \mathrm{Is}_A(P_{\mathbb{Q}}, Q_{\mathbb{Q}})$ and g' is any element of $\mathrm{Is}_A(P'_{\mathbb{Q}}, Q'_{\mathbb{Q}})$ which makes the stated equality valid, and the bijective map ι_{A_F} is defined similarly. Since $\det_{A_F}(\mu) = 1$ the commutativity of (1.1.7) implies that $\iota_{A_F}\big(\det_{A_F}(\phi)\big) = \det_A(\phi') \in \delta^+_A(P'_{\mathbb{Q}}, Q'_{\mathbb{Q}})$. The commutativity of (1.1.8) now implies that $\det_{A_F}(\phi) \in \delta^+_A(P_{\mathbb{Q}}, Q_{\mathbb{Q}})$, and this completes the proof of (ii). □

Choose $\phi \in \mathrm{Is}_A(Q_{\mathbb{Q}}, P_{\mathbb{Q}})$ and $\tau \in \delta_{A_F}(P_F, Q_F)$. Then $\tau \otimes_{\zeta(A_F)} \det_A(\phi) \in \delta_{A_F}(Q_F, Q_F) = \zeta(A_F)$ and $\tau \in \delta_A(P, Q)$ if and only if $\tau \otimes_{\zeta(A_F)} \det_A(\phi)$ is fixed by the natural action of $\mathrm{Gal}(F/\mathbb{Q})$ on $\zeta(A_F) = \zeta(A) \otimes_{\mathbb{Q}} F$. In addition, if E' is any field extension of E_i, then there is a natural $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-equivariant inclusion

$$E_i \otimes_{\mathbb{Q}} F \subseteq E' \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \prod_{\Sigma(E')} \mathbb{C}$$

where $\Sigma(E')$ denotes the set of embeddings of E' into $\mathbb{C}$. For any $\lambda = (\lambda_\sigma)_\sigma \in E' \otimes_{\mathbb{Q}} \mathbb{C}$ and $\omega \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ one has $(\omega\lambda)_\sigma = \omega(\lambda_{\omega^{-1} \circ \sigma})$ for each $\sigma \in \Sigma(E')$. Putting this all together gives the following useful criterion:

LEMMA 1.1.3. *Let E' be any subfield of $\mathbb{C}$ which contains splitting fields for the central simple E_i-algebra A_i for each $i \in I$, and F any subfield of $\mathbb{C}$. Let*

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$\tau \in \delta_{A_F}(P_F, Q_F)$ *and* $\phi \in \mathrm{Is}_A(Q_{\mathbb{Q}}, P_{\mathbb{Q}})$, *and set*

$$\tau(\phi) := \tau \otimes_{\zeta(A_F)} \det_A(\phi) \in \delta_{A_F}(Q_F, Q_F) \xrightarrow{\sim} \zeta(A_F) = \prod_{i\in I} E_i \otimes_{\mathbb{Q}} F \subseteq \prod_{i\in I} E' \otimes_{\mathbb{Q}} \mathbb{C}.$$

Then $\tau \in \delta_A(P_{\mathbb{Q}}, Q_{\mathbb{Q}})$ *if and only if* $\tau(\phi)_{i,\omega\circ\sigma} = \omega(\tau(\phi)_{i,\sigma})$ *for each* $i \in I$, $\sigma \in \Sigma(E')$ *and* $\omega \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. □

1.2. REFINED EULER CHARACTERISTICS

In this section we review the Euler characteristic construction introduced in [3]. We let R be a Dedekind domain with quotient field E, and $\mathcal{A}$ an R-order in a finite dimensional semisimple E-algebra A.

Let $C^\bullet$ be a complex of A-modules. In each degree i we write $B^i(C^\bullet)$, $Z^i(C^\bullet)$ and $H^i(C^\bullet)$ for the A-modules of coboundaries, cocycles and cohomology, and write $d^i(C^\bullet)\colon C^i \to B^{i+1}(C^\bullet)$ for the corresponding differential. In each degree i there are tautological exact sequences

$$0 \to Z^i(C^\bullet) \xrightarrow{\subseteq} C^i \xrightarrow{d^i(C^\bullet)} B^{i+1}(C^\bullet) \to 0$$

$$0 \to B^i(C^\bullet) \xrightarrow{\subseteq} Z^i(C^\bullet) \xrightarrow{\pi^i(C^\bullet)} H^i(C^\bullet) \to 0,$$

and so, after choosing A-equivariant sections σ^i and μ^i to $d^i(C^\bullet)$ and $\pi^i(C^\bullet)$ respectively, one obtains an isomorphism

$$\theta(\sigma^i, \mu^i)\colon B^{i+1}(C^\bullet) \oplus H^i(C^\bullet) \oplus B^i(C^\bullet) \xrightarrow{\sim} C^i, \quad (x, y, z) \mapsto \sigma_i(x) + \mu_i(y) + z.$$

For any field extension F of E, we set

$$C_F^o := \bigoplus_{i\in\mathbb{Z}} C_F^{2i+1} \qquad C_F^e := \bigoplus_{i\in\mathbb{Z}} C_F^{2i},$$
$$H^o(C_F^\bullet) := \bigoplus_{i\in\mathbb{Z}} H^{2i+1}(C_F^\bullet) \quad \text{and} \quad H^e(C_F^\bullet) := \bigoplus_{i\in\mathbb{Z}} H^{2i}(C_F^\bullet).$$

For each morphism of complexes of A_F-modules $\theta\colon C_F^\bullet \to D^\bullet$ we set

$$H^o(\theta) := \prod_{i\in\mathbb{Z}} H^{2i+1}(\theta) \quad \text{and} \quad H^e(\theta) := \prod_{i\in\mathbb{Z}} H^{2i}(\theta).$$

For each $\phi \in \mathrm{Is}_{A_F}(H^o(C_F^\bullet), H^e(C_F^\bullet))$ we let $\phi(\sigma^\bullet, \mu^\bullet)\colon C_F^o \xrightarrow{\sim} C_F^e$ denote the A_F-module

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isomorphism which is obtained as the composite of

$$\bigoplus_{i\in\mathbb{Z}}\theta(\sigma^{2i+1},\mu^{2i+1})_F^{-1}\colon C_F^o \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} B^{2i+2}(C_F^\bullet)\oplus H^{2i+1}(C_F^\bullet)\oplus B^{2i+1}(C_F^\bullet),$$

$$\bigoplus_{i\in\mathbb{Z}} B^{2i+2}(C_F^\bullet)\oplus H^{2i+1}(C_F^\bullet)\oplus B^{2i+1}(C_F^\bullet)\xrightarrow{\sim}\left(\bigoplus_{j\in\mathbb{Z}} B^j(C_F^\bullet)\right)\oplus H^o(C_F^\bullet)$$

$$1\oplus\phi\colon\left(\bigoplus_{j\in\mathbb{Z}} B^j(C_F^\bullet)\right)\oplus H^o(C_F^\bullet)\xrightarrow{\sim}\left(\bigoplus_{j\in\mathbb{Z}} B^j(C_F^\bullet)\right)\oplus H^e(C_F^\bullet)$$

$$\left(\bigoplus_{j\in\mathbb{Z}} B^j(C_F^\bullet)\right)\oplus H^e(C_F^\bullet)\xrightarrow{\sim}\bigoplus_{i\in\mathbb{Z}} B^{2i+1}(C_F^\bullet)\oplus H^{2i}(C_F^\bullet)\oplus B^{2i}(C_F^\bullet)$$

and

$$\bigoplus_{i\in\mathbb{Z}}\theta(\sigma^{2i},\mu^{2i})_F\colon\bigoplus_{i\in\mathbb{Z}} B^{2i+1}(C_F^\bullet)\oplus H^{2i}(C_F^\bullet)\oplus B^{2i}(C_F^\bullet)\xrightarrow{\sim} C_F^e$$

(where the second and fourth listed isomorphisms are the obvious ones).

We assume now that $C^\bullet$ is a bounded complex of finitely generated A-modules. Then for any element τ of $\delta^+_{A_F}(H^o(C_F^\bullet), H^e(C_F^\bullet))$ there is a unique element $\tau(C_F^\bullet)$ of $\delta^+_{A_F}(C_F^o, C_F^e)$ which is equal to $\det_{A_F}(\phi(\sigma^\bullet,\mu^\bullet))$ for any $\phi\in\Phi(\tau)$ and any choice of sections $\{\sigma^i,\mu^j\}_{i,j}$ as above (cf. [3], Lem. 1.1.3). If τ_1 and τ_2 both belong to $\delta^+_{A_F}(H^o(C_F^\bullet), H^e(C_F^\bullet))$, then one has

$$\tau_1(C_F^\bullet)^{-1}\circ\tau_2(C_F^\bullet)=\tau_1^{-1}\circ\tau_2\in\zeta(A_F)^\times. \tag{1.2.1}$$

In particular, if $C^\bullet$ is acyclic, then there is a unique element $\iota(C_F^\bullet)$ of $\delta^+_{A_F}(C_F^o, C_F^e)$ which is obtained in this manner. We recall that if A is commutative, then these constructions can be interpreted in terms of the determinantal formalism of Grothendieck, Knudsen and Mumford as introduced in [21] (cf. [3], Rem. 1.1.4).

For any ring Λ we let $\mathfrak{D}(\Lambda)$ denote the derived category of the homotopy category of bounded complexes of Λ-modules, and we write $\mathfrak{D}^{perf}(\Lambda)$ for the full triangulated subcategory of $\mathfrak{D}(\Lambda)$ which consists of those complexes which are perfect. We say that a Λ-module X is 'perfect' if the associated complex $X[0]$ belongs to $\mathfrak{D}^{perf}(\Lambda)$.

Any bounded complex of finitely generated projective $\mathcal{A}$-modules $P^\bullet$ has a natural Euler characteristic $\chi_{\mathcal{A}}P^\bullet$ in $K_0(\mathcal{A})$ which, following [6, 7, 8], we normalise in the following way

$$\chi_{\mathcal{A}}P^\bullet:=\sum_{i\in\mathbb{Z}}(-1)^{i+1}(P^i)\in K_0(\mathcal{A}).$$

The mapping cone of any quasi-isomorphism is acyclic, and so the assignment $P^\bullet\mapsto\chi_{\mathcal{A}}P^\bullet$ induces a well-defined map from $\mathfrak{D}^{perf}(\mathcal{A})$ to $K_0(\mathcal{A})$.

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We now recall the definition and basic properties of the Euler characteristic construction of [3].

PROPOSITION 1.2.1. *Let $P^\bullet$ be a bounded complex of finitely generated projective $\mathcal{A}$-modules. Let F be a field extension of E, and τ an element of $\delta^+_{A_F}(H^o(P^\bullet_F), H^e(P^\bullet_F))$. Set $\chi_{\mathcal{A},F}(P^\bullet, \tau) := [P^o, \tau(P^\bullet_F), P^e] \in K_0(\mathcal{A}, F)$.*

(i) $\partial^0_{\mathcal{A},F}(\chi_{\mathcal{A}}(P^\bullet, \tau)) = \chi_{\mathcal{A}} P^\bullet \in K_0(\mathcal{A})$.

(ii) *For any element τ' of $\delta^+_{A_F}(H^o(P^\bullet_F), H^e(P^\bullet_F))$, one has $\chi_{\mathcal{A},F}(P^\bullet, \tau') - \chi_{\mathcal{A},F}(P^\bullet, \tau) = \hat{\partial}^1_{\mathcal{A},F}(\tau^{-1} \circ \tau')$.*

(iii) *Let $Q^\bullet$ be any bounded complex of finitely generated projective $\mathcal{A}$-modules and $\rho: P^\bullet \to Q^\bullet$ an $\mathcal{A}$-equivariant quasi-isomorphism. If τ_ρ denotes the unique element of $\delta^+_{A_F}(H^o(Q^\bullet_F), H^e(Q^\bullet_F))$ which is equal to $\det_{A_F}\big(H^e(\rho_F) \circ \phi \circ H^o(\rho_F)^{-1}\big)$ for any $\phi \in \Phi(\tau)$, then $\chi_{\mathcal{A},F}(P^\bullet, \tau) = \chi_{\mathcal{A},F}(Q^\bullet, \tau_\rho)$.*

(iv) *If $E = \mathbb{Q}$ and $F \subseteq \mathbb{R}$, then the following conditions are equivalent.*

(a) τ belongs to the subspace $\delta_A\big(H^o(P^\bullet_{\mathbb{Q}}), H^e(P^\bullet_{\mathbb{Q}})\big)$ of $\delta_{A_F}\big(H^o(P^\bullet_F), H^e(P^\bullet_F)\big)$.

(b) $\tau \circ \det_{A_F}(\phi_F) \in \zeta(A)$ for any choice of isomorphism $\phi \in \mathrm{Is}_A(H^e(P^\bullet_{\mathbb{Q}}), H^o(P^\bullet_{\mathbb{Q}}))$.

(c) $\chi_{\mathcal{A},F}(P^\bullet, \tau) \in K_0(\mathcal{A}, \mathbb{Q})$.

Proof. Claim (i) is obvious, and (iii) is an immediate consequence of ([3], Th. 1.2.1(ii)). The equivalence of (iv)(a) and (iv)(b) follows from the remarks made prior to Lemma 1.1.3, and that of (iv)(a) and (iv)(c) follows from Lemma 1.1.2(ii) and the fact that $\tau(P^\bullet_F)$ is constructed using sections σ^i and μ^i which are defined over $\mathbb{Q}$. We note finally that claim (ii) follows from the equalities

$$\begin{aligned}
&\chi_{\mathcal{A},F}(P^\bullet, \tau') - \chi_{\mathcal{A},F}(P^\bullet, \tau)\\
&\quad = [P^o, \tau'(P^\bullet_F), P^e] - [P^o, \tau(P^\bullet_F), P^e]\\
&\quad = [P^o, \tau'(P^\bullet_F), P^e] + [P^e, \tau(P^\bullet_F)^{-1}, P^o]\\
&\quad = [P^o, \tau(P^\bullet_F)^{-1} \circ \tau'(P^\bullet_F), P^o]\\
&\quad = \hat{\partial}^1_{\mathcal{A},F}(\tau(P^\bullet_F)^{-1} \circ \tau'(P^\bullet_F))\\
&\quad = \hat{\partial}^1_{\mathcal{A},F}(\tau^{-1} \circ \tau'),
\end{aligned}$$

where the last equality here is a consequence of (1.2.1). □

Let $C^\bullet$ be an object of $\mathfrak{D}^{perf}(\mathcal{A})$ and τ an element of $\delta^+_{A_F}(H^o(C^\bullet_F), H^e(C^\bullet_F))$. Then Proposition 1.2.1(iii) allows one to unambiguously define an element $\chi_{\mathcal{A},F}(C^\bullet, \tau)$ of $K_0(\mathcal{A}, F)$. We refer to the pair $(C^\bullet, \tau)$ as a 'trivialised perfect complex', and to $\chi_{\mathcal{A},F}(C^\bullet, \tau)$ as its 'refined Euler characteristic'. Given an isomorphism $\theta \in \mathrm{Is}_{A_F}(H^o(C^\bullet_F), H^e(C^\bullet_F))$ we shall often write $(C^\bullet, \theta)$ in place of $(C^\bullet, \det_{A_F}(\theta))$.

We now suppose given trivialised perfect complexes $(C^\bullet_j, \tau_j)$ for $j \in \{1, 2, 3\}$, and a distinguished triangle in $\mathfrak{D}^{perf}(\mathcal{A})$

$$C^\bullet_1 \to C^\bullet_2 \to C^\bullet_3. \tag{1.2.2}$$

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Let $\Delta_F^\bullet$ denote the long exact sequence of cohomology which is associated to the triangle in $\mathfrak{D}^{perf}(A_F)$ obtained by applying the exact functor $-\otimes_R F$ to (1.2.2). Set $H^i_{j,F} := H^i(C_j^\bullet)_F$, $H^o_{j,F} := H^o(C_j^\bullet)_F$ and $H^e_{j,F} := H^e(C_j^\bullet)_F$ for each $i \in \mathbb{Z}$ and $j \in \{1, 2, 3\}$. We regard $\Delta_F^\bullet$ as an acyclic complex, with $H^0_{1,F}$ placed in degree 0. Then there are natural isomorphisms of A_F-spaces

$$\Delta_F^o \cong H^o_{1,F} \oplus H^e_{2,F} \oplus H^o_{3,F}, \qquad \Delta_F^e \cong H^e_{1,F} \oplus H^o_{2,F} \oplus H^e_{3,F}$$

and these isomorphisms together induce an identification of $\zeta(A_F)$-lines

$$\delta_{A_F}(\Delta_F^o, \Delta_F^e) \cong \delta_{A_F}(H^o_{1,F}, H^e_{1,F}) \otimes \delta_{A_F}(H^e_{2,F}, H^o_{2,F}) \otimes \delta_{A_F}(H^o_{3,F}, H^e_{3,F}).$$

If, with respect to this identification, one has

$$\iota(\Delta_F^\bullet) = \tau_1 \otimes \tau_2^{-1} \otimes \tau_3, \tag{1.2.3}$$

then we say that there exists a 'distinguished triangle of trivialised perfect complexes'

$$(C_1^\bullet, \tau_1) \to (C_2^\bullet, \tau_2) \to (C_3^\bullet, \tau_3) \tag{1.2.4}$$

which refines the triangle (1.2.2).

PROPOSITION 1.2.2 (cf. [3], Th. 1.2.7). *For any distinguished triangle of trivialised perfect complexes* (1.2.4) *one has*

$$\chi_{\mathcal{A},F}(C_1^\bullet, \tau_1) - \chi_{\mathcal{A},F}(C_2^\bullet, \tau_2) + \chi_{\mathcal{A},F}(C_3^\bullet, \tau_3) = 0 \in K_0(\mathcal{A}, F). \qquad \square$$

Remark 1.2.3. In certain cases the compatibility condition (1.2.3) is straightforward to verify. For example, if each complex $C^\bullet_{j,F}$ is acyclic outside degrees 0 and 1 and $\Delta_F^\bullet$ takes the form

$$0 \to H^0_{1,F} \xrightarrow{\alpha} H^0_{2,F} \xrightarrow{\beta} H^0_{3,F} \xrightarrow{0} H^1_{1,F} \xrightarrow{\gamma} H^1_{2,F} \xrightarrow{\delta} H^1_{3,F} \to 0,$$

then the trivialisations τ_j satisfy (1.2.3) if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{1,F} & \xrightarrow{\gamma} & H^1_{2,F} & \xrightarrow{\delta} & H^1_{3,F} & \longrightarrow & 0 \\ & & \downarrow{\psi_1} & & \downarrow{\psi_2} & & \downarrow{\psi_3} & & \\ 0 & \longrightarrow & H^0_{1,F} & \xrightarrow{\alpha} & H^0_{2,F} & \xrightarrow{\beta} & H^0_{3,F} & \longrightarrow & 0 \end{array}$$

with $\psi_j \in \Phi(\tau_j)$ for each $j \in \{1, 2, 3\}$.

If $\mathcal{A} = \mathbb{Z}[G]$ for a finite group G, then we shall always abbreviate $\partial^i_{\mathcal{A},F}(-)$, $\hat{\partial}^i_{\mathcal{A},F}(-)$, $\chi_{\mathcal{A},F}(-,-)$ and $t_{\mathcal{A}}(-)$ to $\partial^i_{G,F}(-)$, $\hat{\partial}^i_{G,F}(-)$, $\chi_{G,F}(-,-)$ and $t_G(-)$ respectively. If $F = \mathbb{Q}$, then we shall further abbreviate $\partial^i_{G,\mathbb{Q}}(-)$, $\hat{\partial}^i_{G,\mathbb{Q}}(-)$ and $\chi_{G,\mathbb{Q}}(-,-)$ to $\partial^i_G(-)$, $\hat{\partial}^i_G(-)$ and $\chi_G(-,-)$ respectively.

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2. h^0(Spec K)

Let L/K be a finite Galois extension of number fields. In this section we apply the constructions of Section 1 to complexes arising from étale cohomology of the constant sheaf $\mathbb{Z}$ in order to define a canonical element of $K_0(\mathbb{Z}[\mathrm{Gal}(L/K)], \mathbb{R})$. We then reinterpret conjectures of Stark, of Chinburg, of Gruenberg, Ritter and Weiss and of the author in terms of this element.

For any $\mathbb{Q}$-algebra A we write $\mathcal{M}_K(A)$ for the full subcategory of $\mathcal{M}_K$ consisting of those motives which admit a left action of A.

At the outset we fix a finite group G, set $A := \mathbb{Q}[G]$ and write $A = \prod_{i\in I} A_i$ for the associated Wedderburn decomposition. For each index $i \in I$ we fix an identification of $\zeta(A_i)$ with a subfield E_i of $\mathbb{Q}^c$. Having fixed such identifications each component A_i corresponds to a unique irreducible E_i-valued character χ_i of G in the following way: if E_i' is any field extension of E_i such that the algebra $A_i' := A_i \otimes_{E_i} E_i'$ is split, and e_i' is any indecomposable idempotent of A_i', then the character χ_i of the simple $E_i'[G]$-module $V_i := A_i'e_i'$ is E_i-valued and independent of the choices of both E_i' and e_i'. For any irreducible $\mathbb{Q}^c$-character χ of G there is a unique index i such that $\chi = \alpha \circ \chi_i$ for some $\alpha \in \Gamma(\mathbb{Q})$. In particular, if E' is any subfield of $\mathbb{Q}^c$ which contains splitting fields E_i' for each index i, then for each $\mathbb{Q}^c$-character χ there is an E'-space V_χ which realises χ. To each such space V_χ there is associated an object $[\chi]$ of $\mathcal{M}_K(E')$ (depending, to within isomorphism, only on χ) as described explicitly in ([14], 5.3). Each motive $[\chi]$ has pure weight 0 and Hodge type $(0, 0)$.

We now assume that $\mathrm{Gal}(L/K)$ can be identified with G. Having fixed such an identification, we fix an embedding $\iota: L \to K^c$ and let $\pi: \Gamma(K) \to G$ denote the surjection defined by $\gamma\iota(\lambda) = \iota(\pi(\gamma)(\lambda))$ for all $\lambda \in L$ and $\gamma \in \Gamma(K)$. Then the motive $\mathbb{Q}(0)_L := h^0(\mathrm{Spec}\ L)$ has an induced structure as an object of $\mathcal{M}_K(A)$. Moreover, if $A^\#$ denotes the $G \times \Gamma(K)$-module which consists of A with $G \times 1$ acting via left multiplication and $1 \times \Gamma(K)$ via the surjection π and the contragredient right action of G on A, then the actions of $G \times \Gamma(K)$ on each of the realisations of $\mathbb{Q}(0)_L$ can be described explicitly in terms of $A^\#$. For example, the l-adic realisation $H_l(\mathbb{Q}(0)_L)$ can be identified with $\mathbb{Q}_l[G]$ as a left G-module. In this identification the (left) action of $\gamma \in \Gamma(K)$ is given by

$$\gamma(x) := x\pi(\gamma^{-1}), \quad x \in \mathbb{Q}_l[G]. \tag{2.0.1}$$

The decomposition $A = \prod_{i\in I} A_i$ in turn induces a decomposition $\mathbb{Q}(0)_L = \oplus_{i\in I}\mathbb{Q}(0)_{L,i}$ with $\mathbb{Q}(0)_{L,i}$ an object of $\mathcal{M}_K(A_i)$ such that there is a natural identification in $\mathcal{M}_K(E')$

$$e_i'(\mathbb{Q}(0)_{L,i} \otimes_{E_i} E') = [\overline{\chi_i}], \tag{2.0.2}$$

where here $\overline{\chi_i}$ denotes the contragredient of χ_i (and occurs because $\Gamma(K)$ acts via (2.0.1)).

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For any number field F we write $S(F)$, $S_\infty(F)$ and $S_f(F)$ for the set of all places, all Archimedean places and all non-Archimedean places of F respectively. For each $v \in S(K)$ we write $S_v(L)$ for the set of places of L which lie above v.

Let S be a finite subset of $S(K)$ which contains $S_\infty(K)$. To each object of $\mathcal{M}_K(A)$, one can associate an S-truncated A-equivariant L-function (cf. ([14], 2.12) and [3]). We now recall the explicit description of this function for the motive $\mathbb{Q}(0)_L$.

For each $v \in S_f(K)$ we fix an embedding $K^c \to K_v^c$ and use this to regard $\Gamma(K_v)$ as a subgroup of $\Gamma(K)$. We let K_v^{un} denote the maximal unramified extension of K_v in K_v^c, and write G_v and I_v for the images of $\Gamma(K_v)$ and $\Gamma(K_v^{un})$ in G under the surjection $\pi\colon \Gamma(K) \to G$. We write f_v for the image of the (arithmetic) Frobenius element in G_v/I_v. For each $v \in S_f(K)$ and $s \in \mathbb{C}$ we set

$$\varepsilon_v(s) := \det_{\mathbb{C}[G]}\big(1 - f_v^{-1}(Nv)^{-s} \mid H^0(I_v, A^\# \otimes_{\mathbb{Q}} \mathbb{C})\big).$$

If V is any $\mathbb{Q}[G]$-module, then we set $V_v^0 := H^0(G_v, V)$ and $V_v^1 := H^0(I_v, V)$. For each $\mathbb{Q}[f_v]$-module U we write $U^\#$ for the contragredient module (that is, $U^\# = U$ as $\mathbb{Q}$-module but $f_v(u^\#) = f_v^{-1}(u)$ for each $u \in U$).

For any function $g(s)$ of a complex variable s which has algebraic order d at a point s_0 we set $g^*(s_0) := \lim_{s\to s_0}(s - s_0)^{-d} g(s)$.

LEMMA 2.0.1. *For each $v \in S_f(K)$ and $s \in \mathbb{C}$ one has*

$$\varepsilon_v(s) = \prod_{i\in I} \det_{E'\otimes_{\mathbb{Q}}\mathbb{C}}\big(1 - f_v(Nv)^{-s} \mid V^1_{i,v} \otimes_{\mathbb{Q}} \mathbb{C}\big) \in \zeta(\mathbb{C}[G])^\times.$$

This function (of the complex variable s) satisfies

$$\varepsilon_v^*(0) = \prod_{i\in I} (\log(Nv))^{\dim_{E'}(V^0_{i,v})} \det_{E'}\big(1 - f_v \mid V^1_{i,v}/V^0_{i,v}\big) \in \zeta(\mathbb{C}[G])^\times.$$

Proof. For each index i we let $\varepsilon_v(s)_i$ denote the A_i-component of $\varepsilon_v(s)$ according to the decomposition $\mathbb{Q}(0)_L = \oplus_{i\in I}\mathbb{Q}(0)_{L,i}$. Then for each $s \in \mathbb{C}$ one has

$$\begin{aligned}\varepsilon_v(s)_i &= \det_{E'\otimes_{\mathbb{Q}}\mathbb{C}}(1 - f_v^{-1}(Nv)^{-s} \mid \mathrm{Hom}_G(V_i, \mathrm{ind}_{G_v}^G E'[G_v]_v^{1\#}) \otimes_{\mathbb{Q}} \mathbb{C})\\ &= \det_{E'\otimes_{\mathbb{Q}}\mathbb{C}}\big(1 - f_v^{-1}(Nv)^{-s} \mid \mathrm{Hom}_{G_v}(V_i, E'[G_v]_v^{1\#}) \otimes_{\mathbb{Q}} \mathbb{C}\big)\\ &= \det_{E'\otimes_{\mathbb{Q}}\mathbb{C}}\big(1 - f_v(Nv)^{-s} \mid V^1_{i,v} \otimes_{\mathbb{Q}} \mathbb{C}\big)\end{aligned}$$

where the last equality follows because the $E'[f_v]$-spaces $\mathrm{Hom}_{G_v}(V_i, E'[G_v]_v^{1\#})$ and $(V^1_{i,v})^\#$ are naturally isomorphic. This explicit formula proves the first assertion. In conjunction with the tautological exact sequence

$$0 \to V^0_{i,v} \to V^1_{i,v} \to V^1_{i,v}/V^0_{i,v} \to 0$$

it also implies that

$$\varepsilon_v(s)_i = (1 - (Nv)^{-s})^{\dim_{E'\otimes_{\mathbb{Q}}\mathbb{C}}(V^0_{i,v})} \det_{E'\otimes_{\mathbb{Q}}\mathbb{C}}\big(1 - f_v(Nv)^{-s} \mid (V^1_{i,v}/V^0_{i,v}) \otimes_{\mathbb{Q}} \mathbb{C}\big).$$

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By considering Laurent expansions of these expressions at $s = 0$ we obtain the stated formula for $\varepsilon_v^*(0)$. □

For each finite subset S of $S(K)$ which contains $S_\infty(K)$, the S-truncated G-equivariant L-function of the motive $\mathbb{Q}(0)_L$ is defined by the formal product

$$L_S(s) := \prod_{v \in S(K)\setminus S} \varepsilon_v(s)^{-1}.$$

From Lemma 2.0.1 it follows that

$$L_S(s) = \prod_{i\in I} \prod_{\sigma\in\Sigma(E_i)} L_S(\sigma \circ \chi_i, s) \in \prod_{i\in I} E_i \otimes_{\mathbb{Q}} \mathbb{C}, \tag{2.0.3}$$

where for each $\chi \in R_{\mathbb{C}}(G)$ we write $L_S(\chi, s)$ for the Artin L-function of χ as defined in ([27], Chap. 0,§4) and then truncated by removing Euler factors corresponding to places in S. (The formula (2.0.3) is compatible with (2.0.2) because each function $L_S(\chi, s)$ is defined using the elements f_v rather than f_v^{-1}).

We end this preliminary section with some comments concerning linear duality. For any commutative ring R the ring $R[G]$ is Gorenstein and so if X is a projective $R[G]$-module, then $\mathrm{Hom}_R(X, R)$ is a projective $R[G]$-module (when endowed with the contragredient action of G). The functor $\mathbb{R}\mathrm{Hom}_R(-, R)$ thus restricts to give a functor on $\mathfrak{D}^{perf}(R[G])$, and we write $\mathbf{D}$(-) and $\mathbf{D}_p(-)$ for the restricted functor with respect to $R = \mathbb{Z}$ and $R = \mathbb{Z}_p$ respectively. We write ψ_G^*, ψ_G^* and $\psi_{G,p}^*$ for the involutions on $K_0(\mathbb{Z}[G], \mathbb{R})$, $K_0(\mathbb{Z}[G], \mathbb{Q})$ and $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ which are induced by $\mathbf{D}$(-), $\mathbf{D}$(-) and $\mathbf{D}_p(-)$ respectively. For example, if $[P_1, \phi, P_2] \in K_0(\mathbb{Z}[G], \mathbb{R})$, then

$$\psi_G^*([P_1, \phi, P_2]) = [\mathrm{Hom}_{\mathbb{Z}}(P_1, \mathbb{Z}), \phi^D, \mathrm{Hom}_{\mathbb{Z}}(P_2, \mathbb{Z})] \tag{2.0.4}$$

with $\phi^D := \mathrm{Hom}_{\mathbb{R}}(\phi, \mathbb{R})^{-1}$.

Assume now that R is a field. Then $\mathbb{R}\mathrm{Hom}_R(-, R) = \mathrm{Hom}_R(-, R)$ and whenever R is clear from context we write this more simply as $-^*$. Let V_1 and V_2 be finitely generated isomorphic $R[G]$-spaces. Then for each element $\tau \in \delta^+_{R[G]}(V_1, V_2)$ we write τ^* and τ^D for the elements of $\delta^+_{R[G]}(V_2^*, V_1^*)$ and $\delta^+_{R[G]}(V_1^*, V_2^*)$ which are equal to $\det_{R[G]}(\phi^*)$ and $\det_{R[G]}(\phi^D)$ for any (and therefore every) isomorphism $\phi \in \Phi(\tau)$.

Let E be a field extension of R which splits $R[G]$, and write $x \mapsto x^\#$ for the E-linear involution of $E[G]$ for which $g^\# = g^{-1}$ for each $g \in G$. If V is any finitely generated $R[G]$-space, then for each idempotent e of $E[G]$ there is a canonical identification of E-spaces

$$\mathrm{Hom}_{E[G]}(E[G]e, (E \otimes_R V)^*) = \mathrm{Hom}_{E[G]}(E[G]e^\#, E \otimes_R V)^*.$$

These identifications imply that for each $\tau \in \delta^+_{R[G]}(V) \subseteq \zeta(R[G])^{\times +}$ one has

$$\tau^* = \tau^\#, \quad \tau^D = (\tau^{-1})^\#. \tag{2.0.5}$$

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2.1. THE DEFINITION OF $T\Omega(L/K, 0)$

For any finite subset S of $S(K)$ we let Y_S denote the free Abelian group on the set of places of L lying above those in S, and we write X_S for the kernel of the natural augmentation map $Y_S \to \mathbb{Z}$. Both Y_S and X_S have natural G-actions. If S contains $S_\infty(K)$, then we let $\mathcal{O}_{L,S}$ denote the subring of L consisting of those elements which are integral at all places which do not lie above those in S, and we write U_S for the unit group of $\mathcal{O}_{L,S}$. In case $S = S_\infty(K)$ we write U, Y and X in place of U_S, Y_S and X_S respectively. We say that a finite subset S of $S(K)$ is 'admissible for L/K' if it contains $S_\infty(K)$ together with those places which ramify in L/K and is in addition sufficiently large that $\mathrm{Pic}(\mathcal{O}_{L,S})$ is trivial.

For each $v \in S_f(K)$ we set $Y_v := Y_{\{v\}}$ and let $C_v^\bullet$ denote the complex

$$H^0(I_v, \mathbb{Z}[G]) \xrightarrow{1-f_v^{-1}} H^0(I_v, \mathbb{Z}[G])$$

where here the modules are placed in degrees 0 and 1. If $w \in S_v(L)$ is the place which is induced by the fixed embeddings $L \to K^c \to K_v^c$, then for each $i \in \{0, 1\}$ there is a natural isomorphism of G-modules

$$H^i(C_v^\bullet) \xrightarrow{\iota_v^i} Y_v \tag{2.1.1}$$

which is induced by $\iota_v^0(x) = (\#G_v)^{-1}xw$ for each $x \in H^0(G_v, \mathbb{Z}[G])$ and $\iota_v^1(x) = (\#I_v)^{-1}xw$ for each $x \in H^0(I_v, \mathbb{Z}[G])$.

For any complex of Abelian groups C we write $C^\vee$ for its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$. We write $\hat{\mathbb{Z}}$ for the profinite completion of $\mathbb{Z}$.

The following result is proved in [7] by using the Artin–Verdier Duality Theorem.

PROPOSITION 2.1.1 (cf. Prop. 3.1 and 3.2 of **[7]**). *If S is admissible for L/K, then there exists an exact sequence of G-modules*

$$0 \to U_S \to \Psi_S^0 \to \Psi_S^1 \to X_S \to 0 \tag{2.1.2}$$

with the following properties:

(i) *Ψ_S^0 and Ψ_S^1 are $\mathbb{Z}[G]$-perfect.*

(ii) *Let $\tilde{\Psi}_S^\bullet$ denote the complex $\Psi_S^0 \to \Psi_S^1 \to X_{S,\mathbb{Q}}$, where the maps are induced by those in (2.1.2) and Ψ_S^0 is placed in degree 0. Then there is a map $\tilde{\Psi}_S^\bullet \to R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z})^\vee[-3]$ in $\mathfrak{D}(\mathbb{Z}[G])$ which induces an isomorphism on $H^i(-)$ for $i \neq 0$ and the inclusion of U_S into $U_S \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ for $i = 0$.*

(iii) *Let $\Psi_S^\bullet$ denote the naive truncation in degree 1 of $\tilde{\Psi}_S^\bullet$. If T is any finite set of places which is disjoint from S, then there is a distinguished triangle in $\mathfrak{D}(\mathbb{Z}[G])$*

$$\Psi_S^\bullet \to \Psi_{S\cup T}^\bullet \to \bigoplus_{v\in T} C_v^\bullet \tag{2.1.3}$$

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whose long exact cohomology sequence splits into two short exact sequences

$$0 \to U_S \to U_{S\cup T} \xrightarrow{v} Y_T \to 0 \tag{2.1.4}$$

$$0 \to X_S \to X_{S\cup T} \to Y_T \to 0 \tag{2.1.5}$$

in which all maps are the natural ones (in particular, v is given by taking valuations). Here we have used the maps (2.1.1) *together with* (2.1.2) *(for S and $S \cup T$) to identify the cohomology modules of* (2.1.3). □

For each $w \in S(L)$ we let $| - |_w$ denote the canonically normalised absolute value of w (cf. [27], Chap. 0, 0.2). For any finite subset S of $S(K)$ which contains $S_\infty(K)$ we let $R_S: U_{S,\mathbb{R}} \xrightarrow{\sim} X_{S,\mathbb{R}}$ denote the $\mathbb{R}[G]$-equivariant isomorphism which satisfies $R_S(u) = -\sum_{w\in S(L)} \log |u|_w \cdot w$ for each $u \in U_S$. (The normalisation of R_S chosen here is motivated by the following result and the computations of [8, 18].)

THEOREM 2.1.2. (i) *There exists an element λ of $\zeta(\mathbb{Q}[G])^\times$ such that*

$$\lambda \cdot L_S^*(0)^\# \in \zeta(\mathbb{R}[G])^{\times +} \tag{2.1.6}$$

for all finite subsets S of $S(K)$ which contain $S_\infty(K)$. For any such λ the element

$$T\Omega(L/K, \lambda, 0) := \psi_G^*(\chi_{G,\mathbb{R}}(\Psi_S^\bullet, R_S^{-1}) + \hat{\partial}_{G,\mathbb{R}}^1(\lambda L_S^*(0)^\#)) \in K_0(\mathbb{Z}[G], \mathbb{R})$$

is independent of the choice of S which is admissible for L/K.

(ii) *Let $x \in \zeta(\mathbb{Q}[G])^\times$ and, for each prime p, let x_p denote the image of x in $\zeta(\mathbb{Q}_p[G])^\times = \mathrm{Im}(\mathrm{nr}_{\mathbb{Q}_p[G]})$. Then the element $T_G(x) := \prod_p \hat{\partial}_{\mathbb{Z}_p[G]}^1(x_p)$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$. If λ satisfies* (2.1.6)*, then the element*

$$T\Omega(L/K, 0) := T\Omega(L/K, \lambda, 0) - \psi_G^*(T_G(\lambda)) \in K_0(\mathbb{Z}[G], \mathbb{R})$$

depends only upon the extension L/K.

Proof. Let τ denote complex conjugation. One has $\zeta(\mathbb{C}[G]) = \prod_{i\in I}\prod_{\sigma\in\Sigma(E_i)} \mathbb{C}$ and, with respect to this decomposition, an element $x = \prod_{i\in I}\prod_{\sigma\in\Sigma(E_i)} x_{i,\sigma}$ of $\zeta(\mathbb{C}[G])^\times$ belongs to $\zeta(\mathbb{R}[G])^\times$, respectively $\zeta(\mathbb{R}[G])^{\times +}$, if and only if $\tau(x_{i,\sigma}) = x_{i,\tau\circ\sigma}$ for all i and σ, respectively $x \in \zeta(\mathbb{R}[G])^\times$ and $x_{i,\sigma}$ is a strictly positive real number whenever χ_i is symplectic.

For each finite subset S of $S(K)$ which contains $S_\infty(K)$, each character $\chi \in R_\mathbb{C}(G)$ and each $s \in \mathbb{R}$, one has $L_S(\tau \circ \chi, s) = \tau L_S(\chi, s)$. The formula (2.0.3) therefore implies that $L_S^*(0)^\# \in \zeta(\mathbb{R}[G])^\times$. The existence of an element $\lambda \in \zeta(\mathbb{Q}[G])^\times$ which satisfies (2.1.6) for any given set S is therefore a simple consequence of the Weak Approximation Theorem.

We now choose $v' \in S(K) \setminus S$ and set $S' := S \cup \{v'\}$. Combining Lemma 2.0.1 with the equality $L_{S'}(s) = L_S(s)\varepsilon_{v'}(s)$ gives

$$\left(L^*_{S'}(0)^{\#}/L^*_S(0)^{\#}\right)_{i,\sigma} = (\log(Nv'))^{dim_{E'}(V^0_{i,v'})}\sigma(\det_{E'}\left(1 - f_{v'}^{-1} \mid V^1_{i,v'}/V^0_{i,v'}\right)) \tag{2.1.7}$$

for each $i \in I$ and $\sigma \in \Sigma(E_i)$. Also, if V_i is symplectic, then the eigenvalues of $f_{v'}$ on $V^1_{i,v'}/V^0_{i,v'}$ are equal to -1 or else occur in complex conjugate pairs. Hence (2.1.7) implies that $L^*_{S'}(0)^{\#}/L^*_S(0)^{\#} \in \zeta(\mathbb{R}[G])^{\times +}$. This implies that any element λ chosen as above automatically satisfies (2.1.6) for all finite subsets S of $S(K)$ which contain $S_\infty(K)$.

To prove the rest of (i) we assume that S is admissible for L/K and verify that $T\Omega(L/K, \lambda, 0)$ does not change if one replaces S by S'. After taking into account (2.1.7), we must prove that

$$\begin{aligned}&\chi_{G,\mathbb{R}}(\Psi^\bullet_{S'}, R^{-1}_{S'}) - \chi_{G,\mathbb{R}}(\Psi^\bullet_S, R^{-1}_S)\\ &\quad = \hat{\partial}^1_{G,\mathbb{R}}\left(\prod_{i\in I}(\log(Nv'))^{-dim_{E'}(V^0_{i,v'})}\det_{E'}\left(1 - f_{v'}^{-1} \mid V^1_{i,v'}/V^0_{i,v'}\right)^{-1}\right).\end{aligned} \tag{2.1.8}$$

To compute the left-hand side of this expression we use the following lemma.

LEMMA 2.1.3. *Set $C^\bullet_0 := C^\bullet_{v'}$ and $Y_0 := Y_{v'}$. Choose $w' \in S_{v'}(L)$ and let ψ_0: $H^1(C^\bullet_{0,\mathbb{R}}) \to H^0(C^\bullet_{0,\mathbb{R}})$ denote the isomorphism which is induced by the identifications* (2.1.1) *and scalar multiplication by* $(\log(Nw'))^{-1}$ *on $Y_{0,\mathbb{R}}$. Then there is a distinguished triangle of trivialised perfect complexes*

$$\left(\Psi^\bullet_S, R^{-1}_S\right) \to \left(\Psi^\bullet_{S'}, R^{-1}_{S'}\right) \to \left(C^\bullet_0, \psi_0\right) \tag{2.1.9}$$

which refines the triangle (2.1.3) *with* $T = \{v'\}$.

Proof. This result follows by using the criterion of Remark 1.2.3 in conjunction with the natural morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_{S,\mathbb{R}} & \longrightarrow & X_{S',\mathbb{R}} & \longrightarrow & Y_{0,\mathbb{R}} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle R_S^{-1}} & & \downarrow{\scriptstyle R_{S'}^{-1}} & & \downarrow{\scriptstyle \cdot(log(Nw'))^{-1}} & & \\ 0 & \longrightarrow & U_{S,\mathbb{R}} & \longrightarrow & U_{S',\mathbb{R}} & \longrightarrow & Y_{0,\mathbb{R}} & \longrightarrow & 0, \end{array}$$

where the upper and lower rows are induced by (2.1.5) and (2.1.4) with $T = \{v'\}$ respectively. □

Set $\tau_0 := \det_{\mathbb{R}[G]}(\psi_0)$ and $\hat{\tau}_0 := \tau_0(C^\bullet_{0,\mathbb{R}})$. Then, since $C^0_0 = C^1_0 (= \mathbb{Z}[G])$, one has $\chi_{G,\mathbb{R}}\left(C^\bullet_0, \psi_0\right) = \hat{\partial}^1_{G,\mathbb{R}}(\hat{\tau}_0)$. Applying Proposition 1.2.2 to (2.1.9) therefore gives an equality

$$\chi_{G,\mathbb{R}}\left(\Psi^\bullet_{S'}, R^{-1}_{S'}\right) - \chi_{G,\mathbb{R}}\left(\Psi^\bullet_S, R^{-1}_S\right) = \hat{\partial}^1_{G,\mathbb{R}}(\hat{\tau}_0). \tag{2.1.10}$$

To compute this we set $e_0 := \sum_{g\in G_w} g$ for some place $w \in S_{v'}(L)$. Then

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$\mathbb{R}[G] = \mathbb{R}[G]e_0 \oplus \mathbb{R}[G](1-e_0)$ and, with respect to this decomposition (and the identifications (2.1.1)), the fibre $\Phi(\hat{\tau}_0)$ contains the map ϕ_0 which is defined by

$$(x, y) \mapsto (\#G_{v'}(\log(Nw'))^{-1}x, (1-f_{v'}^{-1})^{-1}y) = ((\log(Nv'))^{-1}x, (1-f_{v'}^{-1})^{-1}y).$$

Now $\hat{\tau}_0 = \det_{\mathbb{R}[G]}(\phi_0)$ and a straightforward computation shows that

$$\det_{\mathbb{R}[G]}(\phi_0) = \prod_{i\in I}(\log(Nv'))^{-dim_{E'}(V^0_{i,v'})}\det_{E'}\left(1-f_{v'}^{-1} \mid V^1_{i,v'}/V^0_{i,v'}\right)^{-1} \in \zeta(\mathbb{R}[G])^\times.$$

The required equality (2.1.8) is therefore a consequence of (2.1.10). This completes the proof of (i).

The proof of (ii) is now easy. Since $x_p \in \zeta(\mathbb{Z}_p[G])^\times = \ker(\hat{\partial}^1_{\mathbb{Z}_p[G]})$ for almost all p it is clear that $T_G(x)$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$. To prove that $T\Omega(L/K, 0)$ depends only on L/K, it suffices to show it does not depend on the choice of λ. But if λ' is any other element of $\zeta(\mathbb{Q}[G])^\times$ which satisfies (2.1.6), then $\lambda'\lambda^{-1} \in \zeta(\mathbb{Q}[G])^{\times +}$ and hence

$$\begin{aligned}\hat{\partial}^1_{G,\mathbb{R}}(\lambda' L^*_S(0)^\#) - \hat{\partial}^1_{G,\mathbb{R}}(\lambda L^*_S(0)^\#) &= \hat{\partial}^1_G(\lambda'\lambda^{-1})\\ &= T_G(\lambda'\lambda^{-1})\\ &= T_G(\lambda') - T_G(\lambda).\end{aligned}$$ □

The element $T\Omega(L/K, 0)$ can in fact be described naturally in terms of Fitting Ideals of finite perfect $\mathbb{Z}[G]$-modules (cf. [1]), but we shall have no need of this description here. In subsequent sections we use $T\Omega(L/K, 0)$ to reinterpret and refine several well known conjectures, but to end this section we now describe its basic functorial properties.

We let R be any Dedekind domain, and F a field extension of the quotient field of R. For each subgroup H of G we let ρ^G_H denote the restriction of scalars functor from $\mathfrak{P}(R[G])$ to $\mathfrak{P}(R[H])$, and we let $\rho^1_{G,H}: K_1(F[G]) \to K_1(F[H])$ and $\rho^0_{G,H}: K_0(R[G], F) \to K_0(R[H], F)$ denote the morphisms which are induced by the assignments $(V, \phi) \mapsto (\rho^G_H V, \rho^G_H \phi)$ and $[P, \phi, Q] \mapsto [\rho^G_H P, \rho^G_H \phi, \rho^G_H Q]$, respectively. If J is a normal subgroup of G, then we write X^J for the (projective) $R[G/J]$-module $H^0(J, X)$, and we let $\pi^1_{G,G/J}: K_1(F[G]) \to K_1(F[G/J])$ and $\pi^0_{G,G/J}: K_0(R[G], F) \to K_0(R[G/J], F)$ denote the morphisms which are induced by the assignments $(V, \phi) \mapsto (V^J, \phi^J)$ and $[P, \phi, Q] \mapsto [P^J, \phi^J, Q^J]$, respectively.

PROPOSITION 2.1.4. *Let H be a subgroup of G.*

(i) $\rho^0_{G,H}(T\Omega(L/K, 0)) = T\Omega(L/L^H, 0) \in K_0(\mathbb{Z}[H], \mathbb{R})$.
(ii) *If H is normal in G, then* $\pi^0_{G,G/H}(T\Omega(L/K, 0)) = T\Omega(L^H/K, 0) \in K_0(\mathbb{Z}[G/H], \mathbb{R})$.

Proof. We first give interpretations of the maps $\rho^i_{G,H}$ and $\pi^i_{G,G/H}$ for $i \in \{0, 1\}$ in terms of the diagram (1.1.1).

Let F be an algebraically closed field of characteristic 0, and for each finite group Γ let $\mathrm{Ir}_F(\Gamma)$ denote the set of irreducible F-valued characters of Γ. Then $\zeta(F[\Gamma])$ can be

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naturally identified with $\prod_{\mathrm{Ir}_F(\Gamma)} F$. For any subgroup Δ of Γ we let $\rho^{\Gamma}_{\Delta}: \zeta(F[\Gamma])^{\times} \to \zeta(F[\Delta])^{\times}$ denote the composite homomorphism

$$\zeta(F[\Gamma])^{\times} = \prod_{\mathrm{Ir}_F(\Gamma)} F^{\times} \xrightarrow{\kappa^{\Gamma}_{\Delta}} \prod_{\mathrm{Ir}_F(\Delta)} F^{\times} = \zeta(F[\Delta])^{\times},$$

where $\left(\kappa^{\Gamma}_{\Delta}(x)\right)_{\phi} = \prod_{\chi | \mathrm{ind}^{\Gamma}_{\Delta}\phi} x_{\chi}$ for each $x = (x_{\chi})_{\chi \in \mathrm{Ir}_F(\Gamma)}$ and $\phi \in \mathrm{Ir}_F(\Delta)$. This homomorphism is equivariant with respect to the action of $\mathrm{Gal}(F/E)$ for any subfield E of F, and hence maps $\zeta(E[\Gamma])^{\times}$ to $\zeta(E[\Delta])^{\times}$. If Δ is a normal subgroup of Γ we let e_{Δ} denote the idempotent $\frac{1}{\#\Delta}\sum_{\delta\in\Delta}\delta \in \zeta(\mathbb{C}[\Gamma])$ and write $\pi^{\Gamma}_{\Gamma/\Delta}: \zeta(\mathbb{C}[\Gamma])^{\times} \to \zeta(\mathbb{C}[\Gamma/\Delta])^{\times}$ for the surjective homomorphism induced by multiplication by e_{Δ}.

LEMMA 2.1.5. *Let R denote either $\mathbb{Z}$ or $\mathbb{Z}_p$, and let F be a field extension of the quotient field of R. Let H, respectively J, be a subgroup, respectively normal subgroup, of G. Then there are commutative diagrams*

$$\begin{array}{ccc} K_0(R[G],F) & \xrightarrow{\rho^0_{G,H}} & K_0(R[H],F) \\ \uparrow{\scriptstyle \partial^1_{R[G],F}} & & \uparrow{\scriptstyle \partial^1_{R[H],F}} \\ K_1(F[G]) & \xrightarrow{\rho^1_{G,H}} & K_1(F[H]) \\ \downarrow{\scriptstyle \mathrm{nr}_{F[G]}} & & \downarrow{\scriptstyle \mathrm{nr}_{F[H]}} \\ \zeta(F[G])^{\times} & \xrightarrow{\rho^G_H} & \zeta(F[H])^{\times} \end{array} \qquad \begin{array}{ccc} K_0(R[G],F) & \xrightarrow{\pi^0_{G,G/J}} & K_0(R[G/J],F) \\ \uparrow{\scriptstyle \partial^1_{R[G],F}} & & \uparrow{\scriptstyle \partial^1_{R[G/J],F}} \\ K_1(F[G]) & \xrightarrow{\pi^1_{G,G/J}} & K_1(F[G/J]) \\ \downarrow{\scriptstyle \mathrm{nr}_{F[G]}} & & \downarrow{\scriptstyle \mathrm{nr}_{F[G/J]}} \\ \zeta(F[G])^{\times} & \xrightarrow{\pi^G_{G/J}} & \zeta(F[G/J])^{\times}. \end{array}$$

Proof. Commutativity is clear except for the lower square of the left-hand diagram. In this case the commutativity can be proved by reducing to the case that F is algebraically closed, and then using the arguments of ([13], (52.9), (52.22)). □

We now need to be precise about field extensions and so shall write $\Psi^{\bullet}_{L/K,S}$, $U_{L,S}$, $X_{L,S}$ and $R_{L,S}$ in place of $\Psi^{\bullet}_S$, U_S, X_S and R_S respectively.

To prove (i) we let S' denote the set of places of L^H which lie above those in S. It is then clear that S' is admissible for L/L^H, and that the obvious identifications $\rho^G_H\Psi^{\bullet}_{L/K,S} = \Psi^{\bullet}_{L/L^H,S'}$ and $\rho^G_H R_{L,S} = R_{L,S'}$ combine to give an equality

$$\rho^0_{G,H}(\chi_{G,\mathbb{R}}(\Psi^{\bullet}_{L/K,S}, R^{-1}_{L,S})) = \chi_{H,\mathbb{R}}(\Psi^{\bullet}_{L/L^H,S'}, R^{-1}_{L,S'}). \tag{2.1.11}$$

On the other hand, by using the left-hand diagram of Lemma 2.1.5, one has

$$\begin{aligned} \rho^0_{G,H}(\hat{\partial}^1_{G,\mathbb{R}}(\lambda L^*_S(0)^{\#})) - \rho^0_{G,H}(T_G(\lambda)) &= \hat{\partial}^1_{H,\mathbb{R}}\left(\rho^G_H(\lambda)\rho^G_H(L^*_S(0)^{\#})\right) - T_H(\rho^G_H(\lambda)) \\ &= \hat{\partial}^1_{H,\mathbb{R}}\left(\rho^G_H(\lambda)L^*_{S'}(0)^{\#}\right) - T_H(\rho^G_H(\lambda)). \end{aligned}$$

Here we write $L^*_{S'}(0)$ for the leading term at $s=0$ of the S'-truncated L-function of $h^0(\mathrm{Spec}\ L)$ considered as an object of $\mathcal{M}_{L^H}(\mathbb{Q}[H])$. The last displayed equality follows because $\rho^G_H(L^*_S(0)) = L^*_{S'}(0)$ as a consequence of the inductivity property of L-functions, and implies in particular that $\rho^G_H(\lambda)L^*_{S'}(0)^{\#} \in \zeta(\mathbb{R}[H])^{\times +}$ (that is,

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$\rho^G_H(\lambda)$ satisfies (2.1.6) for the extension L/L^H). Proposition 2.1.4(i) now follows upon combining this equality with (2.1.11).

To prove Proposition 2.1.4(ii) we let J be a normal subgroup of G and set $Q := G/J$. Taking advantage of Theorem 2.1.2, we increase S if necessary so that it is also admissible for L^H/K. Writing $L_{S,J}(s)$ for the S-truncated L-function of $h^0(\mathrm{Spec}\, L^J)$ considered as an object of $\mathcal{M}_K(\mathbb{Q}[Q])$, one has $e_J L^*_S(0) = L^*_{S,J}(0)$. The second commuting diagram of Lemma 2.1.5 therefore implies that

$$\pi^0_{G,Q}(\hat{\partial}^1_{G,\mathbb{R}}(\lambda L^*_S(0)^\#)) - \pi^0_{G,Q}(T_G(\lambda)) = \hat{\partial}^1_{Q,\mathbb{R}}(e_J\lambda.L^*_{S,J}(0)^\#) - T_Q(e_J\lambda)$$

(so that, in particular, $e_J\lambda.L^*_{S,J}(0)^\# \in \zeta(\mathbb{R}[Q])^{\times +}$). On the other hand, since $\Psi^\bullet_{L/K,S}$ consists of perfect $\mathbb{Z}[G]$-modules one has

$$\pi^0_{G,Q}(\chi_{G,\mathbb{R}}(\Psi^\bullet_{L/K,S}, R^{-1}_{L,S})) = \chi_{Q,\mathbb{R}}(\mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, \Psi^\bullet_{L/K,S}), (R^{-1}_{L,S})^J),$$

and hence it suffices to prove that

$$\chi_{Q,\mathbb{R}}(\mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, \Psi^\bullet_{L/K,S}), (R^{-1}_{L,S})^J) = \chi_{Q,\mathbb{R}}(\Psi^\bullet_{L^J/K,S}, R^{-1}_{L^J,S}). \tag{2.1.12}$$

Set $T_J := \sum_{g\in J} g \in \mathbb{Z}[G]$. Following ([27], Chap. I, §6.5) we embed $X_{L^J,S}$ as a submodule of $X_{L,S}$ via the mapping induced by $w \mapsto T_J w'$ for each $v \in S$, $w \in S_v(L^J)$ and $w' \in S_w(L)$. Via this embedding one has $X_{L^J,S} = T_J X_{L,S}$. Using the fact that S is admissible for both L/K and L^J/K one computes that $\mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, \Psi^\bullet_{L/K,S})$ is acyclic outside degrees 0 and 2 and has cohomology $(U_{L,S})^J$ and $(T_J X_{L,S}) \otimes \mathbb{Q}/\mathbb{Z}$ in these respective degrees. Furthermore, in the course of proving ([7], Lem. 11) it is shown that there exists a $\mathfrak{D}(\mathbb{Z}[Q])$-isomorphism

$$\tilde{\theta}: \mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, \tilde{\Psi}^\bullet_{L/K,S}) \xrightarrow{\sim} \tilde{\Psi}^\bullet_{L^J/K,S}$$

which induces the natural identifications $(U_{L,S})^J = U_{L^J,S}$ and $(T_J X_{L,S}) \otimes \mathbb{Q}/\mathbb{Z} = X_{L^J,S} \otimes \mathbb{Q}/\mathbb{Z}$ on cohomology. Let F denote either L or L^J and set $H := \mathrm{Gal}(F/K)$. Since $X_{F,S} \otimes \mathbb{Q}$ is an injective $\mathbb{Z}[H]$-module and $H^1(\tilde{\Psi}^\bullet_{F/K,S}) = 0$ there is a canonical identification

$$\mathrm{Hom}_{\mathfrak{D}(\mathbb{Z}[H])}((X_{F,S} \otimes \mathbb{Q})[-2], \tilde{\Psi}^\bullet_{F/K,S}) = \mathrm{Hom}_H(X_{F,S} \otimes \mathbb{Q}, H^2(\tilde{\Psi}^\bullet_{F/K,S}))$$

(cf. [7], Lem. 7(b)). In particular, there exists a distinguished triangle in $\mathfrak{D}(\mathbb{Z}[H])$

$$(X_{F,S} \otimes \mathbb{Q})[-2] \to \tilde{\Psi}^\bullet_{F/K,S} \to \Psi^\bullet_{F/K,S} \tag{2.1.13}$$

in which the first morphism corresponds to the natural projection map $X_{F,S} \otimes \mathbb{Q} \to X_{F,S} \otimes \mathbb{Q}/\mathbb{Z}$. Consider now the diagram in $\mathfrak{D}(\mathbb{Z}[Q])$

$$\begin{array}{ccccc}
(X_{L,S} \otimes \mathbb{Q})^J[-2] & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, \tilde{\Psi}^\bullet_{L/K,S}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, \Psi^\bullet_{L/K,S}) \\
\downarrow & & \downarrow{\scriptstyle \tilde{\theta}} & & \\
(X_{L^J,S} \otimes \mathbb{Q})[-2] & \longrightarrow & \tilde{\Psi}^\bullet_{L^J/K,S} & \longrightarrow & \Psi^\bullet_{L^J/K,S}.
\end{array}$$

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The rows of this diagram are the distinguished triangles coming from (2.1.13), and the left hand vertical map is the composite $(X_{L,S}\otimes\mathbb{Q})^J=(T_J X_{L,S})\otimes\mathbb{Q}=X_{L^J,S}\otimes\mathbb{Q}$. The square commutes on the level of cohomology and hence in $\mathfrak{D}(\mathbb{Z}[Q])$. By completing the diagram to a morphism of triangles one deduces that there exists a Q-equivariant quasi-isomorphism $\theta\colon \mathrm{Hom}_{\mathbb{Z}[J]}(\mathbb{Z},\Psi^{\bullet}_{L/K,S})\tilde{\to}\Psi^{\bullet}_{L^J/K,S}$ such that the following diagram commutes

$$\begin{array}{ccc}(X_{L,S}\otimes\mathbb{R})^J & \xrightarrow{(R^{-1}_{L,S})^J} & (U_{L,S}\otimes\mathbb{R})^J\\ \downarrow{\scriptstyle H^1(\theta)_{\mathbb{R}}} & & \downarrow{\scriptstyle H^0(\theta)_{\mathbb{R}}}\\ X_{L^J,S}\otimes\mathbb{R} & \xrightarrow{R^{-1}_{L^J,S}} & U_{L^J,S}\otimes\mathbb{R}.\end{array}$$

The required equality (2.1.12) now follows by applying Proposition 1.2.1(iii) to the morphism θ. □

2.2. $T\Omega(L/K,0)$ AND THE STRONG STARK CONJECTURE

In this section we use $T\Omega(L/K,0)$ to reinterpret the Stark Conjecture (as formulated by Tate in [27]) and the Strong Stark Conjecture (formulated by Chinburg in [10]). Throughout we regard L/K as fixed, and write $T\Omega$ for $T\Omega(L/K,0)$.

For each character $\chi\in R^+_{\mathbb{C}}(G)$ we choose a $\mathbb{C}[G]$-space V_χ which affords χ. For any finite subset S of $S(K)$ which contains $S_\infty(K)$ we choose an injective G-morphism $\varphi_S\colon X_S\to U_S$, and for each $\chi\in R^+_{\mathbb{C}}(G)$ we set

$$A_{\varphi_S}(\chi):=\frac{\det_{\mathbb{C}}\big(R_S\circ\varphi_{S,\mathbb{R}}\mid \mathrm{Hom}_G(V_{\bar\chi},X_{S,\mathbb{C}})\big)}{L^*_S(\chi,0)}\in\mathbb{C}^\times.$$

CONJECTURE 2.2.1 (The Stark Conjecture, ([27], Chap. I, 5.1)). *For each $\chi\in R^+_{\mathbb{C}}(G)$ and $\omega\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ one has $A_{\varphi_S}(\omega\circ\chi)=\omega\big(A_{\varphi_S}(\chi)\big)$.*

We write $\mathcal{O}$ for the ring of integers of E'. By replacing E' with a larger field if necessary, we may assume that for each $\chi\in R^+_{\mathbb{C}}(G)$ there exists an $\mathcal{O}[G]$-lattice T_χ such that $V_\chi=T_\chi\otimes_{\mathcal{O}}\mathbb{C}$. If Conjecture 2.2.1 is true, then each complex number $A_{\varphi_S}(\chi)$ belongs to E' and, assuming this to be the case, the Strong Stark Conjecture predicts the fractional $\mathcal{O}$-ideals which are generated by $A_{\varphi_S}(\chi)$ for any set S which is admissible for L/K. To recall the precise conjecture, we let $C^\bullet_{\varphi_S}(T_\chi)$ denote the complex of $\mathcal{O}$-modules

$$H_0(G,\mathrm{Hom}_{\mathcal{O}}(T_\chi,X_S\otimes_{\mathbb{Z}}\mathcal{O}))\to H^0(G,\mathrm{Hom}_{\mathcal{O}}(T_\chi,U_S\otimes_{\mathbb{Z}}\mathcal{O}))$$

where the modules are placed in degrees 0 and 1, and the differential is induced by the composite morphism $\varphi_S\circ(\sum_{g\in G}g)$. We set

$$q_{\varphi_S}(\chi):=\mathrm{char}_{\mathcal{O}}(H^0(C^\bullet_{\varphi_S}(T_\chi)))^{-1}\mathrm{char}_{\mathcal{O}}(H^1(C^\bullet_{\varphi_S}(T_\chi)))$$

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where, for any finite $\mathcal{O}$-module N we let $\mathrm{char}_{\mathcal{O}}(N)$ denote the ideal of $\mathcal{O}$ with the property that $(\mathrm{char}_{\mathcal{O}}(N))_{\wp} = (\wp\mathcal{O}_{\wp})^{length_{\mathcal{O}_{\wp}} N_{\wp}}$ for each maximal ideal $\wp$ of $\mathcal{O}$.

CONJECTURE 2.2.2 (The Strong Stark Conjecture, ([10], Conj. 2.2)). *Assume the validity of* Conjecture 2.2.1. *If S is admissible for L/K, then $A_{\varphi_S}(\chi)\mathcal{O} = q_{\varphi_S}(\overline{\chi})$ for each $\chi \in R_{\mathbb{C}}^{+}(G)$.*

Remarks 2.2.3. (i) The Stark and Strong Stark Conjectures are formulated in terms of complex numbers $A'_{\varphi_S}(\chi)$ which are defined just as $A_{\varphi_S}(\chi)$ but with R_S replaced by $-R_S$. It is however easy to check that the original conjectures involving $A'_{\varphi_S}(\chi)$ are equivalent to the above stated conjectures.

(ii) Conjecture 2.2.1 is independent of the choice of set S and embedding φ_S ([27], Chap. I, 7.3 and 6.2). Conjecture 2.2.2 is independent of the choices of S, φ_S and each lattice T_χ (cf. [10] or ([23], §7)). Conjecture 2.2.2 was first motivated by the partial verification of Conjecture 2.2.1 described by Tate in ([27], Chap. II, 6.8), and is related to an earlier conjecture of Lichtenbaum (cf. [22]).

(iii) Since $\mathcal{O}$ is a regular ring, each ideal $q_{\varphi_S}(\chi)$ can be described using the determinantal formalism of [21]. Indeed, for any finite $\mathcal{O}$-module N one has $\mathrm{char}_{\mathcal{O}}(N) = \det_{\mathcal{O}} N[0]$ and, hence, $q_{\varphi_S}(\chi) = \det_{\mathcal{O}} C^{\bullet}_{\varphi_S}(T_\chi)$ for each $\chi \in R_{\mathbb{C}}^{+}(G)$ (where here we have normalised the determinant functor as in [6,7,8] rather than [21]).

We are now ready to state the main result of this section.

THEOREM 2.2.4. *$T\Omega$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if* Conjecture 2.2.1 *is true for L/K. If this is the case, then $T\Omega$ has finite order if and only if* Conjecture 2.2.2 *is true for L/K.*

Before proving Theorem 2.2.4, we introduce a little more notation.

Recall that $\mathbb{Q}[G]$ has a Wedderburn decomposition $\prod_{i\in I} A_i$. For each index i let $L_S^*(0)_i^{\#}$ denote the component of $L_S^*(0)^{\#}$ which corresponds to the A_i-component $\mathbb{Q}(0)_{L,i}$ of $\mathbb{Q}(0)_L$. If $A_i X_{S,\mathbb{Q}} = 0$, then ([27], Chap. I, 3.4 and Chap. III, §1) together imply that $L_S^*(0)_i^{\#} \in E_i^{\times} \subset (E' \otimes \mathbb{C})^{\times}$. Taking this into account, we shall henceforth suppose that the element $\lambda = \prod_{i\in I} \lambda_i$ in (2.1.6) is chosen so that $\lambda_i L_S^*(0)_i^{\#} = 1$ if $A_i X_{S,\mathbb{Q}} = 0$. For any such λ, one has $\lambda L_S^*(0)^{\#} \in \delta^{+}_{\mathbb{R}[G]}(X_{S,\mathbb{R}})$, and we set $\tau_S(\lambda) := \det_{\mathbb{R}[G]}(R_S^{-1}) \circ (\lambda L_S^*(0)^{\#}) \in \delta^{+}_{\mathbb{R}[G]}(X_{S,\mathbb{R}}, U_{S,\mathbb{R}})$. If S is admissible for L/K, then Proposition 1.2.1(ii) implies

$$T\Omega(L/K, \lambda, 0) := \psi_G^*(\chi_{G,\mathbb{R}}(\Psi_S^{\bullet}, \tau_S(\lambda))). \tag{2.2.1}$$

Using this equality, the first assertion of Theorem 2.2.4 follows from Proposition 1.2.1(iv) together with the following lemma:

LEMMA 2.2.5. *Let S be any finite subset of $S(K)$ which contains $S_\infty(K)$, and λ any element of $\mathbb{Q}[G]^{\times}$ as above. Then* Conjecture 2.2.1 *is true for L/K if and only if $\tau_S(\lambda) \in \delta^{+}_{\mathbb{Q}[G]}(X_{S,\mathbb{Q}}, U_{S,\mathbb{Q}})$.*

Proof. In $\zeta(\mathbb{R}[G])^\times$ one has equalities

$$\begin{aligned}
&\tau_S(\lambda)^{-1} \otimes_{\zeta(\mathbb{R}[G])} \det_{\mathbb{R}[G]}(\varphi_{S,\mathbb{R}}) \\
&\quad = (\lambda L_S^*(0)^{\#})^{-1}\det_{\mathbb{R}[G]}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid X_{S,\mathbb{R}}\big) \\
&\quad = \prod_{i\in I}\prod_{\sigma\in\Sigma(E_i)} \frac{\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(A_i'e_i' \otimes_{E',\sigma} \mathbb{C}, X_{S,\mathbb{C}})\big)}{\sigma(\lambda_i)L_S^*(0)^{\#}_{i,\sigma}} \\
&\quad = \prod_{i\in I}\prod_{\sigma\in\Sigma(E_i)} \frac{\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(V_{\sigma\circ\chi_i}, X_{S,\mathbb{C}})\big)}{\sigma(\lambda_i)L_S^*(\sigma\circ\overline{\chi_i},0)}
\end{aligned}$$

where the last equality follows from (2.0.3) and the fact that, for each $i \in I$ and $\sigma \in \Sigma(E_i)$, the $\mathbb{C}[G]$-module $A_i'e_i' \otimes_{E',\sigma} \mathbb{C}$ is isomorphic to $V_{\sigma\circ\chi_i}$. Taken together, Lemmas 1.1.2(i) and 1.1.3 therefore imply that $\tau_S(\lambda)^{-1} \in \delta^+_{\mathbb{Q}[G]}(U_{S,\mathbb{Q}}, X_{S,\mathbb{Q}})$ if and only if

$$\begin{aligned}
&\frac{\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(V_{\omega\circ\sigma\circ\chi_i}, X_{S,\mathbb{C}})\big)}{\omega\circ\sigma(\lambda_i)L_S^*(\omega\circ\sigma\circ\overline{\chi_i},0)} \\
&\quad = \omega\left(\frac{\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(V_{\sigma\circ\chi_i}, X_{S,\mathbb{C}})\big)}{\sigma(\lambda_i)L_S^*(\sigma\circ\overline{\chi_i},0)}\right)
\end{aligned}$$

for each $i \in I$, $\sigma \in \Sigma(E_i)$ and $\omega \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. These equalities are obviously equivalent to the truth of Conjecture 2.2.1. □

We now assume that $T\Omega$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$ and for each prime p we let $T\Omega_p$ denote its projection into $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$. We say that Conjecture 2.2.2$_p$ is true if the support of the fractional $\mathcal{O}$-ideal $A_{\varphi_S}(\chi)^{-1}q_{\varphi_S}(\overline{\chi})$ is coprime to p for each $\chi \in R^+_{\mathbb{C}}(G)$.

Conjecture 2.2.2 is true if and only if Conjecture 2.2.2$_p$ is true for all primes p, and $T\Omega$ has finite order if and only if $T\Omega_p$ has finite order for all primes p. To prove the second assertion of Theorem 2.2.4 we may therefore restrict to consider p-primary behaviour for each prime p. In this way, the proof of Theorem 2.2.4 is completed by the following two lemmas.

For each prime p we let $\mathcal{C}_p(L/K)$ denote the set of extensions L'/K' such that $K \subseteq K' \subseteq L' \subseteq L$, L/K' is cyclic and $p \nmid [L' : K']$.

LEMMA 2.2.6. (i) *$T\Omega_p$ has finite order if and only if $T\Omega(L'/K', 0)_p = 0$ for all extensions L'/K' in $\mathcal{C}_p(L/K)$.*

(ii) Conjecture 2.2.2$_p$ *is true for L/K if and only if it is true for all extensions in $\mathcal{C}_p(L/K)$.*

Proof. This type of reduction step is certainly well known, but we shall nevertheless quickly sketch a proof.

Let Z be a $\Gamma(\mathbb{Q}_p)$-module which is $\mathbb{Z}$-torsion-free. If $f \in \mathrm{Hom}_{\Gamma(\mathbb{Q}_p)}(R_{\mathbb{Q}_p^c}(G), Z)$, then Artin's Induction Theorem on characters implies that $f = 0$ if and only if $f(\chi) = 0$ for all characters of the form $\chi = \mathrm{ind}_H^G \mathrm{inf}_{H/C}^H \psi$ with C and H subgroups

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of G such that $C \leqslant H$, H is cyclic and $p \nmid [H\colon C]$, and $\psi \in R^+_{\mathbb{Q}^c}(H/C)$ (cf. the proof of ([23], Prop. 11(b))). We refer to this general criterion as 'Artin reduction'.

To prove (i) we let $\mathbb{Z}_p^c$ denote the valuation ring of $\mathbb{Q}_p^c$ and set $Z := \mathbb{Q}_p^{c\times}/\mathbb{Z}_p^{c\times}$. The isomorphism $h_{G,p}$ of (1.1.4) restricts to give a surjection from $\mathrm{Hom}_{\Gamma(\mathbb{Q}_p)}(R_{\mathbb{Q}_p^c}(G), \mathbb{Z}_p^{c\times})$ to the torsion subgroup of $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ and so elements of the torsion-free quotient of $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ can be parametrised by elements of $\mathrm{Hom}_{\Gamma(\mathbb{Q}_p)}(R_{\mathbb{Q}_p^c}(G), Z)$. Claim (i) follows by applying Artin reduction to an element of $\mathrm{Hom}_{\Gamma(\mathbb{Q}_p)}(R_{\mathbb{Q}_p^c}(G), Z)$ chosen to represent $T\Omega_p$ modulo torsion, and then using Proposition 2.1.4 and the functorial behaviour of the isomorphisms $h_{G,p}$ as G varies (cf. the proof of ([8], Lem. 4(iii))).

To prove (ii) we fix an embedding $j\colon \mathbb{Q}^c \to \mathbb{Q}_p^c$ and let $\mathcal{O}_j$ denote the valuation ring of the completion E'_j of E' at j. For each $\chi \in R^+_{\mathbb{Q}^c}(G)$ we set $\chi_j := j \circ \chi \in R^+_{\mathbb{Q}_p^c}(G)$. We let Z denote the group of fractional $\mathcal{O}_j$-ideals and define $f \in \mathrm{Hom}_{\Gamma(\mathbb{Q}_p)}(R_{\mathbb{Q}_p^c}(G), Z)$ by setting $f(\chi_j) := j\big(A_{\varphi_S}(\chi)^{-1} q_{\varphi_S}(\overline{\chi})\big)\mathcal{O}_j$ for each $\chi \in R^+_{\mathbb{Q}^c}(G)$. The function f behaves functorially with respect to change of extension (cf. [27], Chap. II, proof of Th. 6.8) and so (ii) can be proved by applying Artin reduction to f. □

LEMMA 2.2.7. *If G is abelian and $p \nmid \#G$, then $T\Omega_p = 0$ if and only if* Conjecture $2.2.2_p$ *is true for L/K.*

Proof. We regard p as fixed, and write $\hat{\partial}^1(-)$ and $\chi(-,-)$ in place of $\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}(-)$ and $\chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(-,-)$ respectively. For any finitely generated G-module N we set $N_p := N \otimes \mathbb{Z}_p$, and we use similar notation for complexes of G-modules.

We fix a finite set of places S which is admissible for L/K and a G-equivariant injection $\varphi_S\colon X_S \to U_S$, and set $\varphi_{S,p} := \varphi_S \otimes \mathbb{Q}_p$. Since G is Abelian the condition (2.1.6) is satisfied by $\lambda = 1$, and we set $\tau := \tau_S(1) \otimes \mathbb{Q}_p$. Let $H^\bullet$ denote the complex $[U_{S,p} \xrightarrow{0} X_{S,p}]$ where the modules are placed in degrees 0 and 1. Since $\mathbb{Z}_p[G]$ is a regular ring, there is a $\mathbb{Z}_p[G]$-equivariant quasi-isomorphism $\theta\colon \Psi^\bullet_{S,p} \to H^\bullet$ which induces the identity map on cohomology. Combining equality (2.2.1) (with $\lambda = 1$) together with Proposition 1.2.1(iii) (applied to θ) and Proposition 1.2.1(ii) we obtain

$$\begin{aligned}\psi^*_{G,p}(T\Omega_p) &= \chi(\Psi^\bullet_{S,p}, \tau)\\ &= \chi(H^\bullet, \tau)\\ &= \chi(H^\bullet, \varphi_{S,p}) + \hat{\partial}^1(\det_{\mathbb{Q}_p[G]}(\varphi_{S,p})^{-1} \circ \tau),\end{aligned}$$

and, hence, $T\Omega_p = 0$ if and only if $\chi(H^\bullet, \varphi_{S,p}) = \hat{\partial}^1(\tau^{-1} \circ \det_{\mathbb{Q}_p[G]}(\varphi_{S,p}))$.

We now revert to the notation introduced in the proof of Lemma 2.2.6(ii), and write $I(E'_j)$ for the group of fractional $\mathcal{O}_j$-ideals. We set $G^* := \mathrm{Hom}(G, (E')^\times)$. For each $\psi \in G^*$, we let $\mathcal{O}_\psi$, respectively $\mathcal{O}_{\psi_j}$, denote the G-module consisting of $\mathcal{O}$, respectively $\mathcal{O}_j$, upon which G acts via ψ, respectively ψ_j. Since $p \nmid \#G$, the product functor $\prod_{\phi \in G^*}(- \otimes_{\mathcal{O}_j[G]} \mathcal{O}_{\phi_j})$ induces an isomorphism $\mathcal{O}_j[G] \cong \prod_{G^*} \mathcal{O}_j$. In addition, in this case the map δ in diagram (1.1.1) implies that the functor $\det_{\mathcal{O}_j}(-)$ induces a

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natural isomorphism $K_0(\mathcal{O}_j, \mathbb{Q}_p) \cong I(E'_j)$. Hence one has a composite injection

$$\iota\colon K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \to K_0(\mathcal{O}_j[G], \mathbb{Q}_p) \cong \prod_{G^*} I(E'_j),$$

where the first morphism is induced by scalar extension $-\otimes_{\mathbb{Z}_p}\mathcal{O}_j$ (and is injective since $p \nmid \#G$). For each $\phi \in G^*$ we write x_ϕ for the ϕ-component of any element x of $\prod_{G^*} I(E'_j)$.

Now for each $\phi \in G^*$ one has $\iota(\hat{\partial}^1(\tau^{-1} \circ \det_{\mathbb{Q}_p[G]}(\varphi_{S,p})))_\phi = j(A_{\varphi_S}(\overline{\phi}))\mathcal{O}_j$, whilst $\iota(\chi(H^\bullet, \varphi_{S,p}))_\phi$ is equal to the $\mathcal{O}_j$-determinant of the complex

$$H^0(G, \mathrm{Hom}_{\mathcal{O}_j}(\mathcal{O}_{\phi_j}, X_{S,p} \otimes_{\mathbb{Z}_p} \mathcal{O}_j)) \xrightarrow{\varphi_S} H^0(G, \mathrm{Hom}_{\mathcal{O}_j}(\mathcal{O}_{\phi_j}, U_{S,p} \otimes_{\mathbb{Z}_p} \mathcal{O}_j))$$

with the modules placed in degrees 0 and 1. In addition, $\mathcal{O}_\phi \otimes_{\mathcal{O}} \mathbb{C} = V_\phi$ and, since $p \nmid \#G$, the above complex is naturally isomorphic to $C^\bullet_{\varphi_S}(\mathcal{O}_\phi) \otimes_{\mathcal{O}} \mathcal{O}_j$ in $\mathfrak{D}^{perf}(\mathcal{O}_j)$. Hence $\iota(\chi(H^\bullet, \varphi_{S,p}))_\phi = (\det_{\mathcal{O}} C^\bullet_{\varphi_S}(\mathcal{O}_\phi))\mathcal{O}_j$, and this is equal to $q_{\varphi_S}(\phi)\mathcal{O}_j$ by Remark 2.2.3(iii).

We have now proved that $T\Omega_p = 0$ if and only if $q_{\varphi_S}(\phi)\mathcal{O}_j = j(A_{\varphi_S}(\overline{\phi}))\mathcal{O}_j$ for each $\phi \in G^*$ and each embedding $j\colon \mathbb{Q}^c \to \mathbb{Q}_p^c$, and it is clear that this is true if and only if Conjecture $2.2.2_p$ is true for L/K. □

This completes the proof of Theorem 2.2.4.

2.3. $T\Omega(L/K, 0)$ AND THE CONJECTURES OF GRUENBERG, RITTER AND WEISS, AND OF CHINBURG

In this section we reinterpret the conjectures of [10, 11, 17, 18] in terms of the element $T\Omega(L/K, 0)$. We continue to use the notation of Section 2.2.

In the remainder of this article we shall assume, often without explicit comment, that Conjecture 2.2.1 is true for the extension L/K. Following Theorem 2.2.4 and Lemma 2.2.5, it follows that $T\Omega(L/K, 0)$ and $T\Omega(L/K, \lambda, 0)$ belong to $K_0(\mathbb{Z}[G], \mathbb{Q})$, and that $\tau_S(\lambda) \in \delta^+_{\mathbb{Q}[G]}(X_{S,\mathbb{Q}}, U_{S,\mathbb{Q}})$ for each finite subset S of $S(K)$ which contains $S_\infty(K)$. We shall henceforth write $T\Omega$ and $T\Omega(\lambda)$ in place of $T\Omega(L/K, 0)$ and $T\Omega(L/K, \lambda, 0)$ respectively.

To recall the central conjecture of [17,18] we fix a finite set of places S which is admissible for L/K. We then fix an exact sequence of finitely generated $\mathbb{Z}[G]$-modules

$$0 \to U_S \xrightarrow{i} Q \xrightarrow{d} P \xrightarrow{\pi} X_S \to 0 \qquad (2.3.1)$$

in which P is projective, Q is perfect, and the sequence represents the canonical element $c_S(L/K)$ of $\mathrm{Ext}^2_G(X_S, U_S)$ defined by Tate in [26]. We write B for the image of d, make a choice of G-equivariant injective endomorphisms α and β of B which

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are each homotopic to 0, and form push out and pull back diagrams

$$\begin{array}{ccccccccccc} B & \xrightarrow{\subseteq} & P & \xrightarrow{\pi} & X_S & & U_S & \longrightarrow & B \oplus U_S & \longrightarrow & B \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\beta}} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} \\ B & \longrightarrow & B \oplus X_S & \longrightarrow & X_S & & U_S & \xrightarrow{i} & Q & \longrightarrow & B \end{array}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are defined so that both diagrams commute. For each G-equivariant injection $\varphi_S: X_S \to U_S$ we write $\tilde{\varphi}_S$ for the composite injection

$$P \xrightarrow{\tilde{\beta}} B \oplus X_S \xrightarrow{1_B \oplus \varphi_S} B \oplus U_S \xrightarrow{\tilde{\alpha}} Q.$$

The cokernel of $\tilde{\varphi}_S$ is finite and $\mathbb{Z}[G]$-perfect, and we set

$$\Omega_{\varphi_S} := t_G(\mathrm{cok}(\tilde{\varphi}_S)) - \partial_G^1((B_{\mathbb{Q}}, \alpha \circ \beta)) \in K_0(\mathbb{Z}[G], \mathbb{Q}).$$

We let $J_\infty(\mathbb{Q}^c)$ and $J_f(\mathbb{Q}^c)$ denote the subgroups of $J(\mathbb{Q}^c)$ consisting of those ideles for which all non-Archimedean components and all Archimedean components are trivial respectively, and we write π_∞, π_f and π for the diagonal embeddings of $\mathbb{Q}^{c\times}$ into $J_\infty(\mathbb{Q}^c)$, $J_f(\mathbb{Q}^c)$ and $J(\mathbb{Q}^c)$ respectively. With respect to the direct product decomposition $J(\mathbb{Q}^c) = J_\infty(\mathbb{Q}^c) \times J_f(\mathbb{Q}^c)$ one therefore has $\pi = (\pi_\infty, \pi_f)$. In this section we regard $\mathbb{Q}^c$ as a subfield of $\mathbb{C}$, and assume that $E'/\mathbb{Q}$ is Galois. We may therefore identify $\Sigma(E')$ with $\mathrm{Gal}(E'/\mathbb{Q})$.

Conjecture 2.2.1 implies that there is an element $\hat{A}_{\varphi_S}$ of $\mathrm{Hom}_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J(\mathbb{Q}^c))$ such that

$$\hat{A}_{\varphi_S}(\chi) := \pi\big(A_{\varphi_S}(\bar{\chi})\big) \in J(\mathbb{Q}^c)$$

for each $\chi \in R^+_{\mathbb{Q}^c}(G)$. After choosing a function $W \in \mathrm{Hom}_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J_\infty(\mathbb{Q}^c))$ such that $W\hat{A}_{\varphi_S} \in \mathrm{Hom}^+_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J(\mathbb{Q}^c))$ we set

$$\omega_{\varphi_S} := \Omega_{\varphi_S} - h_G(W\hat{A}_{\varphi_S}) \in K_0(\mathbb{Z}[G], \mathbb{Q}).$$

CONJECTURE 2.3.1 (The Lifted Root Number Conjecture, [17,18]). $\omega_{\varphi_S} = 0 \in K_0(\mathbb{Z}[G], \mathbb{Q})$.

Remark 2.3.2. (i) It is possible to specify a canonical choice of W in terms of Artin root numbers (cf. [11]).

(ii) Since the 2-extension (2.3.1) is chosen to represent the class $c_S(L/K)$ one has

$$\partial_G^0(\Omega_{\varphi_S}) = (P) - (Q) = -\Omega(L/K, 3) \in K_0(\mathbb{Z}[G])$$

where $\Omega(L/K, 3)$ is the element defined by Chinburg in [10,11]. On the other hand, it is easily shown that $-\partial_G^0(h_G(W\hat{A}_{\varphi_S}))$ is equal to the so-called 'Cassou–Noguès–Fröhlich class' (cf. for example [18], App. A, following (A.3)) and so the equality

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$\partial^0_G(\omega_{\varphi_S}) = 0 \in K_0(\mathbb{Z}[G])$ is equivalent to the conjecture formulated by Chinburg in loc. cit.

(iii) Some explicit evidence in support of Conjecture 2.3.1 is described in [24].

We are now ready to state the main result of this section.

THEOREM 2.3.3. *$T\Omega = \psi^*_G(\omega_{\varphi_S}) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ for each choice of S and φ_S as above. In particular, one has $T\Omega = 0$ if and only if* Conjecture 2.3.1 *is true for L/K.*

Remark 2.3.4. In conjunction with Theorem 2.1.2 and Proposition 2.1.4 the equality $T\Omega = \psi^*_G(\omega_{\varphi_S})$ leads to alternative proofs of Theorems 1, 2′ and 3′ of [18].

To prove Theorem 2.3.3 we must first reformulate the definition of Ω_{φ_S} in the spirit of Section 1.2. Since Q is $\mathbb{Z}[G]$-perfect there exists a short exact sequence

$$0 \to P^{-1} \overset{\subseteq}{\to} P^0 \overset{\nu}{\to} Q \to 0$$

in which P^0 and P^{-1} are finitely generated projective G-modules. In this way we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & P^{-1} & = & P^{-1} & & & & \\ & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} & & & & \\ 0 & \longrightarrow & \ker(\kappa) & \overset{\subseteq}{\longrightarrow} & P^0 & \overset{\kappa}{\longrightarrow} & \operatorname{cok}(\tilde{\varphi}_S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle=} & & \\ 0 & \longrightarrow & P & \overset{\tilde{\varphi}_S}{\longrightarrow} & Q & \longrightarrow & \operatorname{cok}(\tilde{\varphi}_S) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \qquad (2.3.2)$$

where first κ, and then ψ, is defined to make the diagram commutative. Since P is projective one can choose a section $\eta: P \to \ker(\kappa)$ to ψ and the diagram then implies that

$$t_G(\operatorname{cok}(\tilde{\varphi}_S)) = \left[P^{-1} \oplus P, (1_{P^{-1}} \oplus \eta)_{\mathbb{Q}}, P^0\right] \in K_0(\mathbb{Z}[G], \mathbb{Q}). \qquad (2.3.3)$$

The next lemma provides the key step in comparing this element to $T\Omega(\lambda)$.

We let $P^\bullet$ denote the complex $P^{-1} \overset{\subseteq}{\longrightarrow} P^0 \overset{d\circ\nu}{\longrightarrow} P$ (with P^{-1} placed in degree -1).

LEMMA 2.3.5. *Using* (2.1.2) *to identify the cohomology modules of $\Psi^\bullet_S$, there exists an isomorphism $\iota: \Psi^\bullet_S \tilde{\to} P^\bullet$ in $\mathfrak{D}^{perf}(\mathbb{Z}[G])$ such that $H^i(\iota)$ is the identity map in each degree i.*

Proof. It suffices to show that the 2-extension (2.1.2) represents the canonical class $c_S(L/K)$, and this follows from the proof of ([7], Th. 3.2). □

Combining the equality (2.2.1) with Lemma 2.3.5 and Proposition 1.2.1(iii) gives an equality

$$\psi_G^*(T\Omega(\lambda)) = [P^{-1} \oplus P, \phi, P^0]$$

for any $\phi \in \mathrm{Is}_{\mathbb{Q}[G]}((P^{-1} \oplus P)_{\mathbb{Q}}, P^0_{\mathbb{Q}})$ such that $\det_{\mathbb{Q}[G]}(\phi) = \tau_S(\lambda)(P^\bullet_{\mathbb{Q}})$. In conjunction with (2.3.3) and the argument which proves Proposition 1.2.1(ii) this implies that

$$t_G(\mathrm{cok}(\tilde{\varphi}_S)) - \psi_G^*(T\Omega(\lambda)) = \hat{\partial}_G^1\big(\det_{\mathbb{Q}[G]}(\phi^{-1} \circ [1_{P^{-1}} \oplus \eta]_{\mathbb{Q}})\big).$$

We may assume that upon restriction to $P^{-1}_{\mathbb{Q}} \subseteq P^0_{\mathbb{Q}}$ the morphism ϕ^{-1} is equal to the identity map and so it induces an isomorphism $\phi_1 \colon Q_{\mathbb{Q}} \xrightarrow{\sim} P_{\mathbb{Q}}$. The last displayed equation therefore implies that

$$\begin{aligned}
&\Omega_{\varphi_S} - \psi_G^*(T\Omega(\lambda))\\
&\quad = t_G(\mathrm{cok}(\tilde{\varphi}_S)) - \psi_G^*(T\Omega(\lambda)) - \partial_G^1\big((B_{\mathbb{Q}}, \alpha \circ \beta)\big)\\
&\quad = \hat{\partial}_G^1\big(\det_{\mathbb{Q}[G]}(\phi^{-1} \circ [1_{P^{-1}} \oplus \eta]_{\mathbb{Q}})\big) - \partial_G^1\big((B_{\mathbb{Q}}, \alpha \circ \beta)\big)\\
&\quad = \hat{\partial}_G^1(\det_{\mathbb{Q}[G]}(\phi_1 \circ \tilde{\varphi}_{S,\mathbb{Q}})) - \hat{\partial}_G^1(\det_{\mathbb{Q}[G]}(\alpha \circ \beta))\\
&\quad = \hat{\partial}_G^1(\det_{\mathbb{Q}[G]}\big(\phi_1 \circ \tilde{\varphi}_{S,\mathbb{Q}}\big)\det_{\mathbb{Q}[G]}(\alpha \circ \beta)^{-1}).
\end{aligned}$$

Theorem 2.3.3 now follows from this last equality together with the following two lemmas.

Let A_{φ_S} denote the element of $\zeta(\mathbb{C}[G])^\times$ such that $(A_{\varphi_S})_{i,\sigma} = A_{\varphi_S}(\sigma \circ \chi_i)$ for each $i \in I$ and $\sigma \in \Sigma(E')$.

LEMMA 2.3.6. $\det_{\mathbb{Q}[G]}(\phi_1 \circ \tilde{\varphi}_{S,\mathbb{Q}})\det_{\mathbb{Q}[G]}(\alpha \circ \beta)^{-1} = \lambda^{-1} A^{\#}_{\varphi_S} \in \zeta(\mathbb{R}[G])^\times$.

Proof. Choose $\psi \in \mathrm{Is}_{\mathbb{Q}[G]}(U_{S,\mathbb{Q}}, X_{S,\mathbb{Q}})$ with $\det_{\mathbb{Q}[G]}(\psi) = \tau_S(\lambda)^{-1}$. To obtain an isomorphism ϕ_1 as above, one combines ψ with a choice of G-equivariant sections σ and μ to the morphisms $Q_{\mathbb{Q}} \xrightarrow{d\otimes\mathbb{Q}} B_{\mathbb{Q}}$ and $P_{\mathbb{Q}} \xrightarrow{\pi\otimes\mathbb{Q}} X_{S,\mathbb{Q}}$ respectively. Since the precise choice of σ and μ is irrelevant, we shall for convenience use the sections defined by $\sigma(b) = \tilde{\alpha}_{\mathbb{Q}}(\alpha_{\mathbb{Q}}^{-1}b, 0)$ and $\mu(x) = \tilde{\beta}_{\mathbb{Q}}^{-1}(0, x)$ for all $b \in B_{\mathbb{Q}}$ and $x \in X_{S,\mathbb{Q}}$ respectively. With these choices of σ and μ there is a commutative diagram of G-equivariant morphisms

$$\begin{array}{ccccccc}
Q_{\mathbb{Q}} = U_{S,\mathbb{Q}} \oplus \sigma(B_{\mathbb{Q}}) & \xrightarrow{(1,d)} & U_{S,\mathbb{Q}} \oplus B_{\mathbb{Q}} & \xrightarrow{(\psi,1)} & X_{S,\mathbb{Q}} \oplus B_{\mathbb{Q}} & \xrightarrow{(\mu,1)} & P_{\mathbb{Q}}\\
\big\downarrow{\scriptstyle ((\varphi_S\otimes\mathbb{Q})\circ\psi,\theta)} & & \big\downarrow{\scriptstyle ((\varphi_S\otimes\mathbb{Q})\circ\psi,\beta)} & & \big\downarrow{\scriptstyle (1,\beta)} & & \big\downarrow{\scriptstyle =}\\
Q_{\mathbb{Q}} = U_{S,\mathbb{Q}} \oplus \sigma(B_{\mathbb{Q}}) & \xleftarrow{\tilde{\alpha}} & U_{S,\mathbb{Q}} \oplus B_{\mathbb{Q}} & \xleftarrow{(\varphi_S,1)} & X_{S,\mathbb{Q}} \oplus B_{\mathbb{Q}} & \xleftarrow{\tilde{\beta}} & P_{\mathbb{Q}}
\end{array}$$

where θ denotes the automorphism of $\sigma(B_{\mathbb{Q}})$ given by

$$\tilde{\alpha}_{\mathbb{Q}}(\alpha_{\mathbb{Q}}^{-1}b, 0) \mapsto \tilde{\alpha}_{\mathbb{Q}}(\alpha_{\mathbb{Q}}^{-1}((\alpha \circ \beta)b, 0))$$

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for each $b \in B_{\mathbb{Q}}$. We can take ϕ_1 to be defined by the top row of this diagram and, on the other hand, the bottom row is equal to $\tilde{\varphi}_{S,\mathbb{Q}}$. It follows that, with respect to the decomposition $Q_{\mathbb{Q}} = U_{S,\mathbb{Q}} \oplus \sigma(B_{\mathbb{Q}})$, one has $\tilde{\varphi}_{S,\mathbb{Q}} \circ \phi_1 = (\varphi_{S,\mathbb{Q}} \circ \psi, \theta)$. Hence

$$\begin{aligned}\det_{\mathbb{Q}[G]}(\phi_1 \circ \tilde{\varphi}_{S,\mathbb{Q}}) &= \det_{\mathbb{Q}[G]}(\tilde{\varphi}_{S,\mathbb{Q}} \circ \phi_1)\\ &= \det_{\mathbb{Q}[G]}(\varphi_{S,\mathbb{Q}} \circ \psi) \cdot \det_{\mathbb{Q}[G]}(\theta)\\ &= \lambda^{-1} A^{\#}_{\varphi_S} \cdot \det_{\mathbb{Q}[G]}(\alpha \circ \beta)\end{aligned}$$

as required. □

LEMMA 2.3.7. $\hat{\partial}^1_G(\lambda^{-1} A^{\#}_{\varphi_S}) = h_G(W\hat{A}_{\varphi_S}) - T_G(\lambda)$.

Proof. To prove this we work in terms of the description (1.1.5). More precisely, after choosing elements θ_1 and θ_2 of $\mathrm{Hom}^+_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J(\mathbb{Q}^c))$ such that $h_G(\theta_1) = \hat{\partial}^1_G(\lambda^{-1} A^{\#}_{\varphi_S})$ and $h_G(\theta_2) = T_G(\lambda)$ we show that $\theta_1^{-1}\theta_2^{-1} W\hat{A}_{\varphi_S} \in \ker(h_G)$.

By explicitly interpreting $\hat{\partial}^1_G$ and $\hat{\partial}^1_{Z_p[G]}$ in terms of h_G and $h_{G,p}$ one finds that the homomorphisms

$$\left[\sigma \circ \chi_i \longmapsto \pi\left(\frac{\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(V_{\sigma\circ\chi_i}, X_{S,\mathbb{C}})\big)}{\sigma(\lambda_i) L^*_S(\sigma \circ \overline{\chi_i}, 0)}\right)\right]_{i,\sigma}$$

and $\big[\sigma \circ \chi_i \longmapsto \pi_f(\sigma(\lambda_i))\big]_{i,\sigma}$ are suitable choices for θ_1 and θ_2 respectively (where here i and σ run over I and $\Sigma(E')$). Note that whilst this choice of θ_2 obviously belongs to $\mathrm{Hom}^+_{\Gamma(\mathbb{Q})}(R_{\mathbb{Q}^c}(G), J(\mathbb{Q}^c))$, the same is true for θ_1 since $\lambda = \prod_i \lambda_i$ satisfies (2.1.6) and $\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(V_{\sigma\circ\chi_i}, X_{S,\mathbb{C}})\big)$ is a strictly positive real number if χ_i is symplectic. Indeed, if χ_i is symplectic, then $\det_{\mathbb{C}}\big(R_S \circ \varphi_{S,\mathbb{R}} \mid \mathrm{Hom}_G(V_{\sigma\circ\chi_i}, X_{S,\mathbb{C}})\big)$ is the reduced norm of an invertible element of a simple algebra component of $\mathrm{End}_{\mathbb{R}[G]}(X_{S,\mathbb{R}})$ which is a matrix algebra over the quaternions.

By using the above choices for θ_1 and θ_2, one computes that for each index i and embedding σ

$$\left[\theta_1^{-1} \hat{A}_{\varphi_S} \theta_2^{-1} W\right](\sigma \circ \chi_i) = W(\sigma \circ \chi_i)\pi_\infty(\sigma(\lambda_i)) \in J_\infty(\mathbb{Q}^c).$$

If χ_i is symplectic, then as a consequence of the precise conditions which W and λ are chosen to satisfy, each component of the last expression is a strictly positive real number. Hence $\theta_1^{-1}\theta_2^{-1} W\hat{A}_{\varphi_S} \in \mathrm{Det}_{\mathbb{R}[G]}(K_1(\mathbb{R}[G])) \subseteq \ker(h_G)$ as required. □

This completes the proof of Theorem 2.3.3.

2.4. $T\Omega(L/K, 0)$ AND THE EQUIVARIANT TAMAGAWA NUMBER CONJECTURE

We continue to use the notation of Section 2.3, and assume that Conjecture 2.2.1 is true for L/K. For each prime p we write $T\Omega_p$ and $T\Omega(\lambda)_p$ for the projections of $T\Omega$ and $T\Omega(\lambda)$ into $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$.

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In this section we describe $T\Omega_p$ in terms of p-adic étale cohomology. As a consequence, we show that Conjecture 2.3.1 is equivalent to the Equivariant Tamagawa Number Conjecture for the pair $(\mathbb{Z}[G], \mathbb{Q}(0)_L)$ as formulated in [3]. This equivalence gives new insight into Conjectures 2.2.2 and 2.3.1 and, in particular, provides a canonical interpretation in terms of étale cohomology of each p-primary component of Conjecture 2.3.1. This interpretation (which assumes only that Conjecture 2.2.1 is true for L/K) is likely to prove crucial to the study of Conjecture 2.3.1 and cannot be obtained using the methods of [17,18]. In conjunction with the results of Sections 2.1, 2.2 and 2.3 the result proved here also provides further clarification of the conjectures formulated in [3] and, in more concrete terms, it allows one to interpret the extensive body of existing work which supports Conjectures 2.2.2 and 2.3.1 as evidence for the general conjectural equality (1). This is important since there are still very few examples for which (1) has been completely verified (see [1] for some recent results in this direction).

We first quickly recall the central conjecture of [3] (as applied to the pair $(\mathbb{Z}[G], \mathbb{Q}(0)_L)$). To do this we fix a finite subset S of $S(K)$ which is admissible for L/K, and set $S_p := S \cup S_p(K)$ and $S_{p,f} := S_p \cap S_f(K)$. For any ($p$-adic) étale sheaf $\mathfrak{F}$ on Spec $\mathcal{O}_{L,S_p}$ we set $H^i_c(\mathfrak{F}) := H^i R\Gamma_c(\mathcal{O}_{L,S_p}, \mathfrak{F})$ for each integer i. We regard p as fixed, and write $\chi(-,-)$, $\hat{\partial}^1(-)$ and ψ^* in place of $\chi_{\mathbb{Z}_p[G],\mathbb{Q}_p}(-,-)$, $\hat{\partial}^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}(-)$ and $\psi^*_{G,p}$ respectively.

The complex $R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Z}_p)$ belongs to $\mathfrak{D}^{perf}(\mathbb{Z}_p[G])$ and so can be represented by a bounded complex of finitely generated projective $\mathbb{Z}_p[G]$-modules $P^\bullet$. For any element λ of $\zeta(\mathbb{Q}[G])^\times$ as specified immediately prior to (2.2.1) we write $\tau(\lambda^\#)$ for the element $\tau_p(K, \mathbb{Q}(0)_L, S, \lambda^\#)$ of $\delta^+_{\mathbb{Q}_p[G]}(H^o_c(\mathbb{Q}_p), H^e_c(\mathbb{Q}_p))$ which is defined in ([3], §3.2.2) (this trivialisation will be described more explicitly a little later). For each place $v \in S_f(K)$ we set $D^\bullet_v := \mathrm{Hom}_{\mathbb{Q}_p}(C^\bullet_{v,\mathbb{Q}_p}, \mathbb{Q}_p)$ where the complex $C^\bullet_v$ is as defined at the beginning of Section 2.1. We let τ_v denote the reduced determinant of the $\mathbb{Q}_p[G]$-equivariant isomorphism $H^1 C^\bullet_{v,\mathbb{Q}_p} \to H^0 C^\bullet_{v,\mathbb{Q}_p}$ which is induced by the identifications (2.1.1). We set

$$\varepsilon_v := \tau_v(C^\bullet_{v,\mathbb{Q}_p})^D \in \delta^+_{\mathbb{Q}_p[G]}(D^{-1}_{v,\mathbb{Q}_p}, D^0_{v,\mathbb{Q}_p}) \subseteq \zeta(\mathbb{Q}_p[G])^{\times +},$$

and then define

$$T\Omega^p := \chi(P^\bullet, \tau(\lambda^\#)) + \hat{\partial}^1(\lambda^\#) - \sum_{v \in S_{p,f}} \hat{\partial}^1(\varepsilon_v) \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p).$$

This element is equal to the element $T\Omega^p(\mathbb{Z}[G], \mathbb{Q}(0)_L)$ introduced in ([3], Th. 4.1.1(ii)), and so the Equivariant Tamagawa Number Conjecture of (loc cit., Conj. 4.1.2) asserts that $T\Omega^p = 0$.

In this section we prove the following result.

THEOREM 2.4.1. *For each prime p one has* $T\Omega_p = T\Omega^p$. *In particular,* Conjecture 2.3.1 *is true for* L/K *if and only if the Equivariant Tamagawa Number Conjecture is true for the pair* $(\mathbb{Z}[G], \mathbb{Q}(0)_L)$.

Before proving this result we record an interesting corollary. This generalises the result of ([7], Th. 3.2) to non-Abelian extensions L/K.

Set $T\Omega^p(\lambda^\#) := T\Omega^p - \hat{\partial}^1(\lambda^\#) \in K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ and $T\Omega(\lambda^\#) := \prod_p T\Omega^p(\lambda^\#)$.

COROLLARY 2.4.2. *$T\Omega(\lambda^\#) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ and $\partial^0_G(T\Omega(\lambda^\#)) = -\psi^*_G(\Omega(L/K, 3)) \in K_0(\mathbb{Z}[G])$.*

Proof. Theorem 2.4.1 implies $T\Omega^p(\lambda^\#)$ is equal to the p-component of $T\Omega - T_G(\lambda^\#)$. Since $T\Omega - T_G(\lambda^\#) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ it follows that $T\Omega^p(\lambda^\#) = 0$ for almost all p, and hence that $T\Omega(\lambda^\#)$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$ and indeed is equal to $T\Omega - T_G(\lambda^\#)$.

From Theorem 2.3.3 and Remark 2.3.2(ii) one has

$$\begin{aligned}\partial^0_G(T\Omega(\lambda^\#)) &= \partial^0_G(T\Omega - T_G(\lambda^\#))\\ &= \partial^0_G(\psi^*_G(\Omega_{\varphi_S})) - \partial^0_G(\psi^*_G(h_G(W\hat{A}_{\varphi_S})) + T_G(\lambda^\#))\\ &= -\psi^*_G(\Omega(L/K, 3)) - \partial^0_G(\psi^*_G(h_G(W\hat{A}_{\varphi_S})) + T_G(\lambda^\#)).\end{aligned}$$

It therefore suffices to prove that $h_G(W\hat{A}_{\varphi_S}) + \psi^*_G(T_G(\lambda^\#))$ belongs to the kernel of ∂^0_G. However, as a consequence of (2.0.4) and (2.0.5) one has $\psi^*_G(T_G(\lambda^\#)) = -T_G(\lambda)$, and so the required containment follows directly from Lemma 2.3.7. □

In the remainder of this section we prove Theorem 2.4.1.

As already observed in the proof of Corollary 2.4.2 one has $\psi^*(\hat{\partial}^1(\lambda)) = -\hat{\partial}^1(\lambda^\#)$ and so Theorem 2.4.1 will follow if we show that $T\Omega(\lambda)_p = T\Omega^p(\lambda^\#)$. In proving this equality the key point is the canonical distinguished triangle

$$R\Gamma_c(\mathcal{O}_{L,S_p}, \mathbb{Q}_p) \to R\Gamma_f(L, \mathbb{Q}_p) \to \bigoplus_{v\in S_{p,f}} D^\bullet_v \;\oplus \bigoplus_{w\in S_\infty(L)} R\Gamma(L_w, \mathbb{Q}_p)$$

of ([7], (6)) (cf. also ([3], triangle $\Delta_S(V)$ following (2.2.2.2))). This triangle implies the existence of a complex $K^\bullet$ which lies in canonical distinguished triangles in $\mathfrak{D}^{perf}(\mathbb{Q}_p[G])$

$$K^\bullet \to R\Gamma_f(L, \mathbb{Q}_p) \to \bigoplus_{w\in S_\infty(L)} R\Gamma(L_w, \mathbb{Q}_p)$$

$$P^\bullet_{\mathbb{Q}_p} \to K^\bullet \to \bigoplus_{v\in S_{p,f}} D^\bullet_v. \tag{2.4.1}$$

Using the first of these triangles one computes that $K^\bullet$ is acyclic outside degrees 1 and 2, and that there are canonical identifications $H^1K^\bullet = (X_{\mathbb{Q}_p})^*$ and $H^2K^\bullet = (U_{\mathbb{Q}_p})^*$. In addition, the argument of ([7], Prop. 3.3) proves that there is a $\mathfrak{D}^{perf}(\mathbb{Z}_p[G])$-isomorphism θ: $\mathbf{D}(\Psi^\bullet_{S_p}) \otimes \mathbb{Z}_p \xrightarrow{\sim} P^\bullet[2]$ such that, after identifying $H^1_c(\mathbb{Q}_p)$ and $H^2_c(\mathbb{Q}_p)$ with $(X_{S_p,\mathbb{Q}_p})^*$ and $(U_{S_p,\mathbb{Q}_p})^*$ via $\theta \otimes \mathbb{Q}_p$, the long exact cohomology sequence which is associated to (2.4.1) recovers the $\mathbb{Q}_p$-linear duals of the sequences obtained by applying $\mathbb{Q}_p \otimes_{\mathbb{Z}}$- to the sequences (2.1.4) and (2.1.5) with $S = S_\infty(K)$ and $T = S_{p,f}$.

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Since we are assuming Conjecture 2.2.1 is true for L/K, Lemma 2.2.5 guarantees that $\tau_W(\lambda) \in \delta^+_{\mathbb{Q}[G]}(X_{W,\mathbb{Q}}, U_{W,\mathbb{Q}})$ for any finite subset W of $S(K)$ which contains $S_\infty(K)$. In particular, using the notation of Proposition 1.2.1(iii), we may set

$$\tau'_{S_p}(\lambda) := (\tau_{S_p}(\lambda)^D \otimes \mathbb{Q}_p)_\theta \in \delta^+_{\mathbb{Q}_p[G]}(H^1_c(\mathbb{Q}_p), H^2_c(\mathbb{Q}_p)).$$

Combining the equality (2.2.1) with Theorem 2.1.2(i) and Proposition 1.2.1(iii) (the latter taken with $\mathcal{A} = \mathbb{Z}_p[G]$ and $F = \mathbb{Q}_p$) now gives

$$\begin{aligned} T\Omega(\lambda)_p &= \psi^*(\chi(\Psi^\bullet_S \otimes \mathbb{Z}_p, \tau_S(\lambda) \otimes \mathbb{Q}_p)) \\ &= \psi^*(\chi(\Psi^\bullet_{S_p} \otimes \mathbb{Z}_p, \tau_{S_p}(\lambda) \otimes \mathbb{Q}_p)) \\ &= \chi(\mathbf{D}(\Psi^\bullet_{S_p}) \otimes \mathbb{Z}_p, \tau_{S_p}(\lambda)^D \otimes \mathbb{Q}_p) \\ &= \chi(P^\bullet[2], \tau'_{S_p}(\lambda)) \\ &= \chi(P^\bullet, \tau'_{S_p}(\lambda)). \end{aligned}$$

Applying Proposition 1.2.1(ii) we deduce

$$T\Omega(\lambda)_p - \chi(P^\bullet, \tau(\lambda^\#)) = \hat{\partial}^1(\tau(\lambda^\#)^{-1} \circ \tau'_{S_p}(\lambda)),$$

and hence that

$$T\Omega(\lambda)_p - T\Omega^p(\lambda^\#) = \hat{\partial}^1\left((\tau(\lambda^\#)^{-1} \circ \tau'_{S_p}(\lambda)) \times \prod_{v \in S_{p,f}} \varepsilon_v\right). \tag{2.4.2}$$

It is now time to be more explicit concerning the definition of $\tau(\lambda^\#)$. To do this, we choose $\phi_\infty \in \mathrm{Is}_{\mathbb{Q}[G]}(X_\mathbb{Q}, U_\mathbb{Q})$ with $\det_{\mathbb{Q}[G]}(\phi_\infty) = \tau_{S_\infty(K)}(\lambda)$, and let $\phi'_\infty \in \mathrm{Is}_{\mathbb{Q}[G]}((X_{S_p,\mathbb{Q}})^*, (U_{S_p,\mathbb{Q}})^*)$ be any morphism which fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (Y_{S_{p,f},\mathbb{Q}})^* & \longrightarrow & (X_{S_p,\mathbb{Q}})^* & \longrightarrow & (X_\mathbb{Q})^* & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\phi'_\infty} & & \downarrow{\phi^D_\infty} & & \\ 0 & \longrightarrow & (Y_{S_{p,f},\mathbb{Q}})^* & \longrightarrow & (U_{S_p,\mathbb{Q}})^* & \longrightarrow & (U_\mathbb{Q})^* & \longrightarrow & 0, \end{array}$$

where the upper and lower rows are the $\mathbb{Q}$-linear duals of the exact sequences obtained by applying $\mathbb{Q} \otimes_\mathbb{Z} -$ to (2.1.5) and (2.1.4) respectively (with $S = S_\infty(K)$ and $T = S_{p,f}$). Set $\tau'_\infty(\lambda) := \det_{\mathbb{Q}[G]}(\phi'_\infty)$. Then, by the very definition of $\tau(\lambda^\#)$ in ([3], §3.2.2), one has $\tau(\lambda^\#) = (\tau'_\infty(\lambda) \otimes \mathbb{Q}_p)_\theta$. Hence

$$\begin{aligned} \tau'_{S_p}(\lambda)^{-1} \circ \tau(\lambda^\#) &= \left((\tau_{S_p}(\lambda)^D \otimes \mathbb{Q}_p)_\theta\right)^{-1} \circ (\tau'_\infty(\lambda) \otimes \mathbb{Q}_p)_\theta \\ &= (\tau_{S_p}(\lambda)^* \circ \tau'_\infty(\lambda)) \otimes \mathbb{Q}_p. \end{aligned} \tag{2.4.3}$$

Upon comparing the explicit definitions of $\tau_{S_p}(\lambda)$ and $\tau'_\infty(\lambda)$ (and using (2.0.5)) one

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finds that

$$\tau_{S_p}(\lambda)^* \circ \tau'_\infty(\lambda) = \frac{L^*_{S_p}(0)}{L^*_{S_\infty(K)}(0)} \det_{\mathbb{R}[G]}\left(\mu_{S_{p,f}}\right) \tag{2.4.4}$$

where $\mu_{S_{p,f}}$ denotes the $\mathbb{R}$-linear dual of the automorphism of $Y_{S_{p,f},\mathbb{R}}$ which is induced by $w \mapsto (\log(Nw))^{-1}w$ for each $v \in S_{p,f}$ and $w \in S_v(L)$. Now Lemma 2.0.1 implies

$$\frac{L^*_{S_p}(0)}{L^*_{S_\infty(K)}(0)} = \prod_{v \in S_{p,f}} \left(\prod_{i \in I} (\log(Nv))^{dim_{E'}(V^0_{i,v})} \det_{E'}\left(1 - f_v \mid V^1_{i,v}/V^0_{i,v}\right)\right) \in \zeta(\mathbb{R}[G])^\times,$$

and a direct computation shows that

$$\det_{\mathbb{R}[G]}\left(\mu_{S_{p,f}}\right) = \prod_{v \in S_{p,f}} \left(\prod_{i \in I} (\log(Nw))^{-dim_{E'}(V^0_{i,v})}\right) \in \zeta(\mathbb{R}[G])^\times.$$

Combining the last two equalities together with (2.4.3) and (2.4.4) gives

$$\tau'_{S_p}(\lambda)^{-1} \circ \tau(\lambda^\#) = \prod_{v \in S_{p,f}} \left(\prod_{i \in I} (\#G_v/\#I_v)^{-dim_{E'}(V^0_{i,v})} \det_{E'}\left(1 - f_v \mid V^1_{i,v}/V^0_{i,v}\right)\right). \tag{2.4.5}$$

On the other hand, for each $v \in S_{p,f}$, the definition of $\varepsilon_v = \tau_v(C^\bullet_{v,\mathbb{Q}_p})^D$ together with (2.0.5) and an explicit computation similar to that following (2.1.10) implies

$$\varepsilon_v = \prod_{i \in I} (\#G_v/\#I_v)^{-dim_{E'}(V^0_{i,v})} \det_{E'}\left(1 - f_v \mid V^1_{i,v}/V^0_{i,v}\right) \in \zeta(\mathbb{Q}_p[G])^\times.$$

The required equality $T\Omega(\lambda)_p = T\Omega^p(\lambda^\#)$ now follows by taking the last displayed equation in conjunction with the equalities (2.4.2) and (2.4.5). This completes the proof of Theorem 2.4.1. □

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