FINITE ABELIAN ACTIONS ON SURFACES

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Let G be a finite abelian group of rank m, M an oriented compact connected surface, and $F(G, M)$ the set of all orientation preserving free G-actions on M. Two actions Φ_1, $\Phi_2 \in F(G, M)$ are *equivalent* if there exists an orientation preserving homeomorphism h of M such that

$$h\Phi_1(f)h^{-1} = \Phi_2(f) \quad \text{for all} \quad f \in G.$$

If $F(G, M)$ is nonempty then there is an oriented surface $N \equiv M/G$ such that for each $\Phi \in F(G, M)$, M/Φ is homeomorphic to N. The classification of the elements of $F(G, M)$ has been the subject of much research work (see e.g. **[2]**, **[3]**, **[4]**, **[5]**, **[7]**, **[8]**, **[9]**, **[10]**, and **[11]**). A. Edmonds, in **[2]**, showed that the bordism invariant of such actions defines an injective function $\mathbf{B}: F(G, M)/\sim \rightarrow H_2(G)$. If $m \leq \text{genus}(N)$, then $\mathbf{B}$ is a bijection (see **[2]**). However the characterisation of the set $\text{Im}(\mathbf{B})$ in the remaining cases, i.e. when $\text{genus}(N) < m \leq 2\,\text{genus}(N)$, has remained open. It is only known that $0 \notin \text{Im}(\mathbf{B})$. The aim of this paper is to solve this problem when $G = (\mathbf{Z}_p)^m$ and p is a prime number. The main ingredient of the solution, given in Section 2, is a technical result relating the determinant of a matrix to a recursively defined polynomial in the symplectic products of its rows. Slight modifications of the arguments, in Section 2, lead to a solution of the problem for general primary abelian groups. For nonabelian groups G, C. Livingston **[7]** and B. Zimmerman **[11]** studied the relation between $H_2(G)$ and the stable equivalence classes of G-actions on surfaces.

In Section 1, we reduce the problem of classifying free G-actions on surfaces to that of classifying free actions of the primary decompositions of G, and introduce the symplectic invariant on matrices through which the bordism invariant factors. We also give a simple constructive proof of the injectivity of the symplectic invariant which depends on the original Witt's theorem for symplectic vector spaces over finite fields. Throughout all homeomorphisms are orientation preserving, and all homology groups are taken with integral coefficients. If R is an equivalence relation on a set X then X/R is written as X^*.

1. Let G be a finite abelian group and M an oriented compact connected surface. For any $\Phi \in F(G, M)$, the map $p_\Phi: M \rightarrow M/\Phi$ is a regular G-covering. Let $C(G, N)$ be the set of all regular G-covering projections over N. Two regular G-coverings $p_1, p_2: M \rightarrow N$ are *equivalent* if there exist homeomorphisms F of M, and f of N such that F is a G-map and $fp_1 = p_2F$. If $f = I_N$ then p_1 and p_2 are *isomorphic*. Up to isomorphism, each element $p \in C(G, N)$ is uniquely determined by an epimorphism $\rho_p: H_1(N) \rightarrow G$. For any two groups K and L, let $E(K, L)$ be the set of all epimorphisms from K to L. If $g = \text{genus}(N)$, then $H_1(N) \approx \mathbf{Z}^{2g}$ and the intersection number defines a symplectic product on $H_1(N)$. The set $\mathbf{B}$ of the homology classes of the simple closed curves $\{a_1, b_1, \ldots, a_g, b_g\}$, shown in Figure 1, forms a symplectic basis for $H_1(N)$, which is fixed throughout this paper. Two epimorphisms $\rho_1, \rho_2 \in E(H_1(N), G)$ are said to be *equivalent* if there exists a symplectic automorphism α of $H_1(N)$ such that $\rho_1\alpha = \rho_2$. The sets

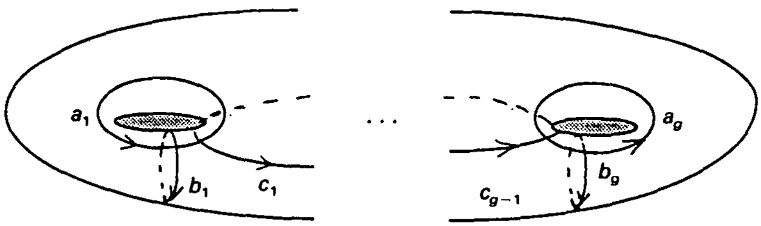

Figure 1.

$F^*(G, M)$, $C^*(G, N)$, and $E^*(H_1(N), G)$ are in bijection with each other (see e.g. [**2**], or [**5**]). A *symplectic* automorphism of $\mathbf{Z}^{2g}$ is one which preserves the symplectic inner product. A symplectic automorphism of $\mathbf{Z}^{2g}$ is said to be *elementary* if and only if

$$\forall x \in \mathbf{B}, \exists y \in \mathbf{B} \cup \{0\} \text{ such that } \alpha(y) = y \text{ and } \alpha(x) = x \pm y.$$

The group $\mathrm{Sp}_{2g}(\mathbf{Z})$ of symplectic automorphisms of $\mathbf{Z}^{2g}$ is generated by $3g-1$ elementary automorphisms which are induced by the "Dehn twists" around the simple closed curves $\{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{g-1}\}$, shown in Figure 1. This is the main factor in the proof of the following proposition.

1.1. Proposition. *If G_1 and G_2 are finite abelian groups of relatively prime order and N is a closed surface, then the map*

$$E^*(H_1(N), G_1 \oplus G_2) \to E^*(H_1(N), G_1) \times E^*(H_1(N), G_2) : [\rho] \to ([\tau_1\rho], [\tau_2\rho])$$

is a bijection, where $\tau_i : G_1 \oplus G_2 \to G_i$ is the natural projection onto the i-th coordinate. (See [**5**].)

1.2. Corollary. *If $G = \bigoplus G_{p_i}$, $i = 1, \ldots, n$, is the primary decomposition of G and N is a closed surface, then there are bijections*

$$\Im : C^*(G, N) \to \prod_{i=1}^{m} C^*(G_{p_i}, N),$$

$$\Im : F^*(G, M) \to \prod_{i=1}^{m} F^*(G_{p_i}, M_i),$$

where M_i is a homeomorphic image of a close surface of genus

$$1 + |G_{p_i}|\,(\text{genus}(M) - 1)/|G|.$$

Let p be a prime number, and $G = \bigoplus \mathbf{Z}_{p^{k_i}}$, where $k_1 \leq \ldots \leq k_m$. We fix as a basis for G the set $\{e_1, \ldots, e_m\}$, where e_i is the vector with 1 in the ith position but zero elsewhere. For any $\Phi \in F(G, M)$ (equivalently $p \in C(G, N)$), let B_Φ be the $m \times 2g$ integral matrix representing

$$\rho_p : H_1(G) \to G$$

with respect to the fixed bases. B_Φ has row rank m and its ith row consists of integers modulo p^{k_i}. Let $M(g, G)$ be the set of all such matrices. Two matrices $B_1, B_2 \in M(g, G)$ are said to be *equivalent* if for some $S \in \mathrm{Sp}_{2g}(\mathbf{Z})$, $B_1 = B_2 S$. Each of $F^*(G, M)$ and $C^*(G, N)$ is in bijection with $M^*(g, G)$.

The column operations that are needed to prove symplectic equivalences can be achieved by a small list of matrices in $\mathrm{Sp}_{2g}(\mathbf{Z})$. The list contains, for $1 \le i,\ j \le g$, the following matrices which are obtained from I_{2g} by the indicated operations, as well as their inverses:

α_i: adding the ith column to the $(i+g)$th column,

β_i: adding the $(i+g)$th column to the ith column,

δ_{ij}: adding the ith column to the jth column, and subtracting the $(j+g)$th column from the $(i+g)$th column,

ξ_{ij}: adding the ith column to the $(j+g)$th column, and the jth column to the $(i+g)$th column,

σ_{ij}: interchanging the ith column with the jth column, and the $(i+g)$th column with the $(j+g)$th column,

τ_{ij}: interchanging the ith column with the $(j+g)$th column, the jth column with the $(i+g)$th column (if $i \ne j$), and then multiplying the $(i+g)$th column and the $(j+g)$th column by -1, and

λ_{ij}: $(\xi_{ij})^t$.

1.3. Lemma. *If $1 \le m < 2g$ and $A \in M(g, G)$, then there is an integer i, $1 \le i \le 2g$, such that A is equivalent to a matrix B whose i-th column is a zero one.*

Proof. Since $\operatorname{rank}(A) = m < 2g$, then, up to symplectic equivalence, there is a column of A which is a linear combination of the previous ones. Let $\mathbf{c}_i$ be the first such column and suppose that

$$\mathbf{c}_i = a_1\mathbf{c}_1 + \ldots + a_{i-1}\mathbf{c}_{i-1}.$$

If $i \le g$, then $B = A(\delta_{1i})^{-a_1} \ldots (\delta_{i-1\,i})^{-a_{i-1}}$ is the required matrix. If $i = g+1$, then $B = A(\alpha_1)^{-a_1}\tau_{11}(\delta_{21})^{-a_2} \ldots (\delta_{g1})^{-a_g}$ is the required matrix. For $i > g+1$, let $k = i - g$ and $d = a_k - a_1a_{g+1} - \ldots - a_{k-1}a_{g+k-1}$. Then

$$B = A(\delta_{1k})^{a_{g+1}} \ldots (\delta_{k-1\,k})^{a_{i-1}}(\xi_{1k})^{-a_1} \ldots (\xi_{k-1\,k})^{-a_{k-1}}(\alpha_k)^{-d}(\xi_{k+1\,k})^{-a_{k+1}} \ldots (\xi_{gk})^{-a_g}$$

is the required matrix.

1.4. Definition. Let $B \in M(g, G)$ be any matrix. For any two integers i and j with $1 \le i < j \le m$, let

$$B_{ij} = \langle r_i(B), r_j(B)\rangle \bmod p^{k_i},$$

where $\langle r_i(B), r_j(B)\rangle$ is the symplectic product of the ith and jth rows of B. The *symplectic invariant* is the function

$$\xi : M(g, G) \to H_2(G)$$

defined for each matrix $B \in M(g, G)$ as:

$$\xi(B) = (B_{12}, \ldots, B_{1m}, B_{23}, \ldots, B_{2m}, \ldots, B_{m-1\,m}).$$

Note that, $H_2(G;\mathbf{Z}) = (\mathbf{Z}_{p^{k_1}})^{m-1} \oplus (\mathbf{Z}_{p^{k_2}})^{m-2} \oplus \ldots \oplus \mathbf{Z}_{p^{k_{m-1}}}$.

The domain of ξ can be extended to the set $M_{m,g}$ of all $m \times 2g$ integral matrices in the obvious way, and sometimes we will refer to this extension. It is obvious that equivalent matrices in $M(g,G)$ have the same symplectic invariant and hence ξ is a function on $M^*(g,H)$. Up to an isomorphism of G, related to the choice of basis for G, the Bordism invariant

$$\mathbf{B}: F(G,M)^* \to H_2(G),$$

defined by Edmonds in [2], factors through ξ. By proving a generalisation of Witt's theorem to symplectic spaces over local rings, Edmonds showed that $\mathbf{B}$ is an injection. In fact, Edmonds proved that ξ is injective. However, we shall now provide a simple, and somehow constructive, proof of the injectivity of ξ in which only the original Witt's theorem for symplectic vector spaces over finite fields is needed. We first prove a lemma which results in the reduction of the problem in the case of general p-groups into that of groups of the form $(\mathbf{Z}_{p^k})^r$.

For any p-group $G = \bigoplus (\mathbf{Z}_{p^{k_i}})$, with $k_1 \le \ldots \le k_r$, let $\mathbf{G} = (\mathbf{Z}_{p^{k_r}})^r$. For any natural number g, the modulo projection $\rho: \mathbf{G} \to G$ induces a function $\rho_g: M(g,\mathbf{G}) \to M(g,G)$.

1.5. Lemma. *Let $1 \le h < k$, $G = \mathbf{Z}_{p^h} \oplus (\mathbf{Z}_{p^k})^m$, $A \in M(g,G)$ any matrix with $1 \le m < 2g$, and let $\xi(A) = (\xi_{12}, \ldots, \xi_{m\,m+1})$. If $\sigma = (s_{12}, \ldots, s_{1\,m+1})$ is any sequence of numbers in $\mathbf{Z}_{p^k}$ such that $s_{1i} \equiv \xi_{1i} \pmod{p^h}$, $i = 1, \ldots, m+1$, then there exists a matrix $\mathbf{A} \in M(g,\mathbf{G})$ such that $\rho_g(\mathbf{A})$ is equivalent to A, and $\xi(\mathbf{A}) = (s_{12}, \ldots, s_{1\,m+1}, \xi_{23}, \ldots, \xi_{m\,m+1})$.*

Consequently, if G is any abelian p-group and $A, B \in M(g,G)$ are two matrices such that $\xi(A) = \xi(B)$, then there exists a matrix $\mathbf{B} \in M(g,\mathbf{G})$ such that $\rho_g(\mathbf{B})$ is equivalent to B, and, as matrices in $M(g,\mathbf{G})$, $\xi(A) = \xi(\mathbf{B})$.

Proof. The proof is by induction on m.

If $m = 1$, then by definition there is a positive integer i such that, up to symplectic equivalence, $2g > i > g$ and $a_{2i} = 1$. If $s_{12} - \xi_{12} = tp^h$, as integers, then the matrix $\mathbf{A} \in M(g,\mathbf{G})$, obtained by replacing $a_{1\,i-g}$ with $a_{1\,i-g} + tp^h$, is the required matrix.

Suppose that the lemma is true for $m = r$ and let $G = \mathbf{Z}_{p^h} \oplus (\mathbf{Z}_{p^k})^{r+1}$. Let A' be the submatrix of A obtained by deleting the 2nd row, and consider the subsequence $\sigma' = (s_{13}, \ldots, s_{1\,m+1})$. Let $\mathbf{A}' \in M(g, \mathbf{G}/\mathbf{Z}_{p^k})$ be the matrix that satisfies the conditions above, and obtained by the induction hypothesis. Let B be the matrix obtained from $\mathbf{A}'$ by inserting the 2nd row of A between its 1st and 2nd row. Obviously, B is of rank $r+2$. Hence, there exists a positive integer i such that, up to symplectic equivalence, $2g > i > g$, $b_{2i} = 1$, and all the entries of B below b_{2i} are zeros. Again, if $s_{12} - \xi_{12} = tp^h$, as integers, then the matrix $\mathbf{A} \in M(g,\mathbf{G})$ obtained from B by replacing $b_{1\,i-g}$ with $b_{1\,i-g} + tp^h$, is the required matrix.

1.6. Proposition. *The symplectic invariant is injective on $M^*(g,G)$ (i.e. if $\xi(A) = \xi(B)$ then A is equivalent to B).*

Proof. By 1.5, one can assume that G is a free $\mathbf{Z}_{p^k}$-module. The proof is divided into 2 cases.

Case (1): $m = 2g$. Let A^{-1} be the inverse of the matrix A considered as a $\mathbf{Z}_{p^k}$ matrix, and consider the $2g \times 2g$ matrix $Q = A^{-1}B$. Let

$$J_{2g} = \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}.$$

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Since $\xi(A)=\xi(B)$, then $AJ_{2g}A^t=BJ_{2g}B^t$. Hence

$$QJ_{2g}Q^t=A^{-1}BJ_{2g}B^t(A^t)^{-1}=A^{-1}AJ_{2g}A^t(A^t)^{-1}=J_{2g},$$

i.e. $Q\in \mathrm{Sp}_{2g}(\mathbf{Z}_{p^k})$. The fact that $AQ=B$ proves the result in this case.

Case (2): $m<2g$. Since A is of rank $m<2g$ then there is a column of A, say the ith, which can be assumed, by 1.3, to be a zero column. Let A' be the matrix whose 1st row is the unit vector $\mathbf{e}_i$ in $(\mathbf{Z}_p)^{2g}$, and for $j>1$ its jth row is the $(j-1)$th row of A. Let $\mathbf{A}'$ and $\mathbf{B}$ be the $\mathbf{Z}_p$ matrices obtained from A' and B, respectively, by reducing all entries mod p. Now, $\mathbf{A}'$ and $\mathbf{B}$ are of rank $m+1$ and m, respectively. Then by the original Witt's theorem for symplectic spaces over $\mathbf{Z}_p$, there is a vector $\mathbf{v}\in(\mathbf{Z}_p)^{2g}$ which is linearly independent of the rows of $\mathbf{B}$ and such that

$$\langle \mathbf{v}, r_j(\mathbf{B})\rangle \equiv \langle \mathbf{e}_i, r_{j+1}(\mathbf{A}')\rangle (\mathrm{mod}\, p), \quad \text{for} \quad j=1,\dots,m.$$

Let B' be the $(m+1)\times 2g$ matrix whose 1st row is the vector $\mathbf{v}\in(\mathbf{Z}_p)^{2g}$, w.r.t. the standard basis, and for $j>1$ its jth row is the $(j-1)$th row of B. Now, A' and B' are two $(m+1)\times 2g$ matrices in $M(g,\mathbf{Z}_p\oplus G)$ such that $\xi(A')=\xi(B')$ in $H_2(\mathbf{Z}_p\oplus G)$. Repeating this process $2g-m$ times produces two matrices A'' and B'' in $M(g,(\mathbf{Z}_p)^{2g-m}\oplus G)$, whose last m rows are those of A and B, respectively, $\xi(A'')=\xi(B'')$ in $H_2((\mathbf{Z}_p)^{2g-m}\oplus G)$. By case (1), and Lemma 1.5, there exists a symplectic $2g\times 2g$ matrix S such that $A''S=B''$. Clearly $AS=B$, and the proof is complete.

1.7. Proposition. *If $1\le m\le g$, then ξ is surjective.*

Proof. Let $\mathbf{v}=(v_{12},\dots,v_{1m},v_{23},\dots,v_{m-1\,m})\in H_2(G)$ be any element. If

$$A=\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & v_{12} & \dots & v_{1m} & 0 & \dots & 0\\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & v_{2m} & 0 & \dots & 0\\ & & \vdots & & & \vdots & & & & \vdots & & & \vdots & \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & 0 & \dots & v_{m-1\,m} & 0 & \dots & 0\\ 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

then, $A\in M(g,G)$ and $\xi(A)=\mathbf{v}$.

1.8. Corollary. *If M is a closed surface of genus g^* and if m is an integer such that $1\le m\le 1+(g^*-1)/|G|$, then there are p^K equivalence classes of free G-actions on M, where*

$$K=\sum_{i=1}^{m}(m-i)k_i.$$

2. This section is devoted to the characterisation of the vectors in $\mathrm{Im}(\xi)$, when $g<m\le 2g$. This completes the solution of the classification of free actions of elementary abelian groups G on closed surfaces M. However, analogous results can be obtained for primary abelian groups by slight modification of the statements and the arguments used. Throughout, all skew-symmetric matrices are assumed to have zero diagonals. First, a technical result, relating the determinant of a $2n\times 2n$ matrix to a recursively defined polynomial function d in the symplectic inner products of its rows, is proved. This yields a condition which must be satisfied by any element in $H_2(G)$ in order to be in $\mathrm{Im}(\xi)$ when

$m=2g$. We then prove that for $g<m<2g$ one of a number of such conditions, in the d function of some $2(m-g)\times 2(m-g)$ matrices, must be satisfied by vectors in $H_2(G)$ in order to be $\mathrm{Im}(\xi)$.

2.1. DEFINITION. Let n be a natural number, $\mathbf{v}\in\mathbf{Z}^{n(n-1)/2}$, and let $I=(i_1,\ldots,i_t)$ be an increasing sequence of natural numbers not exceeding n. By $\mathbf{v}^I$ we mean the subvector of $\mathbf{v}$ obtained by deleting all coordinates in the set

$$\bigcup_{j=1}^{t}\{\{v_{i_j i_j+1},\ldots,v_{i_j n}\}\cup\{v_{k i_j}:1\le k<i_j\}\}.$$

If $I=(1,i)$, then we simply write $\mathbf{v}^i$.

For even numbers n, let

$$d:\mathbf{Z}^{n(n-1)/2}\to\mathbf{Z}$$

be the function defined inductively on n as follows:

$$d((v_{12}))=v_{12},$$

and, for $n>2$,

$$d(v)=\sum_{i=2}^{n}(-1)^i v_{1i}\,d(v^i).$$

If S is a skew-symmetric $2n\times 2n$ matrix, then we define

$$d(S)=d((s_{12},s_{13},\ldots,s_{1\,2n},s_{23},\ldots,s_{2\,2n},\ldots,s_{2n-1\,2n})).$$

If $I=\{i_1,\ldots,i_t\}$ and $J=\{j_1,\ldots,j_t\}$ are sets of distinct natural numbers not exceeding m, and A is an $m\times m$ matrix, then $A(I;J)$ denotes the submatrix of A formed by deleting all rows indexed by I and all columns indexed by J. If $I=J$ then we simply write $A(I;)$. For integers r, t and n with $1<r,\ t\leqslant 2n$, let $n(r,t)=\{\{2,\ldots,2n\}-\{r,t\}\}$, and

$$r[t]=\begin{cases}t+r & \text{if } t>r,\\ t+r-1 & \text{if } t<r.\end{cases}$$

Using these notations, we find that

$$d(S)=\sum_{k=2}^{2n}(-1)^k s_{1k}\,d(S(1,k;)).$$

Moreover, it is not difficult to see that $d(S)$ can be expanded recursively in terms of any row or column. In fact, if $1<r\leqslant 2n$ then

$$d(S)=\sum_{r\neq k=1}^{2n}(-1)^{r[k]}s_{rk}\,d(S(r,k;)),$$

where for $r>k$, $s_{rk}=-s_{kr}$. For any square matrix A, $D(A)$ denotes the determinant of A.

2.2. LEMMA. *Let $n\ge 2$ be an integer, S a $2n\times 2n$ skew-symmetric matrix of integers, and i and j integers such that $2\le i\le j\le 2n$. Then*

(1) $D(S(1,i;1,j))=d(S(1,i;))\,.\,d(S(1,j;))$, *and*

(2) $D(S)=(d(S))^2$.

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Proof. The proof is done by induction on n.

Base case: $n=2$. (1) If $i'=\min\{\{2,3,4\}-\{j\}\}$, $j'=\max\{\{2,3,4\}-\{j\}\}$, $i''=\min\{\{2,3,4\}-\{i\}\}$, and $j''=\max\{\{2,3,4\}-\{i\}\}$, then

$$D(S(1,i;1,j))=s_{i'}s_{j'}s_{i''j''}=d(S(1,j;)) \,.\, d(S(1,i;)).$$

(2) This part follows from the relation

$$D(S)=(s_{12}s_{34}-s_{13}s_{24}+s_{14}s_{23})^2.$$

Induction hypothesis. Assume that both (1) and (2) are true for all $m<n$, and let S be a $2n\times 2n$ matrix.

(1) The case $i=j$ follows from the fact that $S(1,i;)$ is skew-symmetric and the induction hypothesis. Thus we assume that $i<j$.

Expressing $d(S(1,i;))$ in terms of its $(j-2)$th row and $d(S(1,j;))$ in terms of its $(i-1)$th row yields:

$$\begin{aligned} &d(S(1,i;))\,d(S(1,j;)) \\ &\quad=\left(\sum_{k\in n(i,j)}(-1)^{(j-2)[k]}s_{jk}\,d(S(1,i,j,k;))\right).\left(\sum_{h\in n(i,j)}(-1)^{(i-1)[h]}s_{ih}\,d(S(1,i,j,h;))\right) \\ &\quad=\sum_{k\in n(i,j)}\sum_{h\in n(i,j)}(-1)^{(j-2)[k]+(i-1)[h]}s_{jk}s_{ih}\,d((S(1,i,j,k;)))\,d((S(1,i,j,h;))). \end{aligned}$$

But $S(1,i,j,k;)=S(i,j;)(1,k';)$ and $S(1,i,j,h;)=S(i,j;)(1,h';)$, for some k' and h'. Since $S(i,j;)$ is a $2(n-1)\times 2(n-1)$ skew-symmetric matrix, then

$$d(S(1,i,j,k;))\,.\,d(S(1,i,j,h;))=D(S(i,j;)(1,k';1,h'))=D(S(1,i,j,k;1,i,j,h)).$$

Therefore

$$d(S(1,i;))\,d(S(1,j;))=\sum_{k\in n(i,j)}\sum_{h\in n(i,j)}(-1)^{(j-2)[k]+(i-1)[h]}s_{jk}s_{ih}D((S(1,i,j,k;1,i,j,h))).$$

The last expression is $D(S(1,i;1,j;))$ when expanded along its $(i-1)$th column and each resulting minor along its $(j-2)$th row. Note that here the cofactor of s_{ji} is skewsymmetric of odd size and hence it disappears from the expansion of $D(S(1,i;1,j;))$. The result follows by induction.

(2) By definition

$$\begin{aligned} (d(S))^2&=\left(\sum_{k=2}^{2n}(-1)^k s_{1k}\,d(S(1,k;))\right)^2 \\ &=\sum_{k=2}^{2n}\sum_{h=2}^{2n}(-1)^{k+h}s_{1k}s_{1h}\,d(S(1,k;))\,d(S(1,h;)) \\ &=\sum_{k=2}^{2n}\sum_{h=2}^{2n}(-1)^{k+h}s_{1k}s_{1h}D((S(1,k;1,h)). \end{aligned}$$

But the last expression is $D(S)$, when expanded along its 1st row and then each minor along its 1st column.

It was pointed out, by the referee, that the d function is closely related to the pfaffian polynomial.

2.3. Corollary. *If A is a $2g \times 2g$ matrix of integers, then*

$$(D(A))^2 = (d(\xi(A)))^2.$$

Here ξ refers to the extended version.

Proof. This follows from the fact that

$$AJ_{2g}A^t = S,$$

where $S = [s_{ij}]$ is the skew-symmetric $2g \times 2g$ matrix such that s_{ij} is the symplectic product of the ith and jth rows of A.

2.4. Proposition. *Let k and g be integers with $1 \le k < g$. If A is a $(g+k) \times 2g$ matrix over $\mathbf{Z}_p$ of rank $m = g + k$, then up to symplectic equivalence there exists an increasing sequence of natural numbers $I = (i_1, \ldots, i_{g-k})$ not exceeding m, such that:*

(1) *the symplectic products of the rows of the submatrix $A(I; \emptyset)$ are equal to that of the corresponding rows of A, and*

(2) *deleting all the pairs of columns indexed by j and $g+j$, for $1 \le j \le g-k$, yields a $2k \times 2k$ matrix of rank $2k$.*

Proof. Lemma 1.3 shows that, up to symplectic equivalence, A has a zero column. Multiplication on the right by σ_{ij} or τ_{ij}, if necessary, ensures that the $(g+1)$th column of A is a zero column. Let A' be the submatrix of A, obtained by deleting the 1st and $(g+1)$th column. If A' is of rank $m-1$, then some row of A' must be a linear combination of the others. Then the required integer i_1 can be taken to be the smallest index of such row of A'. However, if A' is of rank m then one can take $i_1 = 1$.

Repeating this process $g-k$ times completes the proof.

2.5. Theorem. *Let k and g be integers with $1 \le k \le g$, $m = k + g$, $G = (\mathbf{Z}_p)^m$,*

$$\xi : M^*(g, G) \to H_2(G)$$

the symplectic invariant, defined above, and $\mathbf{v} \in H_2(G)$ any vector. Then $\mathbf{v} \in \operatorname{Im}(\xi)$ if and only if there exists an increasing sequence I of $g-k$ natural numbers not exceeding m such that $d(\mathbf{v}^I) \ne 0 \pmod p$. Note that if $k = g$, then I is empty, and in this case $\mathbf{v}^I$ is taken to be $\mathbf{v}$.

Proof. The necessity follows from Corollary 2.3. and Proposition 2.4. The proof of the sufficiency is divided into 2 cases.

Case (1): $(k = g)$. We use induction on g.

For $g = 1$, and $\mathbf{v} \in H_2(G) - \{0\}$, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & v_{12} \end{bmatrix}.$$

Then $A \in M(g, G)$ and $\xi(A) = \mathbf{v}$.

Assume the result is true for all natural numbers less than g, where $g > 1$. The fact that $d(\mathbf{v}) \ne 0 \pmod p$ implies that there exists an integer r such that $2 \le r \le 2g$ and $d(\mathbf{v}^r) \ne 0 \pmod p$. Without loss of generality we may assume that $r = 2$. By the induction

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hypothesis, there exists a matrix $B \in M(g-1, G/(\mathbf{Z}_p)^2)$ such that $\xi(B) = \mathbf{v}^2$. Let

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_{11} & \cdots & b_{1\,g-1} & v_{23} & b_{1g} & \cdots & b_{1\,2(g-1)} \\ & & \vdots & & & & \vdots & \\ 0 & b_{2(g-1)\,1} & \cdots & b_{2(g-1)\,g-1} & v_{2\,2g} & b_{2(g-1)\,g} & \cdots & b_{2(g-1)\,2(g-1)} \end{bmatrix}$$

and consider the linear system of equations

$$CJ_{2g}\mathbf{x} = (v_{12} \ldots v_{1\,2g})^t.$$

Since $\operatorname{rank}(C) = 2g-1$, then a solution exists. If $\mathbf{u} = (x_1 \ldots x_g x_{g+1} \ldots x_{2g})$, and A is the $2g \times 2g$ matrix whose 1st row is $\mathbf{u}$ and the rest are those of C, then $A \in M(g, G)$ and $\xi(A) = \mathbf{v}$.

The proof in this case, then, follows by induction.

Case (2): $(k < g)$. Let $\mathbf{v} \in H_2(G)$, and let I be an increasing sequence of $g-k$ natural numbers not exceeding m such that $d(\mathbf{v}^I) \neq 0 \pmod p$. Without loss of generality, assume that $I = (1, 2, \ldots, g-k)$. Since $\mathbf{v}^I \in H_2(G/(\mathbf{Z}_p)^{g-k})$, then, by case (1), there exists a $2k \times 2k$ matrix $B \in M(k, G/(\mathbf{Z}_p)^{g-k})$ such that $\xi(B) = \mathbf{v}^I$. If

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & v_{12} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & v_{13} & v_{23} & \cdots & 0 & 0 & \cdots & 0 \\ & & \vdots & & & \vdots & & & & \vdots & & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & v_{1\,g-k} & v_{2\,g-k} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1k} & v_{1\,g-k+1} & v_{2\,g-k+1} & \cdots & v_{g-k\,g-k+1} & b_{1\,k+1} & \cdots & b_{1\,2k} \\ & & \vdots & & & \vdots & & & & \vdots & & & \vdots & \\ 0 & 0 & \cdots & 0 & b_{2k\,1} & \cdots & b_{2k\,k} & v_{1\,g+k} & v_{2\,g+k} & \cdots & v_{g-k\,g+k} & b_{2k\,k+1} & \cdots & b_{2k\,2k} \end{bmatrix}$$

then $A \in M(g, G)$ and $\xi(A) = \mathbf{v}$.

As an example, this means that if $k = 1$ then for any g, $\operatorname{Im}(\xi) = H_2(G) - \{\mathbf{0}\}$. And for $k = g = 2$, $\mathbf{v} = (v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) \in \operatorname{Im}(\xi)$ if and only if

$$v_{12}v_{34} - v_{13}v_{24} + v_{14}v_{23} \neq 0 \pmod p.$$

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