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APPROXIMATION BY GENERALISED POLYNOMIALS WITH INTEGRAL COEFFICIENTS

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Let $C[0, 1]$ be the space of all continuous real valued functions defined in $[0, 1]$ with the supremum norm

(1) $$\|f\|_\infty \equiv \sup_{x \in [0,1]} |f(x)|.$$

The subspace of $C[0, 1]$ consisting of all functions $f(x)$ for which $f(0)$ and $f(1)$ are integers will be denoted by $C_0[0, 1]$. Let $\Lambda = \{\lambda_i\}_0^\infty$ be a sequence of real numbers satisfying:

(2) $$\lambda_0 = 0, \lambda_i \geq 0 \quad (i \geq 1).$$

A Λ-polynomial is a function of the form $\sum_{i=0}^n x^{\lambda_i}$ where a_i are real numbers. It is well known [4, 5] that the Λ-polynomials are dense in $C[0, 1]$ provided

(3) $$\sum_{\lambda_i \neq 0} \lambda_i^{-1} = \infty.$$

There are several results about approximation by algebraic polynomials with integral coefficients. In $C[0, 1]$ the approximation of $f(x)$ is possible if and only if $f(x) \in C_0[0, 1]$ ([1], [2], [3]).

Ferguson [1], and Ferguson and von Golitschek [2] were interested to know if this result is true also for Λ-polynomials in $C_0[0, 1]$, and they proved that the result is true under some restrictions on Λ. Here we remove the unnecessary restrictions.

THEOREM. *Let $\Lambda = \{\lambda_i\}_0^\infty$ be a sequence of real numbers satisfying* (2). *The integral Λ-polynomials are dense in $C_0[0, 1]$ if and only if*

(4) $$\sum_{\lambda_i \neq 0} \lambda_i^{-1} = \infty.$$

In order to prove the theorem we need two lemmas.

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LEMMA 1. *Let $\Lambda=\{\lambda_i\}_0^\infty$ be a sequence of real numbers satisfying*

(i) $0=\lambda_0<\lambda_1<\cdots<\lambda_n\uparrow\infty$

(ii) $\lambda_n\ge n$

(iii) $\sum_{i=1}^\infty \lambda_i^{-1}=\infty$,

then the integral Λ-polynomials are dense in $C_0[0, 1]$.

Proof. This lemma was proved in [2] under the restriction of integral λ_n. The proof is identical as long as $\lambda_n\ge n$.

LEMMA 2. *Let $\lambda=\{\lambda_n\}$ be a strictly increasing sequence satisfying $\lambda_n\to\infty$ and $\sum_{\lambda_n>0}\lambda_n^{-1}=\infty$. Then, there is a subsequence $\{\lambda_{n_i}\}$ which satisfies*

$$\lambda_{n_i}\ge i \tag{5}$$

and still

$$\sum_{i=1}^\infty \lambda_{n_i}^{-1}=\infty. \tag{6}$$

Proof. The construction of $\{\lambda_{n_i}\}$ is made by constructing two sequences of indices. Choose m_1 to be the first index for which $\lambda_{m_1}>0$ and p_1 to be the smallest index which satisfies

$$p_1>m_1,\ \lambda_{p_1}<p_1-m_1. \tag{7}$$

Suppose that we had already chosen $m_k<p_k$. Define m_{k+1} and p_{k+1} to be the smallest indices such that

$$\begin{cases} m_{k+1}>p_k, & \lambda_{m_{k+1}}\ge\sum_{j=1}^k (p_j-m_j) \\ p_{k+1}>m_{k+1}, & \lambda_{p_{k+1}}<\sum_{j=1}^{k+1}(p_j-m_j). \end{cases} \tag{8}$$

If there is no such p_{k+1}, stop the construction and choose $\lambda_{n_i}=\lambda_{m_{k+1}+i}$, $i\ge1$. Here (6) and (7) are obviously satisfied. Hence we can assume that for every $k\ge1$ there are finite m_{k+1} and p_{k+1} as stated in (8). In this case the sequence $\{\lambda_{n_i}\}$ satisfying (5) and (6) will be given by:

$$\begin{cases} \lambda_{m_1+i} & \text{if } 1\le i<p_1-m_1 \\ \lambda_{m_{k+1}+i}-\sum_{j=1}^k (p_j-m_j) & \text{if } \sum_{j=1}^k (p_j-m_j)\le i<\sum_{j=1}^{k+1}(p_j-m_j). \end{cases} \tag{9}$$

It is clear that $\lambda_{n_i} \geq i (i \geq 1)$. Let us check if (6) is satisfied

$$\sum_{i=1}^{\infty} \lambda_{n_i}^{-1} \geq \sum_{k=2}^{\infty} \sum_{j=m_k}^{p_k-1} \lambda_j^{-1}$$

(10)
$$\geq \sum_{k=2}^{\infty} \frac{p_k - m_k}{\sum_{j=1}^{k} (p_j - m_j)}.$$

The quantity q_k defined by

(11)
$$q_k \equiv \sum_{j=1}^{k} (p_i - m_i)$$

tends to ∞ as $k \to \infty$. Then

$$\sum_{j=1}^{\infty} \lambda_{n_i}^{-1} \geq \sum_{k=2}^{\infty} \frac{q_k - q_{k-1}}{q_k} = \sum_{k=2}^{\infty} \left(1 - \frac{q_{k-1}}{q_k}\right).$$

But $\sum_{k=2}^{\infty}(1-(q_{k-1}/q_k))$ diverges if and only if the product $\prod_{k-2}^{\infty}(q_k/q_{k-1})$ diverges:

$$\prod_{k=2}^{l} \left(\frac{q_k}{q_{k-1}}\right) = \frac{q_l}{q_1} \to \infty, \quad \text{as} \quad l \to \infty.$$

Proof of the Theorem. The necessity of the condition (4) follows from the well known Muntz result about Λ-polynomials.

Sufficiency. If the sequence $\{\lambda_l\}$ has a finite limit point, the problem was solved in [2].

If the sequence $\{\lambda_i\}$ has no finite limit points, we can arrange it in an increasing order, and then, by Lemma 2, there exists a subsequence $\{\lambda_{n_i}\}$ which satisfies $\lambda_{n_i} \uparrow$, $\lambda_{n_i} \geq i (i \geq 1)$, and $\sum \lambda_{n_i}^{-1} = \infty$. By Lemma 1, the integral polynomials in 1 and $x^{\lambda_{n_i}}$ will be a dense set in $C_0[0, 1]$.

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