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On 6-Dimensional Nearly Kähler Manifolds

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Abstract. In this paper we give a sufficient condition for a complete, simply connected, and strict nearly Kähler manifold of dimension 6 to be a homogeneous nearly Kähler manifold. This result was announced in a previous paper by the first author.

1 Introduction

An almost Hermitian manifold (M, g, J) is said to be nearly Kähler (NK) if $(\nabla_X J)(X) = 0$ is satisfied for all vector fields X on M, where ∇ denotes the Levi-Civita connection associated with the metric g. An NK manifold is called strict if $\nabla_X J \neq 0$ for any nonvanishing vector field $X \in TM$, where TM denotes the tangent bundle of M.

Nearly Kähler manifolds are characterized as almost Hermitian manifolds such that the canonical Hermitian connection $\bar{\nabla}$ has parallel torsion tensor.

On the other hand, Nagy proved in [10, 11] that, in the compact case, his study amounts to that of quaternion-Kähler manifolds with positive scalar curvature (see Alexandrov, Grantcharov, and Ivanov [1]) and nearly Kähler manifolds of dimension 6. Thus our focus on the study of such manifolds of dimension 6 can be justified by his results.

In dimension 6, the only known examples of compact NK manifolds are the 3-symmetric spaces S^6, $S^3 \times S^3$, $\mathbb{C}P^3$, and the complex flag manifold

$$\mathbb{F}(1,2) = U(3)/[U(1) \times U(1) \times U(1)].$$

Moreover, Butruille [2] proved that there are no other homogeneous examples.

Recently, Moroianu, Nagy, and Semmelmann [9] proved that if a compact NK manifold (M^6, g, J) admits a Killing vector field of constant length, then, up to a finite cover, (M^6, g, J) is isomeric to $S^3 \times S^3$ endowed with its canonical NK structure. We also remark that in dimension 6, NK manifolds are related to the existence of a Killing spinor (see Grunewald [6]).

Motivated by above facts, in this paper we prove the following theorem.

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Main Theorem *Let (M, g, J) be a strict, simply connected, complete nearly Kähler manifold of dimension 6. If $\|\nabla R\|^2 = \frac{S}{15}\|Z\|^2$, then M is a homogeneous nearly Kähler manifold, where R, S, and Z denote the curvature tensor, the scalar curvature, and the concircular tensor of M respectively.*

2 Preliminaries

In this section, we explain our notation and write down some important curvature identities. Let (M, g, J) be a connected almost Hermitian manifold. Then we have $g(JX, JY) = g(X, Y)$ for all $X, Y \in TM$. Throughout this paper we shall assume that (M, g, J) is nearly Kähler, that is, $(\nabla_X J)(X) = 0$ for all $X \in TM$.

Let R denote the curvature tensor defined by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ for any vector fields X and Y in TM. Let $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ denote the value of the curvature tensor for every X, Y, Z, and W in TM. Then we have the following identities (see [3, 5, 12, 13, 15]):

$$(\nabla_X J)(Y) + (\nabla_{JX} J)(JY) = 0; \tag{2.1}$$

$$(\nabla_X J)(JY) + J((\nabla_X J)(Y)) = 0; \tag{2.2}$$

$$R(W, X, Y, Z) - R(W, X, JY, JZ) = g((\nabla_W J)(X), (\nabla_Y J)(Z)), \tag{2.3}$$

and

$$R(W, X, Y, Z) = R(JW, JX, JY, JZ). \tag{2.4}$$

We now define linear transformations R_1 and ${R_1}^*$ by

$$\mathrm{Ric}(X, Y) = g(R_1(X), Y) = \sum_{i=1}^{2n} R(X, e_i, Y, e_i) \quad \text{and}$$

$$\mathrm{Ric}^*(X, Y) = g({R_1}^*(X), Y) = \frac{1}{2}\sum_{i=1}^{2n} R(X, JY, e_i, Je_i)$$

respectively, where $\{e_1, \ldots, e_{2n}\}$ denotes a local orthonormal frame field on M. We shall call Ric the Ricci tensor of the metric and Ric* the Ricci* tensor respectively. Then by using (2.3), the following formulas are easy to prove :

$$R_1 J = JR_1, \qquad R_1^* J = JR_1^*. \tag{2.5}$$

There are two invariants of the curvature tensor of a NK manifold, namely

$$S = \sum_{i=1}^{2n} g(R_1(e_i), e_i), \qquad S^* = \sum_{i=1}^{2n} g(R_1^*(e_i), e_i),$$

called the scalar curvature (resp. the scalar * curvature). Here we also write

$$\|R\|^2 = \sum_{i=1}^{2n} R_{ijkl}^2, \quad \|\mathrm{Ric}\|^2 = \sum_{i=1}^{2n} R_{ij}^2, \quad \|\mathrm{Ric}^*\|^2 = \sum_{i=1}^{2n} {R^*}_{ij}^2, \quad etc.,$$

where $R_{ij} = \mathrm{Ric}(e_i, e_j)$ and $R_{ij}^* = \mathrm{Ric}^*(e_i, e_j)$.

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3 Nearly Kähler Geometry

First, note that $\mathrm{Ric} - \mathrm{Ric}^*$ is given by the formula

$$(3.1)\qquad (\mathrm{Ric} - \mathrm{Ric}^*)(X, Y) = \sum_{i=1}^{2n} g((\nabla_X J)e_i, (\nabla_Y J)e_i)$$

for all vector fields X, Y on M (see [7]). Furthermore, the first author and K. Takamatsu [15] proved that

$$(3.2)\qquad \sum_{i,j=1}^{2n} (\mathrm{Ric} - \mathrm{Ric}^*)(e_i, e_j)(R(X, e_i, Y, e_j) - 5R(X, e_i, JY, Je_j)) = 0$$

(see Gray [5] for another proof).

An object of particular importance is the canonical Hermitian connection defined by

$$(3.3)\qquad \bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY.$$

It is easy to see (Yano [18]) that $\bar{\nabla}$ is the unique linear connection on M, satisfying

$$(3.4)\qquad \bar{\nabla} g = 0, \qquad \bar{\nabla} J = 0,$$

from which in particular, we have

$$(3.5)\qquad \bar{\nabla}(\nabla J) = 0.$$

Therefore, note that the torsion of $\bar{\nabla}$ given by $N(X, Y) = (\nabla_X J)JY$ satisfies

$$(3.6)\qquad \bar{\nabla} N = 0.$$

Nagy remarked in [11] that $\mathrm{Ric} - \mathrm{Ric}^*$ has strong geometric properties. To begin with, we have

$$(3.7)\qquad \bar{\nabla}(\mathrm{Ric} - \mathrm{Ric}^*) = 0.$$

By making use of (3.1) and (3.7), he proved the following theorem.

Theorem 3.1 *Let (M, g, J) be a complete, strict, nearly Kähler manifold. Then it holds that*

(i) *if g is not an Einstein metric, the canonical Hermitian connection has reduced holonomy;*
(ii) *the metric g has positive Ricci curvature, hence M is compact with a finite fundamental group;*
(iii) *the scalar curvature S of the metric g is a positive constant.*

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4 New Curvature Identities in 6-Dimensional NK Manifolds

In lower dimensions, the nearly Kähler manifolds are widely determined. If M is nearly Kählerian with $\dim M \leq 4$, then M is Kählerian. If $\dim M = 6$, then we have the following (see [3, 5, 8, 17]).

Proposition 4.1 *Let (M, g, J) be a 6-dimensional, strict, nearly Kähler manifold. Then we have*

(i) *∇J has constant type [4]; that is,*

$$\|(\nabla_X J)Y\|^2 = \frac{S}{30}\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(JX,Y)^2\} \tag{4.1}$$

for all vector fields X and Y,

(ii) *the first Chern class of (M, J) vanishes, and*

(iii) *M is an Einstein manifold;*

$$\mathrm{Ric} = \frac{S}{6}g, \qquad \mathrm{Ric}^* = \frac{S}{30}g. \tag{4.2}$$

Furthermore, from this proposition we can prove the following lemma (see [3, 5, 17]).

Lemma 4.2 *For vector fields W, X, Y, and Z, we have*

$$g((\nabla_W J)X, (\nabla_Y J)Z) = \frac{S}{30}\{g(W,Y)g(X,Z) - g(W,Z)g(X,Y) - g(W,JY)g(X,JZ) + g(W,JZ)g(X,JY)\} \tag{4.3}$$

and

$$g((\nabla_W \nabla_Z J)X, Y) = \frac{S}{30}\{g(W,Z)g(JX,Y) - g(W,X)g(JZ,Y) + g(W,Y)g(JZ,X)\}. \tag{4.4}$$

On the other hand, it can be easily seen from (4.2) and the proof of Lemma 3.3 in [15] that

$$\sum_{i,j=1}^{2n} g(Je_i, e_j)R(e_i, e_j, X, Y) = \frac{S}{15}g(JX,Y) \tag{4.5}$$

and

$$\sum_{i,j=1}^{2n} g((\nabla_X J)e_i, e_j)R(e_i, e_j, Y, Z) = \frac{S}{15}g((\nabla_X J)Y, Z). \tag{4.6}$$

Operating ∇ to the last equation and using (4.2), we have

$$\sum_{i,j=1}^{2n} g((\nabla_X J)e_i, e_j)(\nabla_Y R)(e_i, e_j, Z, W) = -\frac{S}{15}Z(JX, Y, Z, W), \tag{4.7}$$

where the concircular tensor Z is defined by

$$Z(X,Y,Z,W) = R(X,Y,Z,W) - \frac{S}{30}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}. \tag{4.8}$$

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5 Homogeneity in 6-Dimensional NK Manifolds

In this section, we will use the natural frame as a local frame field and adopt the so-called Einstein summation convention with respect to repeated indices for a long computation using the identities from (2.1) to (4.8).

In the terminology of [14], we shall show that the tensor field T, given by

$$T(X,Y) = \frac{1}{2}(\nabla_X J)JY, \tag{5.1}$$

is a homogeneous structure. Then T has the local components

$$T_{ij}{}^{k} = -\frac{1}{2}(\nabla_i J_j{}^{s})J_s{}^{k}. \tag{5.1$'$}$$

For this purpose, let us consider the tensor field $\bar{\nabla}R$, given by

$$\begin{aligned}(\bar{\nabla}_W R)(X,Y)Z = (\nabla_W R)(X,Y)Z - T(W,R(X,Y)Z)\\ + R(X,Y)T(W,Z) + R(T(W,X),Y)Z + R(X,T(W,Y))Z.\end{aligned} \tag{5.2}$$

Then $\bar{\nabla}R$ has the local components

$$\begin{aligned}\bar{\nabla}_\ell R_{kjih} = \nabla_\ell R_{kjih} + \frac{1}{2}R_{sjih}(\nabla_\ell J_k{}^{a})J_a{}^{s} + \frac{1}{2}R_{ksih}(\nabla_\ell J_j{}^{a})J_a{}^{s}\\ + \frac{1}{2}R_{kjsh}(\nabla_\ell J_i{}^{a})J_a{}^{s} + \frac{1}{2}R_{kjis}(\nabla_\ell J_h{}^{a})J_a{}^{s}.\end{aligned} \tag{5.2$'$}$$

Here we set $\alpha = \frac{S}{30}$. Then by using the identities from (2.1) to (4.8), we have the following four kinds of formulas ((5.3)–(5.6)):

Making use of the Bianchi's identities, (2.1), (4.7), and (4.2), we have

$$\begin{aligned}(\nabla_\ell R_{kjih})R_s{}^{jih}(\nabla^\ell J^k{}_a)J^{as} &= -(\nabla_k R_{j\ell ih} + \nabla_j R_{\ell kih})(\nabla^\ell J^k{}_a)J^a{}_s R^{sjih}\\ &= \frac{1}{2}J_s{}^{a}\nabla_a J^{\ell k}(\nabla_j R_{\ell kih})R^{sjih}\\ &= -\alpha\Big\{R_{sjih} - \frac{S}{30}(g_{ji}g_{sh} - g_{jh}g_{si})\Big\}R^{sjih}\\ &= -\alpha\Big(\|R\|^2 - \frac{S^2}{15}\Big).\end{aligned}$$

In a similar way, we have

$$\begin{aligned}(\nabla_\ell R_{kjih})R_s{}^{jih}(\nabla^\ell J^k{}_a)J^{as} &= (\nabla_\ell R_{kjih})R^{ksih}(\nabla^\ell J^{ja})J_{as}\\ &= (\nabla_\ell R_{kjih})R^{kjsh}(\nabla^\ell J^{ia})J_{as}\\ &= (\nabla_\ell R_{kjih})R^{kjis}(\nabla^\ell J^{ha})J_{as}\\ &= -\alpha\|Z\|^2,\end{aligned} \tag{5.3}$$

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Taking into account (3.1) and (4.2), we have

$$R^{tjih}(\nabla^\ell J^{kb})J_{bt}R_{sjih}(\nabla_\ell J_k{}^a)J_a{}^s = R^{tjih}R^s{}_{jih}(R^{ba} - R^{*ba})J_{bt}J_{as} = 4\alpha\|R\|^2. \tag{5.4}$$

Making use of Lemma 4.2, (2.2), (2.3), and (4.2) , we have

$$\begin{aligned}R_{ksih}R^{tjih}(\nabla^\ell J^{kb})J_{bt}(\nabla_\ell J_j{}^a)J_a{}^s &= R^{tjih}R^{ks}{}_{ih}\nabla^\ell J_{kt}\nabla_\ell J_{js}\\ &= \alpha R^{jiba}R^{kh}{}_{ba}(g_{ki}g_{jh} - g_{kh}g_{ji} - J_{ki}J_{jh} + J_{kh}J_{ji})\\ &= 2\alpha(-\|R\|^2 + S\alpha + 6\alpha^2).\end{aligned}$$

By using a similar method, we have

$$\begin{aligned}R_{ksih}R^{tjih}(\nabla^\ell J^{kb})J_{bt}(\nabla_\ell J_j{}^a)J_a{}^s &= R_{kjth}R^{kjis}(\nabla^\ell J_i{}^b)J_b{}^t(\nabla_\ell J^{ha})J_{as}\\ &= 2\alpha(-\|R\|^2 + S\alpha + 6\alpha^2).\end{aligned} \tag{5.5}$$

By (2.3) and (4.3), we need the following for the proof of formula (5.6),

$$\begin{aligned}R^{jtis}J_i{}^kJ_j{}^hR_{kths} &= R^{tjih}J_h{}^k\{J_i{}^bR_{kjtb} + \alpha(g_{ki}J_{jt} - g_{ji}J_{kt} + J_{ki}g_{jt} - J_{ji}g_{kt})\}\\ &= -\frac{1}{2}\|R\|^2 + 2S\alpha.\end{aligned}$$

Hence, by using (4.3) again, we have

$$\begin{aligned}R^{tjih}R_{kjsh}(\nabla^\ell J^{kb})J_{bt}(\nabla_\ell J_i{}^a)J_a{}^s &= R^{tjih}\nabla_\ell J_{kt}(\nabla^\ell J_{is})R^k{}_j{}^s{}_h\\ &= \alpha R^{jtis}\{g_{ki}g_{jh} - g_{kh}g_{ji} - J_{ki}J_{jh} + J_{kh}J_{ji}\}R^k{}_t{}^h{}_s\\ &= \alpha\left(\frac{1}{2}\|R\|^2 - \frac{S^2}{6} + R^{jtis}J_i{}^kJ_j{}^hR_{kths} + 6\alpha^2\right)\\ &= \alpha\left(-\frac{S^2}{6} + 2S\alpha + 6\alpha^2\right).\end{aligned}$$

In a similar way, we have the following

$$\begin{aligned}R^{tjih}R_{kjsh}(\nabla^\ell J^{kb})J_{bt}(\nabla_\ell J_i{}^a)J_a{}^s &= R^{tjih}R_{kjis}(\nabla^\ell J^{kb})J_{bt}(\nabla_\ell J_h{}^a)J_a{}^s\\ &= R^{ktih}R_{kjsh}(\nabla^\ell J^{jb})J_{bt}(\nabla_\ell J_i{}^a)J_a{}^s\\ &= R^{ktih}R_{kjis}(\nabla^\ell J^{jb})J_{bt}(\nabla_\ell J_h{}^a)J_a{}^s\\ &= \alpha\left(-\frac{S^2}{6} + 2S\alpha - 6\alpha^2\right).\end{aligned} \tag{5.6}$$

Lemma 5.1 *Let M be a strict nearly Kähler manifold with* $\dim M = 6$. *Then we have*

$$\|\bar{\nabla}R\|^2 = \|\nabla R\|^2 - \frac{S}{15}\|Z\|^2 \geq 0. \tag{5.7}$$

Thus by (3.4) and (5.1), the Theorem of Ambrose and Singer (see Tricerri and Vanhecke [14, page 14]) gives the following theorem.

Theorem 5.2 *Let* (M, g, J) *be a strict nearly Kähler manifold with* $\dim M = 6$*. If* $\|\nabla R\|^2 = \frac{S}{15}\|Z\|^2$*, then* M *is locally homogeneous.*

Then by using the result of Nagy (see [11, page 500]) we complete the proof of our Main Theorem in the introduction.

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