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FINITE LINEAR GROUPS OF DEGREE SIX

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1. Introduction. In this paper we classify finite groups G with a faithful, quasiprimitive (see Notation), unimodular representation X with character χ of degree six over the complex number field. There are three gaps in the proof which are filled in by [**16**; **17**]. These gaps concern existence and uniqueness of simple, projective, complex linear groups of order 604800, $|\mathrm{LF}(3, 4)|$, and $|\mathrm{PSL}_4(3)|$. By [**19**], X is a tensor product of a 2-dimensional and a 3-dimensional group, or a subgroup thereof, or X corresponds to a projective representation of a simple group, possibly extended by some automorphisms. The tensor product case is discussed in section 10. Otherwise, we assume that $G/Z(G)$ is simple. We discuss which automorphisms of $G/Z(G)$ extend the representation X (that is, lift to the central extension G and fix the character corresponding to X) just after we find $X(G)$. All cases where the simple groups $G/Z(G)$ have an irreducible projective complex representation of degree 2, 3, 4, 5, or 7 are discussed in section 11, where we use the corresponding classifications of Blichfeldt, Brauer, and Wales. Otherwise, we assume that no such projective representations exist. By section 6, we are allowed to assume that no prime greater than 7 divides $|G|$. By [**18**], if 7^2 divides $|G|$, then G has a nontrivial, normal 7-group, or X is reducible, or G has a subgroup of index 2, all contrary to the simplicity of $G/Z(G)$. The case where 7 does not divide $|G/Z(G)|$ is discussed in section 5. Otherwise, we assume that 7 divides $|G/Z(G)|$ to the first power. As we principally use the degree equation of [**4**] for the prime 7, we break this case up into subcases for $[N(S_7) : C(S_7)]$. These cases are treated in sections 7 and 8.

2. Notation. Let $x, g \in G$, where G is a finite group, let H be a subgroup of G, let π be a set of primes, and let n be an integer. We make the following set of definitions: $|S|$ denotes the cardinality of S, $N(S)$ denotes the normalizer of S, $C(S)$ denotes the centralizer of S, $Z(G)$ or Z denotes the centre of G, $x^g = g^{-1}xg$, $H^g = g^{-1}Hg$, H^G is the smallest normal subgroup of G containing H, $0_\pi(G)$ is the largest normal π-subgroup of G, $0^{\pi'}(G)$ is the smallest normal subgroup of G of index relatively prime to the primes in π, $[g]_\pi$ is the element obtained by projecting g into $0_\pi(\langle g\rangle)$ in $\langle g\rangle = 0_\pi(\langle g\rangle) \times 0_{\pi'}(\langle g\rangle)$, n_π is the largest factor of n divisible only by primes in π, $i_\pi(G) = |G|_\pi/|0_\pi(G)|$.

Further, let X be a representation of G on the vector space V. We define

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$X(G)$ to be $\{X(h)|h \in G\}$, a linear group operating on V. For $v \in V$, let gv be the image of v under $X(g)$. If α is a complex number, define $C_V(\alpha g) = \{v|v \in V, gv = \alpha^{-1}v\}$. We say that W is a homogeneous space of V for G if, for some irreducible representation Y of G, W is the sum of all the spaces on which constituents equivalent to Y act. The number of homogeneous spaces is called the variety of the representation X on G. If X is irreducible and for any normal subgroup N of G, $X|N$ has variety one, then we say that X is quasiprimitive.

We use S_p to denote a p-Sylow subgroup and $\mathscr{S}_n$ to denote a group isomorphic to the symmetric group on n letters. Our use of $\subset$ does not exclude equality. The symbol $\|$ means divides exactly. If p is a prime and $p \parallel |G|$, then π_p denotes a generator of S_p, q_p denotes $[N(S_p) : C(S_p)]$, and

$$t_p = (p - 1)/q_p.$$

After section 3, we assume that G is a finite group with a faithful, quasiprimitive, irreducible, unimodular representation X with character χ of degree six over the complex numbers, and we let ω, ϵ, and β denote primitive third, fifth, and seventh roots of unity, respectively. Finally, $a(X, Y, Z)$ is the coefficient of the conjugacy class containing Z in the product of classes containing X and Y.

We frequently refer to the theorem of Blichfeldt that a quasiprimitive, complex linear group contains no non-scalar element which has an eigenvalue within 60 degrees of all its other eigenvalues. This is [**1**, Theorem 8, page 96]. The statement given there is for primitive groups, but the proof uses only quasiprimitivity.

We must also make extensive use of the classification of linear groups of degrees 2, 3, and 4 given in [**1**]. In [**1**, Chapter III], the linear groups of degree 2 are listed as types (A) through (E) between pages 69 and 73. Of these groups only the icosahedral group, type (E) on page 73, has two noncommuting elements of order 5. The groups of degree 3 are numbered (A) through (J) on pages 105, 109 and 113. Only the groups (H) and (I), A_5 and A_6, contain two non-commuting elements of order 5.

We often refer to a consequence of [**4**, II, Theorem 1] that if Y is an irreducible representation of G with a p-Sylow subgroup $P = \langle x \rangle$ of order p, then the pth roots of unity can be partitioned into disjoint sets S_1 and S_2 such that the elements of S_i occur with the same multiplicity as eigenvalues of $Y(x)$ for $i = 1, 2$. Also, $S_1 = \{1\}$ or there exists a primitive pth root of unity γ such that $S_1 = \{\gamma^{m^r}|r \in \mathbf{Z}\}$ for m equal to the $[(p-1)/[N(P) : C(P)]]$th power of a generator of the multiplicative group of $\mathbf{Z}/p\mathbf{Z}$.

In conjunction with section 4, Lemma 1, we often use [**10**, 53.17] from which it follows that degrees of irreducible characters of G divide $|G/Z|$.

3. Groups of degree six.

THEOREM. *If G has a complex irreducible representation X of degree 6 which is faithful, unimodular, and quasiprimitive, then G is one of the following groups*

(*$|Z|$ is given for the case $G = G'$; G can be extended means that a larger group has a representation of degree* 6).

I. *$G/Z \cong A \times B$ where $A \cong A_5$, A_4, or $\mathscr{S}_4$, and $B \cong$* PSL(2, 7), *A_5, A_6, or H_i for $i = 1, 2$, or 3, where H_3 is the Hessian group in* [**1**] *isomorphic to an extension of $Z_3 \times Z_3$ by* SL(2, 3), *and H_2 and H_1 have indices* 3 *and* 6 *in H_3.*

II. *G/Z is isomorphic to a subgroup of index* 2 *in $\mathscr{S}_4 \times H_i$, $i = 1$ or 2, or to a subgroup of index* 3, 12, *or* 24 *in $A_4 \times H_3$.*

III. *$G/Z \cong A_5$ or $\mathscr{S}_5$, $|Z| = 2$, $|A_5| = 60$.*

IV. *$G/Z \cong A_6 \cong$* PSL(2, 9), *$|Z| = 3$, $|A_6| = 360$; G' has two conjugate characters of degree* 6 *contained in $Q(\omega)$. This group G can be extended by an automorphism of order* 2 *coming from the product of the outer automorphism from* GL(2, 9) *and the field automorphism of* PSL(2, 9).

V. *$G/Z \cong A_6$, $|Z| = 6$; G' has four conjugate characters of degree* 6 *contained in $Q(\omega, \sqrt{2})$.*

VI. *$G/Z \cong A_7$ or S_7, $|Z| = 1$, $|A_7| = 7!/2$.*

VII. *$G/Z \cong A_7$, $|Z| = 3$; G' has two conjugate characters of degree* 6 *contained in $Q(\omega)$.*

VIII. *$G/Z \cong A_7$, $|Z| = 6$; G' has four conjugate characters of degree* 6 *contained in $Q(\omega, \sqrt{2})$.*

IX. *$G/Z \cong$* PSL(2, 7) *or* PGL(2, 7), *$|Z| = 1$, $|$PSL$(2, 7)| = 168$.*

X. *$G/Z \cong$* PSL(2, 7) *or* PGL(2, 7), *$|Z| = 2$, G' has two conjugate characters of degree* 6 *contained in $Q(\sqrt{2})$.*

XI. *$G/Z \cong$* PSL(2, 11), *$|Z| = 2$, $|$PSL$(2, 11)| = 660$; G' has two conjugate characters contained in $Q(\sqrt{-11})$.*

XII. *$G/Z \cong$* PSL(2, 13), *$|Z| = 2$, $|$PSL$(2, 13)| = 13 \cdot 7 \cdot 3 \cdot 2^2$; G' has two conjugate characters of degree* 6 *contained in $Q(\sqrt{13})$.*

XIII. *$G/Z \cong$ $\mathrm{PSU}_4(2)$, $|Z| = 1$, $|\mathrm{PSU}_4(2)| = 2^6 3^4 5$; G' can be extended by an automorphism of order* 2.

XIV. *$G/Z \cong \mathrm{U}_3(3)$, $|Z| = 1$, $|\mathrm{U}_3(3)| = 6048$; G can be extended by a field automorphism of order* 2.

XV. *$G/Z \cong \mathrm{PSU}_4(3) \cong 0_6(3)$, $|Z| = 6$, $|\mathrm{U}_4(3)| = 2^7 3^6 35$; G can be extended by an automorphism of order* 2; *G' has two conjugate characters contained in $Q(\omega)$.*

XVI. *G/Z is isomorphic to the Hall-Janko group, $|Z| = 2$, $|G/Z| = 604800$.*

XVII. *$G/Z \cong$* LF(3, 4), *$|Z| = 6$, $|$LF$(3, 4)| = 2^6 3^2 35$; G' has two conjugate characters contained in $Q(\omega)$; G can be extended by an automorphism of order* 2 *coming from the product of a graph automorphism and a field automorphism.*

The reason for classifying quasiprimitive 6-dimensional rather than primitive 6-dimensional groups is not to achieve greater generality, but to avoid deciding which of the above groups are primitive. For example, the projective representation (IV) of A_6 with centre of order 3 is monomial.

4. The Sylow subgroups. We use our bounds on $G/Z(G)$ to limit the possible degrees of complex representations of G. By [**10**, Theorem 53.17], these degrees divide $|G/Z|$ where $Z = Z(G)$.

Here we establish bounds for the order of $S_p/(S_p \cap Z)$, where S_p is a p-Sylow subgroup of G for $p = 2$ or 3 assuming that G/Z is simple. We also find some general results for S_5 in the case where $X(G)$ is quasiprimitive.

LEMMA 1. *If G/Z is simple and X is a faithful representation of G of degree* 6, *then* $|S_2/(S_2 \cap Z)| \leqq 2^9$ *and* $|S_3/(S_3 \cap Z)| \leqq 3^7$.

Proof. $X(S_2)$ is monomial and S_2 has an abelian subgroup A of index at most 16 in S_2. If $A/(A \cap Z)$ has order as large as 64, then by [**5**, 3D], a proper normal subgroup of G intersects A in a subgroup of index at most $2^{6-1} = 32$, which is a contradiction. The bound for $|S_3|$ is obtained similarly.

LEMMA 2. *If X is a faithful, irreducible, quasiprimitive n-dimensional representation of a finite group G with $n > 4$, then for no $g \in G - Z$ does $X(g)$ have the eigenvalues $\alpha, \ldots, \alpha, \sigma, \tau$ with $g^5 \in Z$.*

Proof. Suppose that $g \in G - Z$, $X(g)$ has eigenvalues $\alpha, \ldots, \alpha, \sigma, \tau$, and that $g^5 = 1$. Let h be conjugate to g but not in $C(g)$; otherwise, $\langle g \rangle^G$ is a normal abelian subgroup of G, contradicting quasiprimitivity. Then as

$$n - \dim C_V(\alpha^{-1}g) \cap C_V(\alpha^{-1}h) \leqq n - \dim C_V(\alpha^{-1}g) + n - \dim C_V(\alpha^{-1}h) \leqq 2 + 2 = 4,$$

$\chi|\langle g, h\rangle = (n-4)\mu + \zeta$, where μ and ζ are characters of $\langle g, h\rangle$ of degree 1 and 4, respectively. If the representation Y corresponding to ζ has two disjoint invariant subspaces of dimension 2, then at least one of them corresponds to a projective representation of A_5. Now, $\langle g, h\rangle$ is isomorphic to a subgroup of the direct product of the irreducible constituents of $X|\langle g, h\rangle$. Then by the subdirect product theorem [**14**, Theorem 5.5.1], we obtain a product of commutators with eigenvalues $-\omega, -\bar{\omega}, 1, 1, \ldots, 1$ or $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1, \ldots, 1$, in contradiction to the eigenvalues result of [**1**]. In particular, $\sigma \neq \tau$; otherwise, $Y(g)$ and $Y(h)$ have eigenvalues $\alpha, \alpha, \sigma, \sigma$ and by [**1**, section 103], Y has two disjoint invariant subspaces of dimension 2, which is a contradiction. Also, $\alpha \neq \sigma$ and $\alpha \neq \tau$, for the same reason.

If Y is irreducible, then by enlarging $Z(G)$ and replacing $X(g)$ and $X(h)$ by scalar multiples of themselves, we may assume that $\det Y(g) = \det Y(h) = 1$, and that Y is unimodular. Then by [**4**], $25 \mid |\langle g, h\rangle|$. Furthermore, Y cannot be irreducible and imprimitive, since in this case $Y(g)$ and $Y(h)$ would permute the spaces of imprimitivity trivially. By Blichfeldt's classification [**1**] of groups of degree 4, Y contains, as a subgroup of index 1 or 2, a tensor product of 2-dimensional representations of a central extension by A_5. Again we get $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1, \ldots, 1$ in contradiction to the eigenvalues result of [**1**]. Therefore, $C_V(\alpha^{-1}g) \cap C_V(\alpha^{-1}h)$ has a 3-dimensional

$\langle g, h\rangle$-invariant complement U on which $\langle g, h\rangle$ irreducibly represents A_5 or A_6 projectively, by the groups of degree 3 of [**1**]. By quasiprimitivity, U is not left invariant by the normal subgroup generated by conjugates of g and in particular, not by k equal to some conjugate of g. Now,

$$n - \dim(C_V(\alpha^{-1}g) \cap C_V(\alpha^{-1}h)) \cap C_V(\alpha^{-1}k) \leqq 3 + 2 = 5.$$

Then if U is contained in a $\langle g, h, k\rangle$-invariant irreducible 4-dimensional space, the remaining constituents are linear and we reach the same contradiction as when we assumed Y to be irreducible. Therefore, $X|\langle g, h, k\rangle$ has an irreducible 5-dimensional constituent W and the remaining constituents are linear. By [**5**, 4E and 9A], because of an element with eigenvalues α, α, α, σ, τ, W is monomial. But then $W(g)$ and $W(h)$, by their eigenvalues, must be diagonal and commute, which is a contradiction.

LEMMA 3. *If X is a faithful, 6-dimensional representation of the finite group G, then a 5-Sylow subgroup S_5 of G is abelian.*

Proof. If this were not true, a non-identity element in $S_5 \cap Z(S_5)$ would have eigenvalues $1, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon$, contrary to Lemma 2.

5. The case G/Z simple and $|G/Z| = 5^c3^b2^a$. Now, until section 10, we assume that G is faithfully and quasiprimitively represented by X on a 6-dimensional space V, G/Z is simple, and $G = G'$. In particular, X is unimodular and $|Z| \mid 6$. Suppose that D is a 5-Sylow intersection group. Further, suppose that $X|D$ has variety 2. Then by Lemma 3, S_5 is abelian, and by Lemma 2 no 5-element has as many as four identical eigenvalues, so we have $g \in D$ with $X(g) = \mathrm{diag}(\epsilon, \epsilon, \epsilon, \bar{\epsilon}, \bar{\epsilon}, \bar{\epsilon})$. Then $C(g)$ does not have a normal 5-Sylow subgroup. By Blichfeldt's classification [**1**] (we mentioned this in section 2) of 2 and 3-dimensional groups, $X|C(g)$ has a 2-dimensional constituent projectively representing A_5, or an irreducible 3-dimensional constituent representing A_5 or A_6 projectively. Suppose that the former is contained in $C_V(\epsilon^{-1}g)$. Then $C(g)$ on $C_V(\epsilon g)$ represents A_5 projectively and irreducibly on the 3-dimensional subspace; otherwise, by the subdirect product theorem we obtain an element with eigenvalues $-\omega, -\bar{\omega}, 1, 1, 1, 1$ or $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1$. Then $X|S_5$ has variety 6 and is centralized by an element, $X(h) = \mathrm{diag}(-1, -1, 1, 1, 1, 1)$ of order 2 corresponding to the centre of the 2-dimensional projective representation of A_5. We may look at a 2-modular representation of G corresponding to X. When $X(S_5)$ is diagonal, $X(h)$ is diagonal and in the kernel, a 2-group by [**5**], of the 2-modular representation, contradicting the simplicity of G/Z. Therefore, the 2-dimensional projective representation of A_5 cannot arise here. Then, as an element $(\epsilon, \bar{\epsilon}, 1, 1, 1, 1)$ is impossible by Lemma 2, $X|C(g)$ has two irreducible 3-dimensional constituents dependently representing A_5 or A_6 projectively. Now, $C(D)/Z(C(D)) \cong A_m$ for $m = 5$ or 6. By unimodularity of X, all 5-elements x of $Z(C(D))$ have $X(x) = \mathrm{diag}(\gamma, \gamma, \gamma, \bar{\gamma}, \bar{\gamma}, \bar{\gamma})$ for some γ, and $[Z(C(D))]_5$

is cyclic. If $25 \mid |Z(C(D))|$, then for some x we have $\gamma = e^{2\pi i/25}$, in contradiction to the eigenvalues result of [**1**]. Then $5 \| \, |Z(C(D))|$ and, as $S_5 \subset C(D)$, $|S_5| = 25$. Also, $C(g)'$ has an element with eigenvalues $1, -1, -1, 1, -1, -1$. Because conjugates of this element generate G we may take a conjugate j not commuting with g. Also, $X|\langle g, j\rangle$ has at least two linear constituents corresponding to $C_V(-j) \cap C_V(\epsilon^{-1}g)$ and $C_V(-j) \cap C_V(\epsilon g)$. On a complementary space, $X(g)$ has eigenvalues $\epsilon, \epsilon, \bar{\epsilon}, \bar{\epsilon}$ and $X(j)$ has eigenvalues $1, 1, -1, -1$. By [**1**, section 103], this reduces at least to 2-dimensional constituents. If either represents A_5 projectively, then we get an element contradicting the eigenvalues result of [**1**], as before. Therefore, these constituents are monomial. Because $|S_5| = 25, S_5 = \langle(\epsilon, \bar{\epsilon}; \epsilon, \bar{\epsilon}; \epsilon, \bar{\epsilon})$; either $(\epsilon, \bar{\epsilon}, \bar{\epsilon}, \epsilon, \epsilon, \bar{\epsilon})$ or $(\epsilon, \bar{\epsilon}, \bar{\epsilon}, \epsilon, \bar{\epsilon}, \epsilon)\rangle$. In the first case the first and fifth linear characters are identical; and in the second case the third and fifth characters are identical. This contradicts variety 6.

Now, assume that $X|D$ has variety 3. The possibilities for $X|C(D)$ where $X|D$ has homogeneous spaces of dimensions 1, 2, and 3 were all eliminated in the case of variety 2. Homogeneous spaces of dimensions 1, 1, and 4 contradict Lemma 2. Therefore, homogeneous spaces have dimensions 2, 2, and 2. Then $C(D)$ is a central extension by A_5 whose image in each 2-dimensional constituent must be non-abelian to avoid the previous contradiction to the eigenvalues result of [**1**]. By Lemma 2 and unimodularity, an element $g \in D$ of order 5 must have eigenvalues $1, 1, \epsilon, \epsilon, \bar{\epsilon}, \bar{\epsilon}$ or with ϵ replaced by ϵ^2 everywhere. Therefore, D is cyclic. Suppose that $h^5 = g$ and $h \in D$. We may take $X(g) = \operatorname{diag}(1, 1, \epsilon, \epsilon, \bar{\epsilon}, \bar{\epsilon})$, $\epsilon = \gamma^5$, $X(h) = \operatorname{diag}(\epsilon^a, \epsilon^a, \gamma, \gamma, \gamma^{-1-5a}, \gamma^{-1-5a})$ where γ is a fixed primitive twentyfifth root of unity. If $a = 0$, then h contradicts the eigenvalues result of [**1**]; if $a = 1$, $X(h^7) = \operatorname{diag}(\gamma^{10}, \gamma^{10}, \gamma^7, \gamma^7, \gamma^8, \gamma^8)$ contradicts [**1**]; if $a = 2$, $X(h^8) = \operatorname{diag}(\gamma^5, \gamma^5, \gamma^8, \gamma^8, \gamma^{12}, \gamma^{12})$; if $a = 3$, $X(h^9) = \operatorname{diag}(\gamma^{10}, \gamma^{10}, \gamma^9, \gamma^9, \gamma^6, \gamma^6)$; if $a = 4$, $X(h^8) = \operatorname{diag}(\gamma^{10}, \gamma^{10}, \gamma^8, \gamma^8, \gamma^7, \gamma^7)$. Therefore, $|D| = 5$ and $|S_5| = 25$.

If $X|D$ has variety as large as 4, then the partition into homogeneous subspaces is a refinement of $6 = 1 + 2 + 3$ and all the possibilities were eliminated in the case of variety 2. Therefore, in the case of nontrivial 5-Sylow intersection, $|S_5| = 25$, $D = \langle(1, 1, \epsilon, \epsilon, \bar{\epsilon}, \bar{\epsilon})\rangle$, $C(D)$ is a central extension by A_5, $C(D)' \cong \mathrm{SL}(2, 5)$, and $\chi|D \times C(D)' = (1)\theta_1 + (\epsilon)\theta_2 + (\bar{\epsilon})\theta_3$. Here, the θ_i are faithful 2-dimensional representations of $\mathrm{SL}(2, 5)$ and (ξ) is the linear character of D taking g to ξ. In particular, the image of the 2-element in the centre of $C(D)'$ is $\operatorname{diag}(-1, -1, -1, -1, -1, -1)$ and for any 6-dimensional representation of G, the centre has even order. Therefore, the skew-symmetric tensors Y of $X \otimes X$ cannot have a constituent of degree 6 and must be irreducible of degree 15, since by [**1**; **5**] G has no representation of degree 2, 3, 4, or 5. Let ϕ be the character of Y. A 5-block B of defect 1 contains ϕ. By [**3**, Corollary 6] if ϕ has four 5-conjugates, then the sum of these conjugates and a character η of degree 15 is a 5-principal indecomposable. Now, $\phi(g) = 5 + 4\epsilon + 4\bar{\epsilon} + \epsilon^2 + \epsilon^{-2}$ so $20 - 10 + \eta(g) = 0$. This is impossible since

$15 < 40$. Therefore, since $\phi(g)$ is irrational, ϕ has two 5-conjugates. Then ϕ is fixed by taking ϵ to $\bar{\epsilon}$ and $\theta_2 = \theta_3$. Another character of odd degree equal to $3^d 5 \equiv \pm 30 \pmod{25}$ is contained in B by [**3**, Corollaries 4 and 5]. The degree equation must be $45 - 15 = 30$, $405 + 15 = 430$, or $3645 - 15 = 3630$. Only $45 = 15 + 30$ is possible. By [**3**, Corollary 6], $\phi +$ (the conjugate of ϕ) $+$ (the character of degree 45) is a principal indecomposable Φ of degree 75. The character $\phi|D \times C(D)'$ contains at least two copies of the principal character of $D \times C(D)'$, one from the skew-symmetric tensors of $(1)\theta_1$, and one from the $(\epsilon)\theta_2(\bar{\epsilon})\theta_2$. Then $\Phi|D \times C(D)'$ contains at least four copies of this principal character. We can write $\Phi|D \times C(D)' = (1)v_0 + (\epsilon)v_1 + (\epsilon^2)v_2 + (\epsilon^3)v_3 + (\epsilon^4)v_4$, where the v_i are characters of $C(D)'$. This expression is 0 at uv, where v is any fixed element in $C(D)'$ and u runs through g, g^2, g^3, and g^4. Therefore, the $v_i(v)$ are identical for all i, and the v_i are identical. Letting $u = 1$ and running through the 5-elements of $C(D)'$ we see that v_0 is a sum of principal 5-indecomposables. Also, v_0 contains the principal character at least four times and $\deg v_0 \geqq 20$, which is a contradiction.

The only remaining case is the trivial intersection case. Here, $(\mathrm{mod}|S_5|)\ 1 \equiv |G|/|N(S_5)|\ |2^9 3^7$, by Lemma 1. Now, $|S_5|\ |25$ and by Brauer's classification [**6**] of simple groups of order $2^a 3^b 5$, we may assume that $|S_5| = 25$. By [**10**, 88.8 and 53.17], G has no 5-block of defect 1. Therefore, the skew symmetric tensors of $X \otimes X$ are reducible and must contain an irreducible constituent of degree 6. Replacing X by this constituent, we may take $|Z|$ as odd. Suppose that the simple group $\bar{G} = G/Z$ contains a p-element h centralizing a 5-element g for $p = 2$ or 3. If $B_0(p)$ is the principal p-block, we have the block orthogonality relation

$$\sum_{\chi_i \in B_0(p)} \chi_i(1)\chi_i(gh) = 0.$$

Because χ_0 contributes 1, we have another term not divisible by q, the other prime 2 or $3 \neq p$. Then, $\chi_i(1) = 5^e p^f$. Also, $e \neq 2$ or χ_i is in a block of 5-defect 0 and $\chi_i(gh) = 0$; $e \neq 1$ or χ_i is in a block of 5-defect 1; and $e \neq 0$ or by [**9**, Lemma 2] χ_i would not be in a p-block of full defect. This is a contradiction, and for any $g \neq 1$ in a 5-Sylow subgroup S_5 of G, $C(g) = S_5 Z$. This is the situation described in [**8**].† Let ζ be an irreducible constituent of $\chi|N(S_5)$ which does not have S_5 in its kernel. ζ is in a proper family F of $24/[N(S_5) : C(S_5)]$ characters of degree $[N(S_5) : C(S_5)]$ of $N(S_5)$. By [**8**, 2C], $\chi|N(S_5)$ takes on all but one of these characters the same number of times m and the other character $m - 1$, m or $m + 1$ times. By [**2**, Lemma 1], $N(S_5)/C(S_5)$ acts without fixed points on the non-principal characters of S_5. Then,

$$[N(S_5) : C(S_5)] \leqq 6$$

†More easily, by the inequality in the proof of [**11**, Lemma 4.1], $||\chi|N(S_5)|| = 1$. Then the argument on $X(J)$ near the end of this section completes the proof.

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as all conjugates of $\zeta|S_5$ in $N(S_5)$ are constituents of $\chi|S_5$. The sum of all but one of the characters of F still has degree as large as $18 > 6$, so $\chi|N(S_5) - \zeta$ contains no characters in F. However, F contains all irreducible characters of $N(S_5)$ with action on Z identical to that of χ and nontrivial action on S_5. Therefore, $\chi|S_5 = \zeta|S_5 +$ (principal characters of S_5). By unimodularity and Lemma 2, an element with eigenvalues $\epsilon, \bar{\epsilon}, 1, 1, 1, 1$ is precluded; therefore, $\deg \zeta \neq 1$ or 2. Then, $\deg \zeta = 3, 4$, or 6. If $\deg \zeta = 3$, then

$$[N(S_5) : C(S_5)] = 3$$

and S_5 is not cyclic. Then, some element has eigenvalues $\epsilon, \bar{\epsilon}, 1, 1, 1, 1$. Suppose that $[N(S_5) : C(S_5)] = \deg \zeta = 6$. Because $|Z|$ is odd, we can find J of order 2 in $N(S_5) - C(S_5)$. Then $X(J)$ permutes the six character of $\chi|S_5$ without fixed points and $X(J)$ may be taken as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

contrary to unimodularity. Suppose $\deg \zeta = 4$. Let S be the set of 5-singular elements of G. Then

$$\begin{aligned} 1 &= ||\chi||^2 \\ &\geq \sum_{x \in S} |\chi(x)|^2/|G| \\ &\geq \Big[\sum_{x \in C(S_5)} |\chi(x)|^2 - 6^2|Z| \Big] / |N(S_5)| \\ &\geq (6 - 36/25)/4, \end{aligned}$$

which is a contradiction.

6. The case where a large prime divides $|G|$. Here again G/Z is assumed to be simple. Suppose that $p^a| \, |G|$ for p a prime greater than 7 and $a \geqq 1$. By [**11**] applied to $S_p \subset G$, G has a normal p-subgroup of order at least p^{a-1}. By simplicity of G/Z, $a = 1$, and by [**4**], if $p \geqq 13$, then $G/Z \cong \mathrm{PSL}(2, 13)$. Assume that $11| \, |G|$. Then $11 \| \, |G|$. Also, $q_{11} \leqq 6$, so $q_{11} = 5$ or 2, for if $q_{11} = 1$, then G has a normal 11-complement. By [**4**], if $q_{11} = 2$ and γ is a primitive 11th root of unity, then we have that $X(\pi_{11})$ has eigenvalues $\gamma, \gamma, \gamma, \bar{\gamma}, \bar{\gamma}, \bar{\gamma}$, contrary to the eigenvalues result of [**1**]. Let ζ be the character corresponding to the skew-symmetric tensors of $X \otimes X$. Now, $\chi(\pi_{11}) = 1 + \alpha$ where $\alpha = \gamma + \gamma^3 + \gamma^9 + \gamma^5 + \gamma^4$. In $\zeta(\pi_{11})$ no eigenvalue 1 occurs, but 1γ and $\gamma^9\gamma$ occur, so $\zeta(\pi_{11}) = 2\alpha + \bar{\alpha}$ or $2\bar{\alpha} + \alpha$. In any event, by [**4**], ζ is reducible. Since 5 divides the degrees of the constituents, ζ has an irreducible character of degree 5 and by [**4**], $G/Z \cong \mathrm{PSL}(2, 11)$.

By [**18**, Theorem], $7^2 \nmid |G|$. By the previous section, we may assume that a higher prime than 5 divides $|G|$. Therefore, when G/Z is simple, we may assume that $|G| = 2^a 3^b 5^c 7$. This case concerns the next two sections.

7. The cases $t_7 = 6$ and $t_7 = 3$. The case $t_7 = 6$ is eliminated by the existence of a normal 7-complement in this case. Assume that $t_7 = 3$ and $|G/Z| = 2^a 3^b 5^c 7$. Now, $X(\pi_7) = \mathrm{diag}(\gamma, \gamma, \gamma, \bar{\gamma}, \bar{\gamma}, \bar{\gamma})$ contradicts the eigenvalues result of [**1**], so by [**4**], $\chi(\pi_7) = -1$ and $C(\pi_7) = S_7 Z$. Furthermore, $B_0(7)$ contains characters of degrees 1, x, and y, with $x - y = \pm 1$; $x, y \equiv \pm 1$ or $\pm 2 \pmod 7$ ($\pm$'s independent here). Since x and y do not have a common prime divisor, one of them, say x, is a power of 2, 3, or 5. We may assume by [**1**; **5**; **29**] that no degree is 2, 3, 4, 5, or 7. All the characters of G with action on Z identical to the action of χ lie in the same 7-block which can contain at most two characters of degree 6. Therefore, χ has at most two 5-conjugates. Then S_5 is elementary abelian, and $\chi|S_5 = \theta + \bar{\theta}$, with θ faithful of degree 3. Because θ is faithful on S_5 elementary abelian, if $|S_5| \geqq 125$, then for some element $x \in G$, $\theta(x) = 1 + 1 + \epsilon$ and $\chi(x) = 1 + 1 + 1 + 1 + \epsilon + \bar{\epsilon}$, contrary to section 4.

If $x = 9$, $y = 8$; $x = 27$, $y = 26$; $x = 243$, $y = 244$; $x = 729$, $y = 730$; $x = 8$, $y = 9$; $x = 16$, $y = 15$; $x = 64$, $y = 65$; $x = 128$, $y = 127$; $x = 512$, $y = 513$. The only possible degree equations are $1 + 8 = 9$ and $1 + 15 = 16$. However, $B_0(7)$ consists of characters of G/Z where S_7 is self-centralizing, and by block separation (see [**9**, Lemmas 2 and 3]) in the first case $|G/Z| = 5^c 504$. Also, $c < 3$ and $\pmod{49}$ $14 \equiv |G/Z| = 5^c 504$. Therefore, $c = 0$ and $G/Z \cong \mathrm{PSL}(2, 8)$ which is dealt with later.

If $x = 16$, $y = 15$, then 5 or 25 divides $|G|$ exactly. If $25 \parallel |G|$, then the character of degree 15 is in a block of 5-defect 1, contrary to block separation. If $5 \parallel |G|$, then by block separation the principal character and the algebraic conjugates of the character of degree 16 lie in the same 5-block. Then this block contains a character of degree $1 + 16 + 16 + 16 = 49$, which is impossible.

8. The case $t_7 = 2$. Assume that G/Z is simple, $t_7 = 2$, and that $|G/Z| = 2^a 3^b 5^c 7$. By [**4**], $\chi(\pi_7) = -1$ or $\beta + \beta + \beta^2 + \beta^2 + \beta^4 + \beta^4$. Suppose that $\chi(G) \not\subset Q(\omega)$. Then we may find a distinct conjugate χ^α with the same action on Z. Also, $\chi^\alpha \bar{\chi}$ has non-principal characters of G/Z as its constituents, entirely. If $\chi(\pi_7) = -1$, then by [**5**, 3F], the image of $\langle \pi_7 \rangle$ in G/Z is self-centralizing, and $\chi^\alpha(\pi_7)\bar{\chi}(\pi_7) = 1$. In the second case, let $C(\pi_7) = \langle \pi_7 \rangle \times V$. Then $\chi|\langle \pi_7 \rangle \times V = (\beta)\theta + (\beta^2)\theta^g + (\beta^4)\theta^{g^2}$, where $g \in N(\langle \pi_7 \rangle) - C(\pi_7)$ and θ is an irreducible character of V. Also, $\theta = \theta^g = \theta^{g^2}$ or χ lies in a 7-block with a character of degree 6 with π_7 in the kernel, which is a contradiction. Furthermore, $\chi\bar{\chi}|\langle \pi_7 \rangle \times V = (1 + 1 + 1 + \beta + \beta^2 + \beta^3 + \beta^4 + \beta^5 + \beta^6)(\theta\bar{\theta})$. Because $\theta\bar{\theta}$ contains the principal character as a constituent, there is a surplus of two $1_{\langle \pi_7 \rangle} 1_V$, at most one corresponding to 1_G, and $\chi\bar{\chi}$ has an irreducible constituent in $B_0(7)$ of G/Z of degree 8, 15, 36, 18, 25, or 32. When $\chi(\pi_7) = -1$, $\chi^\alpha \bar{\chi}$ has such a constituent. In either event, let n be the degree of this constituent. Only for $\chi(\pi_7) = -1$ is $n = 36$ possible. If the four term degree equation $1 + 8 + \ldots = \ldots$ has another term on the left, then by the tree it must be 8. However, $1 + 8 + 8 = 17$, $1 + 15 + 15 = 31$, $1 + 36 + 36 = 73$, $1 + 18 + 18 = 37$, $1 + 25 + 25 = 51$ and $1 + 32 + 32 = 65$ are impossible. If $1 + 8$, $1 + 15$, $1 + 36$, $1 + 18$, $1 + 25$, $1 + 32 = x + y$, we may take $x \leqq 18$ and $x = 6$ or 10. If $x = 6$, we may take X to be the corresponding representation. Then χ is rational, contrary to our supposition. If $x = 10$, then we are in the case of $n = 8$, 15, or 36. It must be the last, since $x = 6$ is contrary to assumption. The degree equation is $1 + 36 = 10 + 27$. Then $\chi(\pi_7) = -1$. As in the previous section, χ has at most three 5-conjugates and $|S_5| \leqq 25$. If $|S_5| = 25$, then 5-7 block separation gives a contradiction, since the character of degree 10 is in a 5-block of defect 1. Therefore, $|S_5| = 5$. By 3-7 block separation, since π_7 is self-centralizing in G/Z, $|G/Z| = 2^a 3^3 35$. Then (mod 7) $1 \equiv [G : N(\langle\pi_7\rangle)] = 2^a 3^2 5 \equiv 2^a 3$, and this is impossible. Therefore, $\chi \subseteq Q(\omega)$ and $\chi(\pi_7) = -1$. Suppose that $3^6 \mid |G/Z|$. Let S_3 be the image of 3-Sylow subgroup under X after adding ωI_6. There exists an abelian subgroup $A \trianglelefteq S_3$ of index at most 9 in S_3 and $|A| \geqq 3^5$. Let ζ be a linear constituent of $\chi|A$. Since $\chi \subseteq Q(\omega)$, $\zeta(g)^9 = 1$ for all $g \in A$, and ζ's lying outside of $Q(\omega)$ occur in triples of conjugate characters. The image of A under such a triple has order 9. If such a triple occurs, then by unimodularity, $|A| \leqq 3^3 9/3$ or $9.9/3$, which is a contradiction. Therefore, $|A| = 3^5$ and $X|A$ consists of all diagonal matrices of order 3 and determinant 1. This shows that S_3 has two non-linear constituents and some element $g \in S_3$ permutes the linear characters of the first constituent Y cyclically and fixes the linear characters of the second constituent, W. Furthermore, $g^3 \in D$ where D is the group of diagonal matrices in S_3, and $Y(g^3) = (\det Y(g))I_3$, so $(\det Y(g))^3 = 1$. Multiplying g by an element in A we may take $\det Y(g) = 1$,

$$Y(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and $W(g) = I_3$. We may reverse the roles of the constituents and $X|S_3$ is determined. Furthermore, ωI_6 occurs as a commutator, $3 \mid |Z|$, and $3^6 \parallel |G/Z|$.

As $\chi \subseteq Q(\omega)$, $\chi|S_5$ is rational and by [**21**],

$$c \leqq [6/(5 - 1)] + [6/5(5 - 1)] + \ldots = 1.$$

If $5 \mid |G|$, then χ is 5-non-exceptional and by [**4**], we may take $S_5 = \langle\pi_5\rangle$ with $X(\pi_5) = \operatorname{diag}(1, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4)$, $C(\pi_5) = S_5 \times V$, and

$$\langle X(V)\rangle = \langle \operatorname{diag}(\alpha^{-5}, \alpha, \alpha, \alpha, \alpha, \alpha)\rangle.$$

Then $\chi(G) \subseteq Q(\omega)$ implies that α is a 6th root of unity, $\alpha^{-5} = \alpha$, and $\langle\pi_5\rangle$ is self-centralizing in G/Z. We have not yet assumed that G has no irreducible representation of degree 7, so this eliminates A_8.

If 5 divides no degree of a character of $B_0(7)$, then $B_0(7)$ contains characters of degrees 8, 32, 64, 256, or 512, and 27, 81, or 729. The degree 8 has already been shown impossible. The other degree is $33 - 27 = \underline{6}$, $33 + 729 = 762$, $65 - 27 = 38$, $65 + 81 = 146$, $65 + 729 = 794$, $257 + 729 = 986$, $257 - 27 = 230$, $513 - 27 = \underline{486}$, $513 + 81 = 594$, and $513 + 729 = 1242$. In the case $1 + 32 = 6 + 27$, block separation implies that $|G/Z| = 2^5 3^3 5^a 7$. By $t = 2$ and a Sylow theorem, $a = 0$. Then $G/Z \cong U_3(3)$, which is studied later. When $1 + 512 = 27 + 486$ is the degree equation, by 3-7 block separation and $3^4 | 486 |\, |G/Z|$, the character of degree 27 is in $B_0(7)$, which is a contradiction.

Now, we may assume that $5 \,\|\, |G|$. Since S_5 does not have a normal complement, $t_5 = 2$ or 1. If $t_5 = 2$, then $B_0(5)$ has a character of degree a power of 2 or 3. The other degree is $32 - 1 = 31$, $64 - 1 = \underline{63}$, $256 + 1 = 257$, $512 - 1 = 511$, $27 - 1 = 26$, $81 + 1 = 82$, or $729 - 1 = 728$. The degree equation is $1 + 63 = 64$. By 2-5 block separation and a Sylow theorem, $|G/Z| = 2^6 3^2 3 5$. This by [**17**] turns out to be LF(3, 4).

We are left with the case $t_5 = 1$ and $B_0(7)$ has a degree divisible by 5. Since $C(S_5) = S_5 Z$ by [**4**], all degrees are $0, \pm 1 \pmod 5$. Then $B_0(7)$ has exactly two degrees divisible by 5. If the last degree is not a prime power, then these degrees are $2^e 5$ and $3^f 5$, and the possible degree equations are $1 + 15 = 6 + 10$, $1 + 64 = 20 + 45$, $1 + 15 + 64 = 80$, $1 + 15 + 144 = 160$, and $1 + 64 + 1215 = 1280$. Note that since $(\beta + \beta^2 + \beta^4)(\beta^3 + \beta^5 + \beta^6) = 2$, the highest power of 2 dividing $\gamma(1)\gamma(\pi_7)$ is at most 2 for γ an irreducible character of degree 45. Only the first two degree equations have a tree and in the second, by 2-7 block separation and $t_7 = 2$, we have $|G/Z| = 2^6 3^2 3 5$ contrary to $t_5 = 1$ and $C(S_5) = S_5 Z$. Suppose that the degree equation for $B_0(7)$ is $1 + 15 = 6 + 10$. Using this character of degree 6 we may assume that $|Z| = 1$ and $|S_3| \leqq [6/2] + [6/6] + \ldots = 4$. The possible orders are $2^3 3^2 3 5$ and $2^5 3^4 3 5$. The first is small enough to conclude that $G = A_7$. In the second case, $\chi | S_3$ has two algebraically conjugate constituents contained in $Q(\omega)$; otherwise, $4 \leqq 2([3/2] + [3/6] + \ldots) = 2$. As in the argument in this section showing that $|S_3| \leqq 3^6$, these constituents are determined; in fact, they are

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \operatorname{diag}(1, 1, \omega) \right\rangle .$$

We have an element T of order 3 in $Z(S_3)$ with $\chi(T) = -3$. Then for $\chi_{15}(1) = 15$ and $\chi_{15} \in B_0(7) \subseteq B_0(3)$, by 3-7 block separation,

$$[G : C(T)]\chi_{15}(\mathrm{T})/15 \equiv [G : C(T)]1/1 \pmod 3,$$

and $\chi_{15}(T) = -3$; otherwise, $\chi_{15}(T) = 6$ and $\chi_{10}(T) = 10$, which is impossible. The character of $C_2(X)$, the skew-symmetric tensors of $X \otimes X$, takes T to $((-3)^2 - (-3))/2 = 6 \neq \chi_{15}(T)$. Therefore, this character is reducible since by [**4**, II, Theorem 1], $B_0(7)$ is the only 7-block of defect 1. It must have irreducible constituents of degrees 1 and 14. The character χ_{21} of $P_2(X)$, the symmetric tensors of $X \otimes X$, takes T to $((-3)^2 + (-3))/2 = 3$ and cannot be $\chi_6 + \chi_{15}$. Since $C_2(X)$ contains χ_0, $P_2(X)$ is irreducible. The character of $C_3(X)$, the skew-symmetric tensors of $X \otimes X \otimes X$, takes T to $9\omega + 9\bar{\omega} + 2 = -7$ and cannot equal $\chi_{10} + \bar{\chi}_{10}$. As a constituent of $(\chi_0 + \chi_{14})\chi_6$ it does not contain χ_0 and must be $\chi_6 + \chi_{14'}$. Then

$$\chi_{14'}(T) = -7 - (-3) = -4 \neq 5 = \chi_{14}(T).$$

Since $C(S_5) = S_5$, $B_0(5)$ consists of χ_0, χ_6, χ_{21}, χ_{14}, and $\chi_{14'}$. If $J \in G$ satisfies $J^{-1}\pi_5 J = \pi_5^{-1}$, then $X(J)$ has eigenvalues $1, -1, 1, -1, \pm 1, \pm 1$, $\chi_0(J) = 1$, $\chi_6(J) = \pm 2$, $\chi_{21}(J) = (2^2 + 6)/2 = 5$, $\chi_{14}(J) = -1 + (2^2 - 6)/2 = -2$, and $\chi_{14'}(J) = \pm(4 + 4 - 12) - (\pm 2) = \pm(-6)$. However,

$$5 = a(J, J, \pi_5) = (|G|/|C(J)|^2)(1 + 2/3 + 25/21 - 4/14 - 36/14) = 0,$$

which is a contradiction.

If a prime power is a degree, it must be 64 or 729. In the first case, the possibilities are $1 + 64 + 64 = 129$ or $1 + 64 = 10 + 55$, $15 + 50$, or $20 + 45$, and these have already been covered. In the second case, we have a degree 2^e5. The possibilities are $1 + 729 = \underline{10 + 720}$, $20 + 710$, $80 + 650$, $160 + 570$, or $\underline{640 + 90}$. By 3-7 block separation, $|G/Z| = 2^a3^635$ and by $t_7 = 2$, we have $a = 7$. The degree equation of $B_0(7)$ of $\mathrm{PSU}_4(3)$ treated in [**17**] is $1 + 729 = 640 + 90$. In the case $1 + 729 = 10 + 720$, let ζ be the character of degree 10. Then, $\zeta\bar{\zeta}(\pi_7) = 2$ and, as $729 > 100$, it must contain the principal character twice, which is a contradiction.

9. The case $t_7 = 1$. Here we assume that G/Z is simple, $|G/Z| = 2^a3^b5^c7$, and $t_7 = 1$. Then, $\chi(\pi_7) = -1$ and $C(\pi_7) = S_7Z$. By unimodularity, if J is a 2-element with $J^{-1}\pi_7 J = \pi_7^{-1}$, then $\chi(J^2) = -I_6$ and $2 \mid |Z|$. Therefore, $\chi\chi$ has no irreducible constituent of degree 6. By [**29**], we assume that G has no irreducible representation of degree 7. The symmetric and skew symmetric tensors of $X \otimes X$ are irreducible or have two irreducible constituents, one of which is linear. Also $X \otimes X$ has irreducible constituents of degrees 15 and 21 or 1, 14, and 21 or 1, 15, and 20. In the latter two cases, replace G by the kernel of the character of degree 1. Then $|Z| = 2$ and χ is real. In the first case $(\chi\bar{\chi}, \chi\bar{\chi}) = (\chi\chi, \chi\chi) = 2$ and $\chi\bar{\chi}$ has irreducible constituents of degrees 1 and 35.

Suppose X has a 5-conjugate Y for all 6-dimensional representations X of G. Then $X \otimes \bar{Y}$ and $X \otimes \bar{X}$ have the same 5-modular constituents and $X \otimes \bar{Y}$ has a constituent in the principal 5-block, $B_0(5)$. Then total degrees of constituents in $B_0(5)$ of $X \otimes \bar{X}$ equals the total for $X \otimes \bar{Y} > 1$, and a character of degree 15, 21, 14, 20, or 35 is in $B_0(5)$ by the last two lines of the above paragraph. If the degree is 35, let U be this constituent of $X \otimes \bar{X}$. By the proof of [**7**, Theorem 1] and the fact that S_5 is abelian, if π_5 is an element of

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order 5, then $U(\pi_5)$ has the eigenvalue 1 a multiple of five times. Let

$$\chi(\pi_5) = a_0 1 + \ldots + a_4\epsilon^4,$$

and let $\sum a_i = 6$. Then $5|(a_0^2 + \ldots + a_4^2 - 1)$. Furthermore, $a_i^2 \equiv 0, \pm 1 \pmod 5$. The possibilities after permuting the a_i are

$$a_0 = 0, a_1^2 \equiv \ldots \equiv a_4^2 \equiv -1; a_0^2 \equiv 1, a_1^2 \equiv \ldots \equiv a_4^2 \equiv 0;$$
$$a_0^2 \equiv a_1^2 \equiv 1, a_2^2 \equiv -1, a_3 \equiv a_4 \equiv 0; \text{ and } a_0^2 \equiv a_1^2 \equiv a_2^2 \equiv 1,$$
$$a_3^2 \equiv a_4^2 \equiv -1.$$

If $a_i^2 \equiv 1$, $a_i = 1$ or 4. If $a_i^2 \equiv -1$, $a_i = 2$ or 3. The first and last cases do not arise since, in these cases, $\sum a_i > 6$. The second case arises only when $\chi(\pi_5) = 1 + 5\epsilon$. In the third case, $\sum a_i > 6$ unless $a_0 = a_1 = 1$, $a_2 = 2$ or 3. Therefore, $\chi(\pi_5) = 1 + 5\epsilon$, contrary to Lemma 2. Therefore, χ is real.

Suppose $5 \| |G|$. Then by [**4**] and unimodularity, $\chi(\pi_5) = 3\epsilon + 3\bar{\epsilon}$ or $2 + 2\epsilon + 2\bar{\epsilon}$; otherwise, $\chi|S_5 \times V = [1 + (\epsilon) + (\bar{\epsilon})]\theta + [1 + (\epsilon^2) + (\epsilon^{-2})]\theta^\sigma$, where $C(S_5) = S_5 \times V$. As θ and θ^σ are real and linear, $1 = \theta^3(\theta^\sigma)^3 = \theta^\sigma\theta^{-1}$ and χ has no 5-conjugate, contrary to assumption. The constituent of $\chi\chi$ in $B_0(5)$ must have degree 14 or 21. Then $X \otimes X$ has irreducible constituents of degree 1, 14, and 21. Also ξ, the character of the symmetric tensors of $X \otimes X$, is irreducible. However, $\xi(\pi_5) = 6\epsilon^2 + 6\epsilon^{-2} + 9$ or $4\epsilon + 4\bar{\epsilon} + 7 + 3\epsilon^2 + 3\epsilon^{-2}$ in contradiction to [**4**].

By arguments in the previous section, χ real implies that $|S_5| \leqq 25$ if S_5 is elementary abelian. We still assume that χ has some 5-conjugate until we state otherwise. As $|Z| = 2$, there are two 7-blocks of defect 1, say B_0 and B_1. Therefore, B_1 is taken to itself by complex conjugation and [**25**, Theorems A and B] apply. Since all characters of degree 6 are real, they appear on the stem and χ has at most four conjugates. Suppose that $k \in G$ has order 25. Then the automorphisms $\gamma \to \gamma^6$ and $\gamma \to \gamma^{-1}$ fix $\chi|S_5$ where $\gamma^5 = \epsilon$. This is impossible as $\chi(k)$ would then have at least ten eigenvalues. Therefore, S_5 is elementary abelian of order 25 and $|G/Z| = 2^a3^b5^27$.

Because χ has a 5-conjugate, $\chi|S_3$ has at most two conjugates and χ is real, $\chi|S_3$ is rational, and $b \leqq [6/(3-1)] + [6/3(3-1)] + \ldots = 4$. Furthermore, $5^23^b2^a \equiv 6 \pmod 7$ and b is odd. If $b = 1$, then $B_0(3)$ has χ_0 and a character of odd degree which by [**29**] must be 7.5 or 7.25, but $35 - 1 = 34$ and $175 + 1 = 176$ are impossible degrees. Therefore, $|G/Z| = 2^a3^35^27$. By section 4, $a = 4$ or 7. If $a = 7$, then by [**15**], G/Z is the Hall-Janko group treated in [**16**]. Otherwise, $|G/Z| = 75600$. By a well known result of algebraic number theory, any automorphism of our 7-modular field R/M lifts to an automorphism of our algebraic number field leaving invariant our local ring R and its maximal ideal M. Then this automorphism induces an automorphism of the tree of any block which it fixes. In particular the automorphism of the 7-modular field of raising elements to their seventh power lifts to an automorphism τ of the algebraic number field containing $Q(\chi_i(G))$, taking 7′-roots of unity to their seventh powers. As S_5 is elementary abelian, τ does not fix χ and must flip the stem. Then $B_1(7)$ has three pairs of characters of equal degree. Because they are faithful on Z, the degrees are even. If there are two pairs of characters of degree 6, then the tree is real with degree equation $6 + 6 + 6 + 6 = 8 + 8 + 8$. Then $S_5 \simeq Z_5 \times Z_5$ is not cyclic, contrary to unimodularity of a 7-modular character of degree 2. Another degree divisible exactly by 2 must be 90, 50, or $3^3 5^2 2$. $3^3 5^2 2$ is too large. This must be in a pair and we are left with $2(90 + 6) = 192$ or $2(50 - 6) = 88$. Degrees not divisible by 5 are 36, 8, 216, and 48. Adding one or a pair of these to the first or second case, we obtain:

$$(192 - 36)/2 = 78,\ (192 - 8)/2 = 92,\ (192 + 216)/2 = 204,$$
$$(192 + 48)/2 = \underline{120},\ 192 - 2.36 = \underline{120},\ 192 - 2.8 = 176,$$
$$192 + 2.216 = 624,\ 192 + 2.48 = 288,\ (88 + 36)/2 = 62,$$
$$(88 + 8)/2 = \underline{48},\ (216 - 88)/2 = 64,\ (88 - 48)/2 = \underline{20},$$
$$88 + 2.36 = \underline{160},\ 88 + 2.8 = 104,\ 2(216) - 88 = 344, \text{ and}$$
$$2.48 - 88 = \underline{8},$$

which has been listed already. The possible degree equations are:

$$6 + 6 + 90 + 90 + 48 = 120 + 120,\ 6 + 6 + 90 + 90 = 36 + 36 + 120,$$
$$6 + 6 + 48 + 48 = 50 + 50 + 8,\ 6 + 6 + 20 + 20 + 48 = 50 + 50, \text{ and}$$
$$6 + 6 + 160 = 36 + 36 + 50 + 50.$$

In the first two cases, $120 + 120 - 90 - 90 = 60 \not\equiv 0 \pmod{25}$ and $90 + 90 - 120 = 60 \not\equiv 0 \pmod{25}$, contrary to 5-7 block separation. In the third case, the stem consists of 6, 8, and 6, since the character of degree 50 has no 5-conjugate, and then $6 + 6 > 8$ gives a contradiction. The fourth case has no tree. The last case is not possible by 5-7 block separation.

Now, we may assume that χ has no 5-conjugate. In particular $25 \nmid |G|$. Suppose that $5 \nmid |G|$. Then no character has degree 15 and by the first paragraph of this section, the skew-symmetric tensors of $X \otimes X$ have irreducible constituents of degrees 1 and 14, χ is real, and $|Z| = 2$. If $\chi|S_3$ is irrational, then the above mentioned automorphism τ taking x to x^7 for x a $7'$-root of unity, induces an automorphism of the tree of $B_1(7)$. Then τ^2 fixes the stem and χ, since χ is real and on the stem. As a primitive 27th root of unity has $9 > 6 = \chi(1)$ images under τ^2, S_3 has exponent 9 or 3. For γ a primitive ninth root of unity, τ^2: $\gamma \rightarrow \gamma^{49}$ and complex conjugation: $\gamma \rightarrow \gamma^{-1}$ generate the automorphism group of $Q(\gamma)$ and fix χ, so $\chi|S_3$ is rational. Then as before, $3^5 \nmid |G|$. Since $[G:S_7Z] \equiv 6 \pmod 7$, 3 divides $|G|$ to the first or third power. Then $|G/Z|\,|2^9 3^3 7 = 96768$. In $B_0(7)$, the degree 288 is too large since $288^2(1 + 1/5) > 96768$. The possible degrees in $B_0(7)$ are 8, 64, 48, 36, 27, and 216. Then there must be three (mod 4) characters of degree 27. It must be three and $|G/Z| = 2^a 3^3 7$, $a = 3$, 6, or 9. Since $216 + 27 + 27 + 27 > 1 + 64 + 64$, the degree 216 is impossible. Only 8, 64, 48, and 36 remain.

Since $27 + 27 + 27 + 48 + 48 > 1 + 64$, the $-1 \pmod 7$ side of the degree equation is $27 + 27 + 27$ or $27 + 27 + 27 + 48$. By the degree equation, taken (mod 3), the other side is $1 + 8 + 36$, $1 + 8 + 36 + 36$, $1 + 64 + 64$, $1 + 64 + 64 + 36$, or $1 + 8 + 8 + 64$. The degree equations are $1 + 8 + 36 + 36 = 27 + 27 + 27$, $1 + 64 + 64 = 27 + 27 + 27 + 48$, and $1 + 8 + 8 + 64 = 27 + 27 + 27$. In the first case, 2-7 block separation implies that $a = 3$, contrary to $\sum x_i^2 > 56.27$. In the second case, $a = 6$, contrary to $\sum x_i > 2^6 3^3 7$. In the last case, 1, 27, 27, and 27 occur in the same 2-block by block separation. The characters of degree 8 must also lie in this 2-block, which is a contradiction.

Suppose that $5 \,\|\, |G|$ and S_5 is not self-centralizing in G/Z. By [**4**] we may write $X(\pi_5) = \operatorname{diag}(1, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4)$ and $[C(\pi_5)]_{5'} = \langle g \rangle$ with $X(g) = \operatorname{diag}(\gamma^{-5}, \gamma, \gamma, \gamma, \gamma, \gamma)$. As G/Z is simple, G is generated by Z and conjugates of the homology $X(g)$. Then the classification in [**20**] of finite primitive collineation groups generated by homologies contradicts the simplicity of G/Z.

Now, we may assume that $C(\pi_5) = S_5 Z$. If $t_5 = 2$, then as in section 8, the degree equation for $B_0(5)$ is $1 + 63 = 64$. By 2-5 block separation, $|G/Z| = 2^6 3^b 35$. Then $b \equiv 10 \pmod{12}$, which is a contradiction. Therefore, $t_5 = 1$. The possibilities for $|G/Z|$ are $2^9 3^4 35$ and $2^7 3^2 35$. Degrees x in $B_0(5)$ satisfy $(x - 1)^2(1 + (1/3)) \leqq 2^9 3^4 35 = 1451520$ and $x \leqq 1043$. The possible degrees in $B_0(5)$ are 1, 64, 14, 56, 224, 896, 384, 21, 84, 336, 36, 126, 504, 216, 189, and 756. Odd degrees are 21 and 189. Degrees not divisible by 3 are 64, 14, 56, 224, and 896. If some degree is 64, then we have $1 + 21 + \ldots = 64 + \ldots$ or $1 + \ldots = 64 + 189 + \ldots$. In the first case, there is another degree divisible at most by 2 which would have to be 21, 189, 14, or 126. Then $42 - 21 = \underline{21}$, $42 + 189 = 231$, $42 + 14 = \underline{56}$, or $126 - 42 = \underline{84}$. In the second case, $64 + 64 + 189 - 1 = 316$ and $64 + 189 + 189 - 1 = 441$, so there is a degree $x \equiv 1 \pmod 5$ with

$$(64 + 189 - 1)/2 \leqq x \leqq 64 + 189 - 1.$$

The possibilities are

$$64 + 189 - 1 - \underline{126} = 126 \text{ and } 64 + 189 - 1 - 216 = \underline{36}.$$

If no degree is 64, then we have another degree not divisible by 7 which must be 384, 216, or 36. We then have characters of degree 1, either 384, 216, or 36, either 21 or 189, and either 14, 56, 224, or 896. Only four of these twenty-four combinations give a degree on our list. The possible degree equations are: $1 + 21 + 21 + 21 = 64$, $1 + 21 + 56 = 14 + 64$, $1 + 21 + 126 = 64 + 84$,

$1 + 126 + 126 = 64 + 189$, $1 + 36 + 216 = 64 + 189$,

$1 + 896 = 189 + 384 + 324$, $1 + 216 + 21 = 14 + 224$,

$1 + 216 = 189 + 14 + 14$, and $1 + 216 + 56 = 189 + 84$.

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In the first five cases 2-7 block separation shows that $2^6 \| |G/Z|$, which is a contradiction. In the last four cases, 216 or $189 | |G/Z|$, $|G/Z| = 2^9 3^4 35$, and the characters of degree 216 and 189 are in a 3-block of defect 1, contrary to 3-7 block separation since $81 \nmid (216 - 189)$.

10. The case where G is a subdirect product of 2 and 3-dimensional groups. This section covers [**19**, Theorem, Case B]. Here $G \subseteq A \times B$, U is a faithful, unimodular, irreducible, 2-dimensional representation of A, V is a faithful, unimodular, irreducible, 3-dimensional representation of B, ζ is the character of U, ξ is the character of V, the representation

$$Y(A \times B) = U(A) \otimes V(B), \quad \text{and} \quad X(G) = Y(G).$$

Also, $\pi_1(G) = A$ and $\pi_2(G) = B$, where π_1 and π_2 are the natural projections of $A \times B$. We may have to alter elements $X(g)$ by scalar multiplication corresponding to changing $U(a)$ and $V(b)$ by scalar multiplication to make them unimodular. After this is done, $Z(A)$ and $Z(B)$ do not have common constituents, so G is a subdirect product instead of just a central, subdirect product. To get $X(g)$ back in its original form, we multiply $U(a)$ and $V(b)$ by unimodular, scalar matrices.

If $V(B)$ is imprimitive, then B has a normal abelian subgroup B_1 of index 3 or 6. Then $H = \pi_2^{-1}(B_1)$ is normal of index 3 or 6 in G and, by quasi-primitivity, $\chi|H = \theta + \theta + \theta$, where θ is a 2-dimensional character of H. Then

$$1 = ||\chi||^2 = \sum_{x \in G} |\chi(x)|^2/|G| \geq \sum_{x \in H} |\chi(x)|^2/6|H| = ||\chi|H||^2/6 \geq 9/6,$$

which is a contradiction. Therefore, $V(B)$ is primitive. Similarly $U(A)$ is primitive. As mentioned in section 2, by [**1**], $A \simeq \mathrm{SL}(2, 5)$, $\mathrm{SL}(2, 3)$, or $\mathrm{GL}(2, 3)$, and $\mathrm{B} \approx \mathrm{PSL}(2, 7)$, A_5, a central extension of Z_3 by A_6, the Hessian group H_3 of order $|\mathrm{SL}(2, 3)|.9.3$, or a subgroup H_2 or H_1 of index 3 or 6 respectively in H_3. If $N \trianglelefteq G$ and $\pi_1(N) \subseteq Z(A)$, then $N = Z_2 \times \pi_2(N)$ or $\pi_2|N$ is faithful. The first case is obtained from the second by adding $-\mathrm{I}_6$. In the first case, if θ is the non-principal linear character of $N = Z_2 \times \pi_2(N)$ with $\pi_2(N)$ in its kernel, then $\chi|N = \theta(\xi\pi_2|N + \xi\pi_2|N)$. Therefore, when $\pi_1(N) \subseteq Z(A)$ or similarly when $\pi_2(N) \subseteq Z(B)$, N does not contradict quasiprimitivity of $X(G)$ since $\pi_2(N) \trianglelefteq B$ and ξ is quasiprimitive.

In the cases where $G = A \times B$, if $N \trianglelefteq G$ and $\pi_1(N) \not\subseteq Z(A)$, since $Z(A/Z(A))$ is trivial the commutator subgroup $[A, N] \not\subseteq Z(A)$, $\pi_1([A, N]) \trianglelefteq A$, and by quasiprimitivity, $\zeta|\pi_1([A, N])$ is irreducible. We may reverse the roles of A and B. Since $[A, N] \times [B, N] \subseteq N$, either

$$\pi_1(N) \subseteq Z(A), \quad \pi_2(N) \subseteq Z(B), \quad \text{or} \quad X|N$$

is irreducible. Therefore, the direct products are quasiprimitive. By the subdirect product theorem, the other possibilities for G are a subgroup of index 60 in $\mathrm{SL}(2, 5) \times A_5$, a subgroup of index 2 in $\mathrm{GL}(2, 3) \times H_i$, $i = 1$ or 2, or a subgroup of index 3, 12, or 24 in $\mathrm{SL}(2, 3) \times H_3$. The first case is covered as a projective representation of A_5. The case where G is of index 2 in a direct product is shown to be quasiprimitive in the same way that the direct products were, since they contain $\mathrm{SL}(2, 3) \times \langle 1\rangle$ and $\langle 1\rangle \times$ (the group of index 2 in H_1). In the case $A \cong \mathrm{SL}(2, 3)$ and $B \cong H_3$, suppose that $N \trianglelefteq G$ and N contradicts quasiprimitivity. Then $\pi_1(N) \not\subseteq Z(A)$ and $\pi_1(N) = A$ or the quaternions, Q, and $\pi_2(N) =$ a normal subgroup of index 1, 3, 12, or 24 in H_3. Then, since $\pi_1(\ker \pi_2|N) \trianglelefteq A$ and $\pi_2(\ker \pi_1|N) \trianglelefteq B$, and N is not a direct product of these groups, N is a subgroup of index 3, 12, or 24 in $A \times B$ or a subgroup of index 4 or 8 in $Q \times H_2$. In any event, N contains a subdirect product M of index 8 in $Q \times H_2$. Then $\pi_2|M$ is faithful. Let

$$L = M \cap \pi_1^{-1}(Z(A)).$$

Then

$$\begin{aligned}\sum_{x\in M} |\chi(x)|^2/|M| &= \sum_{x\in M} |\zeta(\pi_1(x))|^2|\xi(\pi_2(x))|^2/|M| \\ &= \sum_{x\in L} 2^2|\xi(\pi_2(x))|^2/4|L| \\ &= ||\xi|L||^2 = 1,\end{aligned}$$

since $\xi|$(the subgroup of index 4 in H_2) is irreducible. Therefore, $X|N$ is irreducible and $X(G)$ is quasiprimitive and irreducible.

11. Groups of degree 2, 3, 4, 5, **and** 7. Projective representations of $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 8)$, $\mathrm{PSL}(2, 11)$, $\mathrm{PSL}(2, 13)$, A_5, A_7, and their automorphism groups (except for $\mathrm{PSL}(2, 8)$), $\mathrm{PGL}(2, 7)$, $\mathrm{PGL}(2, 11)$, $\mathrm{PGL}(2, 13)$, $\mathscr{S}_5$, and $\mathscr{S}_7$ are classified in [**22**; **26**]. The former reference also treats $A_6 \cong \mathrm{PSL}(2, 9)$ which has outer automorphism group of order 4. The outer automorphisms of $\mathrm{PSL}(2, 9)$ are generated by α, the field automorphism, and β, conjugation by

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$$

where $\gamma = 1 + i$, $i \in \mathrm{GF}(9)$, and $i^2 = -1$. An element F of order 8 in $\mathrm{SL}(2, 9)$ is

$$F = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$

and

$$S_3 = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} |x \in \mathrm{GF}(9)\right\}.$$

Then $F^\alpha = F^3$ and $F^\beta = F$. Let R and T be inverse images of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

in G respectively, the non-trivial central extension of Z_3 by A_6. Then $(R, T)^\alpha = (R, T^{-1}) = (R, T)^{-1}$ and $(R, T)^\beta = (RT, R^{-1}T) = (R, T)^{-1}$. Therefore, $\langle\alpha\beta\rangle$ is the subgroup of outer automorphisms of $\mathrm{PSL}(2, 9)$ with trivial action on $Z(G)$. This automorphism $\alpha\beta$ lifts to G by the uniqueness of the representation group of A_6. The 6-dimensional matrix group $\approx G$ can be enlarged by this automorphism. However, $F^{\alpha\beta} = F^3$ and $\alpha\beta$ permutes the classes of elements of order 8 in $\mathrm{SL}(2, 9)$ and this automorphism cannot enlarge the 6-dimensional matrix group isomorphic to the representation group of A_6. In section 8 we eliminated A_8. By [**29**], the symplectic group $S_6(2)$ has $t_5 = 1$ and a 6-dimensional projective representation would imply that $C(\pi_5)$ is cyclic, which it is not. By [**5**; **15**], $\mathrm{U}_3(3)$ and $\mathrm{PSU}_4(2)$ each have a representation of degree 6.

The degree equation of the 7-block of defect 1 of $\mathrm{U}_3(3)$ is $1 + 32 = 6 + 27$ by [**15**]. Suppose that G is a non-trivial central extension of Z_2 by $\mathrm{U}_3(3)$. Then $B_1(7)$ of G has a character $\chi_{32'}$ faithful and completing the 2-block of defect 1. It contains other degrees not divisible by 3. The block orthogonality equations with $(1, -\pi_7)$ where $-\pi_7$ is of order 14, taken over all 2-blocks containing a character of degree 32 (thus, of defect 1) shows that $B_1(7)$ contains exactly one character of degree 32. 8 is the only other degree not divisible by 3. Since $3 \nmid (32 + 8)$, the degree equation for $B_1(7)$ is $32 + 8 + 8 = 48$. Let c' be the inverse image of c in the character table of $U_3(3)$ in [**15**]. Then $|C(c')| = 9|Z|$, c' is conjugate to c'^{-1}, and c' has order 3. The tree of $B_1(7)$ shows that the characters of degree 8 are algebraic conjugates and the $\chi_8(c')$'s are equal and congruent to $-1 \pmod 3$. Since $|C(c')| = 9|Z|$, $\chi_8(c') = -1$ or 2 and $\chi_{48}(c) = -3$ or 3, which is a contradiction. Therefore, any central extension of Z_2 by $\mathrm{U}_3(3)$ is isomorphic to $Z_2 \times \mathrm{U}_3(3)$.

Now, suppose that G is a non-trivial central extension of Z_3 by $\mathrm{U}_3(3)$. By [**3**, Corollaries 4 and 5], a faithful character of degree 27 is in a 3-block of defect one with its 3-conjugate and a 3-rational character of degree 27, which by its 3-rationality has $Z(G)$ in its kernel. Therefore, a non-principal 7-block has at most one character of degree 27, contrary to the fact that 27 is the only possible odd degree in this block. There can be no projective representation of $\mathrm{U}_3(3)$ with centre of order 6, since Z_2 factors out of any central extension of Z_6 by $\mathrm{U}_3(3)$.

Now, suppose that G is a non-trivial central extension of Z_2 by $\mathrm{PSU}_4(2)$ with a faithful representation of degree 6. We shall make frequent appeal to the character table of $\mathrm{PSU}_4(2)$. There is exactly one degree 64 in $B_0(5)$ and, hence, exactly one degree 64 in $B_1(5)$, the non-principal 7-block of defect 1. By assumption, $\chi_{6'}$ of degree 6 is in $B_1(5)$. There is a degree divisible exactly by 2 since if $\zeta \in B_1(5)$, then $G = G'$, $\det \zeta = 1$, $1 = \det \zeta(-1) = (-1)^{\zeta(1)}$, and $\zeta(1)$ is even. Such degrees must be 6 or 54. Degrees not divisible by 3 are 4 and 16, and there are two characters of degree 4 or one of degree 16. The possibilities are $64 - 6 + 4 + 4 = 66$, $64 - 6 - 16 - 6 = \underline{36}$, and $64 - 6 - 16 + 54 = \underline{96}$. Consider the first degree equation:

$$6 + 6 + 16 + 36 = 64.$$

Take $T \in Z(S_3) \cap S_3'$. Then $\chi_{6'}(T) = -3, -3\omega$, or $-3\bar{\omega}$. In G/Z the class multiplication coefficient

$$a(\bar{T}, \bar{T}^{-1}, \bar{\pi}_5) = \left[\sum_{\chi_i \in B_0(5)} \chi_i(T)\chi_i(T^{-1})\bar{\chi}_i(\pi_5)/\chi_i(1)\right][|G|/|C(T)||C(T^{-1})|]$$
$$= (\text{constant})(1 + 9/6 + 0/81 - 36/24 - 64/64)$$
$$= 0.$$

Therefore, $a(T, T^{-1}, \pi_5) = 0$ in G and the sums of $\chi_i(T)\chi_i(T^{-1})\bar{\chi}_i(\pi_5)/\chi_i(1)$ over $B_0(5) \cup B_1(5)$ and over $B_1(5)$ are also 0. However, $\chi_{64'}(T) = -8$ and $\sum_{B_1(5)} \geqq 9/6 - 64/64 > 0$, which is a contradiction. Suppose that the degree equation for $B_1(7)$ is $6 + 16 + 96 = 54 + 64$. Then $27|\chi_{54}(T)$ implies that $\chi_{54}(T) = 0$ since $27^2 > 648 = |C(T)|/|Z|$, and $\sum_{B_1(5)} \geqq 9/6 - 64/64 > 0$ gives a contradiction.

Now, suppose that G is a non-trivial central extension of Z_3 by $\mathrm{PSU}_4(2)$ with a faithful representation of degree 6. Then $B_1(5)$ has a degree 6 and exactly one degree 81. Another odd degree must be 9. Degrees divisible exactly by 3 are 6, 24, and 96. The possibilities are: $6 + 81 - 9 + 6 = 84$, $78 - 24 = \underline{54}$, and $78 + 96 = 174$. The degree equation is: $6 + 81 = 9 + 24 + 54$. Some involution J in G has $|C(J)| = 576|Z|$. Since the determinant is 1, $\chi_{6'}(J) = \pm 2$; and $\chi_{81'}(J) = 9$. In G/Z,

$$a(\bar{J}, \bar{J}, \bar{\pi}_5) = (\text{constant})(1 + 4/6 + 81/81 - 64/24 - 0/64) = 0.$$

Therefore, since $B_1(5)$ and $B_2(5)$ are conjugate, $\sum_{B_1(5)}$ and $\sum_{B_2(5)}$ are both 0. Now, $8|\chi_{24}(J)$ since $\chi_{24}(J)|G|/|C(J)|\chi_{24}(1)$ is an integer. Then

$$8^2/24 > 4/6 + 81/81$$

shows that $\chi_{24}(J) = 0$. Similarly, $6|\chi_{54}(J)$. Then $3|\chi_9(J)$ since $\sum_{B_1(5)}$ is a local 3-integer. However, $\chi_{6'}(J) + \chi_{81'}(J) = \chi_9(J) + \chi_{24}(J) + \chi_{54}(J)$ gives a contradiction, since 3 divides the right side but not the left.

If χ is a projective representation of $\mathrm{PSU}_4(2)$ with centre of order 6, then the symmetric tensors of $X \otimes X$ have degree 21 and have an irreducible constituent Y of degree $y \equiv 0 \pmod 3$ (using $\det Y([Z(G)]_3) = 1$), and $y \equiv 1 \pmod 5$. Then $y = 6$, which is a contradiction since $\mathrm{PSU}_4(2)$ has no projective 6-dimensional representation with centre of order 3.

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