EARTH TIDE COMPONENTS AND FORCED NUTATIONS

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The relation between the Earth tide components and the forced nutations has been demonstrated in detail in previous publications (Melchior and Georis 1968, Melchior 1971,1973,1976,1977). We will therefore restrict the development to the essential formulae.

One can express the tidal potential of the Moon or the Sun at any point A (r,ϕ,λ) in the interior of the Earth by

$$W(A) = GM \sum_{n=2}^{\infty} \frac{r^n}{d^{n+1}} P_n(\cos z) \quad , \qquad (1)$$

where M is the mass of the disturbing body, d its distance from the centre of mass of the Earth, P_n is the Legendre polynomial of order n, and z is the geocentric zenith distance of the external body at the considered point A.

By expressing the local coordinate z as a function of the equatorial coordinates of the place (ϕ,λ) and of the external body (α,δ),

$$\begin{aligned} \cos z &= \sin\phi \sin\delta + \cos\phi \cos\delta \cos H(A) \\ H(A) &= H - \lambda(A) = \omega t' - \alpha - \lambda(A) \quad , \end{aligned} \qquad (2)$$

where ω is the sidereal velocity of rotation of the Earth, we get the general expression

$$W(A) = \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} W_{\ell m} r^{\ell} P_{\ell}^{m}(\sin\delta) P_{\ell}^{m}(\sin\phi) \cos m \cdot H(A) \quad , \qquad (3)$$

where

$$W_{\ell m} = \frac{2(\ell-m)!\ GM}{(\ell+m)!\ d^{\ell+1}} \quad . \qquad (4)$$

In these expressions d, δ and α are very complicated functions of time. To obtain a purely harmonic development Doodson chose a set of variables which can be considered as linear functions of time during a

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sufficiently short interval (compatible with the duration of the observations):

for the Moon	its mean longitude s
	longitude of the perigee p
	longitude of the node N
	the lunar time τ
for the Sun	its mean longitude h
	longitude of the perigee p_s
	the solar time t

Formulae in terms of these variables were given by Brown for the Moon and by Newcomb for the Sun.

The multiplication of the series so obtained, as needed by equation (3), generates a spectrum of tidal lines of many different frequencies (in the commonly used tidal potential we currently keep 487 waves).

Since $\tau + s = t + h = t'$, sidereal time, one can list the main tidal waves as follows:

Wave symbol	Origin	Argument	Period
$K_1^{☾}$	Moon	$\tau + s$	$23^h 56^m 4^s$
O_1	Moon	$\tau - s$	$25^h 49^m 10^s$
$K_1^{\odot}$	Sun	$t + h$	$23^h 56^m 4^s$
P_1	Sun	$t - h$	$24^h 3^m 57^s$
Q_1	Moon	$(\tau-s) - (s-p)$	$26^h 52^m 6^s$
J_1	Moon	$(\tau+s) + (s-p)$	$23^h 5^m 54^s$
M_1	Moon	$(\tau+s) - (s-p)$	$24^h 49^m 59^s$
π_1	Sun	$(t-h) - (h-p_s)$	$24^h 7^m 56^s$
ψ_1	Sun	$(t+h) + (h-p_s)$	$23^h 52^m 9^s$
S_1	Sun	$(t+h) - (h-p_s)$	$24^h 0^m 0^s$

It is clear that the wave K_1 is partly generated by the Moon (2/3) and partly by the Sun (1/3) and that the two parts, having the same period, cannot be separated by any analysis or observation. It is a luni-solar wave which is precisely associated with the luni-solar precession, as we will see later. The total torque exerted on the planet is

$$\underset{\sim}{N} = \iiint_V (\underset{\sim}{r} \wedge \underset{\sim}{\text{grad}}\, W)\rho \, dv \quad . \tag{5}$$

This integral, extended to the entire volume of the planet, is transformed to

$$\underset{\sim}{N} = -\iiint_V \text{rot}(\rho W \underset{\sim}{r}) dv - \iiint_V (\underset{\sim}{r} \wedge \underset{\sim}{\text{grad}}\, \rho) W \, dv \tag{6}$$

and, using the Ostrogradsky theorem:

$$\underset{\sim}{N} = - \oiint_S (\underset{\sim}{n} \wedge \underset{\sim}{R}) \rho W \, dS - \iiint_V (\underset{\sim}{r} \wedge \underset{\sim}{\text{grad}}\, \rho) W \, dv \quad , \tag{7}$$

where R is the vectorial radius at the external surface and n is the external normal. The first term is zero in the case of a spherical Earth ($\underset{\sim}{n},\underset{\sim}{R}$ parallel) (geometrical ellipticity) while the second term is zero for a density distribution with spherical symmetry ($\underset{\sim}{r}$ parallel to $\underset{\sim}{\text{grad}}\, \rho$) (dynamical ellipticity). A surface integral term exists for every surface of discontinuity of ρ.

The tidal potential has to be introduced into the expression for the torque. Choosing as system of axes the direction of the vernal equinox (Ox_o), the direction of the North pole of the Earth (Oz_o) and the axis perpendicular to the plane x_oOz_o (Oy_o), Melchior and Georis demonstrated that the projections of this torque are

$$\left.\begin{aligned} N_{xo} &= + \sum_\ell J'_\ell \, W_{\ell 1} \, P^1_\ell(\sin \delta) \sin \alpha \quad , \\ N_{yo} &= - \sum_\ell J'_\ell \, W_{\ell 1} \, P^1_\ell(\sin \delta) \cos \alpha \quad , \end{aligned}\right\} \tag{8}$$

where

$$J'_\ell = \frac{1}{2} \ell(\ell+1) a^\ell J_\ell \quad . \tag{9}$$

Thus

$$J'_2 = 3a^2 J_2 = 3a^2 \frac{C-A}{Ma^2} \quad . \tag{10}$$

Introducing the variations of distance, declination and right ascension of the external body with time, we develop the perturbing potential in the form of a sum of simple periodic terms

$$W_{\ell m} P^m_\ell(\sin \delta) \cos mH = K_\ell \sum_i A_{\ell m i} \cos\left[\omega_i t + \frac{1}{2} (\ell - m)\pi\right] , \tag{11}$$

with

$$K_2 = \frac{1}{2} \frac{GM}{c^3} \quad . \tag{12}$$

The tesseral tidal frequency spectrum is symmetric with respect to the central sidereal frequency ω: there are n lines on the left and n lines on the right of ω. Thus we may put

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$$\omega_i = \omega + \Delta\omega_i = 15\overset{\circ}{.}041\ 069 + \Delta\omega_i \tag{13}$$

with

$$\Delta\omega_i = -\Delta\omega_{-i} \quad . \tag{14}$$

This leads to the very simple expressions (for $\ell = 2$, $m = 1$)

$$\begin{aligned} N_{xo} &= - \sum_{i=-n}^{i=+n} K\ J'\ A_i \cos(\Delta\omega_i t) \quad , \\ N_{yo} &= + \sum_{i=-n}^{i=+n} K\ J'\ A_i \sin(\Delta\omega_i t) \quad . \end{aligned} \tag{15}$$

A first important remark can be made now: precession and nutations are movements of the axis of figure of the Earth described in an inertial system of fixed axes, while tides are observed at points fixed with respect to the Earth, rotating with the angular velocity

$$\omega = \frac{2\pi}{t'} = 15\overset{\circ}{.}041\ 069 \text{ per hour} \quad . \tag{16}$$

Thus, the frequency of a nutation ($\Delta\omega_i$) can be directly deduced from the frequency of the corresponding tide by simple subtraction of the "sidereal frequency" (15°.041 per hour of universal time): $\omega_i - \omega = \Delta\omega_i$ as is obvious in formulas (15).

But these formulas permit a second statement: Two waves of symmetric frequency with respect to the sidereal frequency form only one and the same wave of nutation; the sum of their amplitudes (major axis) appears in N_{xo} and their difference (minor axis) in N_{yo},

$$\begin{aligned} N_{xo} &= - \sum_{i=0}^{i=+n} K\ J'(A_i + A_{-i})\cos(\Delta\omega_i t) \quad , \\ N_{yo} &= + \sum_{i=0}^{i=+n} K\ J'(A_i - A_{-i})\sin(\Delta\omega_i t) \quad . \end{aligned} \tag{17}$$

Conversely we may consider an elliptic nutation as equivalent to two circular nutations of equal and opposite velocity corresponding to the two symmetrical tidal waves.

The rotations of the system of axes are given by

$$\dot\theta = + \frac{N_{yo}}{C\omega} \quad , \quad (\sin\theta)\cdot\dot\psi = + \frac{N_{xo}}{C\omega} \quad , \tag{18}$$

and using (17) we obtain the nutations in terms of the development in tidal waves:

$$\dot{\theta} = \sum_i \frac{K}{C\omega} J'(A_i - A_{-i})\sin(\Delta\omega_i \cdot t) \quad ,$$
$$\sin\theta \cdot \dot{\psi} = - \sum_i \frac{K}{C\omega} J'(A_i + A_{-i})\cos(\Delta\omega_i \cdot t) \quad . \qquad (19)$$

Let us introduce a dimensionless constant:

$$E = \frac{2}{a^2} D \left(\frac{C-A}{C\omega^2}\right) = \frac{3}{2} \frac{GM}{c^3} \frac{C-A}{C\omega^2} = \frac{KJ'}{C\omega^2} \quad . \qquad (20)$$

Its value, expressed in seconds of arc, is for the Moon

$$E_{☾} = 0''.016\ 4 \quad . \qquad (21)$$

Then

$$\dot{\theta} = +E_{☾}\ \omega \sum_i (A_i - A_{-i})\sin(\Delta\omega_i t) \quad ,$$
$$\dot{\psi} \sin\theta = -E_{☾}\ \omega \sum_i (A_i + A_{-i})\cos(\Delta\omega_i t) \quad . \qquad (22)$$

The K_1 tidal field of force is distributed according to the cos(τ+s) function, i.e., the cosine of the sidereal time or hour angle of the vernal equinox. It therefore permanently points towards the vernal equinox ($\dot{\theta}$ axis), and the torques produced have no resultant component along the $\dot{\theta}$ axis. Instead, they act along the direction 90° away; i.e., $\psi \sin\theta$.

The equations for K_1 give

$$\dot{\psi} = -E_{☾} \frac{\omega A(K_1)}{\sin\theta} \quad , \quad \dot{\theta} = 0 \quad , \qquad (23)$$

and from

$$E = 0''.016\ 44 \quad , \quad \omega = 7.292 \times 10^{-5}\ s^{-1} \quad ,$$
$$\sin^{-1}\theta = 2.512 \quad , \quad A = 0.530\ 5 \quad ,$$

we obtain

$\dot{\psi} = -50''.38$ per year, the luni-solar precession constant.

The nutations are obtained by integration of equations (22):

$$\Delta\theta = -E_{☾} \sum_i \frac{\omega}{\Delta\omega_i} (A_i - A_{-i})\cos(\Delta\omega_i t) \quad ,$$
$$\Delta\psi = -\frac{E_{☾}}{\sin\theta} \sum_i \frac{\omega}{\Delta\omega_i} (A_i + A_{-i})\sin(\Delta\omega_i t) \quad . \qquad (24)$$

The presence of $\Delta\omega_i$ in the denominator shows that the waves give rise to nutations of an amplitude which becomes lower as their frequency diverges from that of the sidereal day (wave K_1), even when the amplitude of the tide is comparable to that of K_1 (this is the case with O_1 versus P_1).

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We observe that tidal waves symmetrical with respect to K_1 and of equal amplitude ($A_i = A_{-i}$) do not cause nutations in obliquity ($\Delta\theta=0$) but only nutations in longitude. This is the case for waves generated by the ellipticity of the orbits:

NO_1 and J_1 for the Moon,

S_1 and ψ_1 for the Sun.

The periods of the nutations associated with the ellipticity of the orbits are evidently a month and a year. In the sense of mechanics it seems unsuitable to classify these components among the "short-period nutations," as they do not practically alter the angle θ and show only a variation of ψ, that is a precession. The two components produced by the ellipticity of the orbits should logically have been named "short-period precessions."

The forced diurnal nutations described inside the Earth and associated with the precession-nutations in space (often called Oppolzer terms) may be deduced as a function of the tidal components by introducing the expressions (22) in the Euler kinematic relations. It is found that they have the same frequencies as the corresponding tidal waves and amplitudes equal to those of these tidal waves multiplied by the factor $(\omega^2a^3/GM)/[(C-A)/C]$, which is nearly unity.

General comments on the Earth tide observations

One of the most important points in tidal observations is the calibration of the instruments. What we are doing indeed is to compare amplitudes and phases measured by dynamometers (pendulums and gravimeters) with a model of tidal forces which basically depends upon the value given to the mass of the Moon. The calibration of the Verbaandert-Melchior quartz horizontal pendulums is presently achieved to 0.5% by the use of a special device invented by Verbaandert. This device allows us to tie the instrument's sensitivity to a well-known spectral line. Here the systematic errors are lower than the accidental errors.

When calibrating the gravimeters it is more difficult to avoid systematic errors and I am not sure that they could not be more important than the accidental errors. In Bruxelles we compared some 30 instruments from different makers (Geodynamics, LaCoste-Romberg, and Askania) calibrated in different ways, and we defined for each instrument a frequency dependent rheological model which is then used for all the data reduction operations.

Another important point for the separation of the different diurnal tidal waves is that the instrument must be made independent of barometric effects. The gravimeters are more or less free from that effect because there is a compensating capsule fixed on the beam (opposite to the mass with respect to the rotating point). A pendulum, however, is very sensitive to deformation of its support. To avoid spurious tilts due to the effect of pressure changes on its base, the instrument's box should not be sealed (which is erroneously often the case). The VM pendulums are constructed in such a way that this barometric effect is completely avoided.

Moreover experience shows that pendulums must be installed at a minimum depth of 50 meters. Unsatisfactory installations are obvious when the results of analysis exhibit a large spurious atmospheric component (called wave S_1). When this is the case all diurnal components are spoiled and cannot be used for our purpose. Repeated calibrations and elimination of pressure and temperature effects are the criteria that I used for the selection of the best tidal series.

Finally, it is clear that to separate very nearby tidal frequencies, longer and longer series of observations are needed. Again, for our purpose a minimum of one year of observations is needed to correctly separate the wave K_1 (precession) from the wave P_1 (semi-annual nutation). In Europe these waves have respective amplitudes of 0".006 and 0".002 in tilt. It is clear that for separating waves of amplitudes smaller than 0".001, several years of registration are needed.

For the very small waves NO_1, ϕ_1 and ψ_1 we have used three series obtained with VM pendulums and covering 10 years each. Unfortunately we could not find sufficient agreement in the gravimeter series to give the ψ_1 wave in this component with sufficient confidence. The results are presented in the following four tables, and clearly fit the Molodensky models very well.

In the Tables 2, 3, and 4, which give the experimental results, one will observe the evident good fit for the main waves $K_1 P_1 O_1$ with the model I. This is true also for the smaller waves, especially $\delta(\phi_1)$ and $\gamma(\psi_1)$, which should lead us to prefer indeed Model I.

Evidently one cannot expect to observe a pure resonance as it is given by the crude models used by Molodensky as well as by Jeffreys and Vicente. The rigidity of the core is not absolutely zero. Viscosity of the core material and consequently friction at the core mantle boundary should produce a damping of the core free nutation, a reduction of the resonance effect mainly on the ψ_1 wave, the nearest to the resonance frequency, and phase lags. We have presently no real possibility of measuring phase lags with sufficient precision, but the amplitude factors observed, particularly $\gamma(\psi_1)$, show no reduction with respect to the Molodensky models. This should mean that the viscosity of the core is very low and that there should be no observable deviations of the nutation amplitudes from the dissipationless values. The damping factor introduced by Sasao, Okamoto and Sakai (1977) is probably much less than what they propose (0.2 year^{-1} instead of 1 year^{-1}).

Table 1. Liquid core effects on Earth tide measurements: Molodensky theoretical models

	Model I		Model II	
	γ	δ	γ	δ
Main waves				
K_1	0.733	1.137	0.726	1.143
P_1	0.700	1.153	0.697	1.158
O_1	0.687	1.160	0.686	1.164
Small waves				
Q_1	0.686	1.160	0.686	1.164
π_1	0.696	1.155	0.694	1.160
ψ_1	0.520	1.242	0.527	1.246
ϕ_1	0.657	1.174	0.658	1.178
J_1	0.684	1.161	0.683	1.166
OO_1	0.685	1.161	0.684	1.165

$\gamma = 1 + k - h$ is the amplitude factor for the horizontal components.

$\delta = 1 + h - (3/2)k$ is the amplitude factor for the vertical components.

Table 2. Earth tides observations: LARGER DIURNAL WAVES - Vertical Component

	Amplitude μgals	$\delta = 1+h-(3/2)k$	α
Mean results from 22 series where the separation of P_1 from K_1 has been made[a]			
K_1	52	1.1436 ± 0.0147	+0.08° ± 0.47°
P_1	17	1.1502 ± 0.0210	+0.25° ± 0.63°
O_1	35	1.1608 ± 0.0086	-0.16° ± 0.42°
Results from 26 other European stations where the group $K_1P_1S_1$ could not be separated			
$K_1P_1S_1$	69	1.150 ± 0.016	+0.15° ± 1.12°
O_1	35	1.160 ± 0.015	-0.21° ± 0.47°

[a]These series are from Western Europe plus Hyderabad (India), Armidale and Alice Springs (Australia) and four series from North America.

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Table 3. Earth Tide Observations: SMALLER DIURNAL WAVES - Vertical Component (arithmetic means)

Wave	Group	Amplitude in μgals	Number of series	δ = 1+h-(3/2)k	α
2Q1	12-21	0.9	8	1.166±0.044	-0.59°±3.70°
σ1	22-32	1.1	9	1.166±0.041	-0.53°±2.51°
Q1	33-52	7.3	21	1.160±0.018	-0.48°±1.15°
P1	53-62	1.3	9	1.159±0.032	+0.10°±1.31°
NO1	89-103	2.3	10	1.150±0.017	-0.42°±1.52°
π1	111-113	0.9	8	1.158±0.054	-0.22°±2.62°
ϕ1	137-143	0.7	9	1.175±0.092	+3.13°±3.35°
J1	152-165	3.2	17	1.171±0.020	+0.24°±1.45°
OO1	173-183	1.8	12	1.162±0.024	+0.01°±1.46°

The nine basic series are those from Frankfurt (3 series), Hannover (1 series), Sèvres (1 series), Bruxelles (2 series), and Ottawa (2 series). Among the other stations, those outside of Europe are Hyderabad (India), Armidale and Alice Springs (Australia) and McDonald (USA). The dashed line indicates the position of the resonance line.

Table 4. Earth Tide Observations: DIURNAL WAVES - Horizontal East West Component[a]

Wave	Group	Amplitude 0".001	Number of series	γ = 1+h-k
K1	124-134	5.7	14	0.7530±0.0079
P1	114-120	1.7	14	0.7227±0.0113
O1	63-78	3.9	14	0.6603±0.0088
		Minor Components		
Q1	33-52	0.7	12	0.618±0.015
NO1	89-103	0.3	3	0.720±0.023
ψ1	135-136	0.02	3	0.552±0.032
ϕ1	137-143	0.07	2	0.680(±0.007)
J1	152-165	0.3	12	0.679±0.033
OO1	173-183	0.2	11	0.669±0.075

[a]Results from 14 European stations (Verbaandert-Melchior quartz pendulums).

For the very small waves the series used are two from Dourbes and one from Walferdange. For γ(ψ1) they give, respectively, 0.585, 0.520 and 0.549. The dashed line indicates the position of the resonance line.

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I personally feel that owing to the results obtained from theoretical models, which are well confirmed by the geophysical observations of the Earth tides as well as by the experimental determination of nutation corrections on the basis of the very long series of classical astronomical measurements, the astronomers can trust the proposed corrections and should accordingly modify their nutation series.

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